In Algebra a radical is an expression containing square-root or cube-root. A *nested radical* contains a radical expression inside another radical expression. For example the following expression is a nested radical:

$$x = \sqrt{1 + x\sqrt{1 + y}}\ .$$

If it continues without an end, it is called an *infinite nested radical*. These infinite radicals are quite fascinating. For example, a simple integer like 2 may be written as an infinite nested radical:

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}\ ,$$

where the $\cdots$ indicate that the nested expression continues for ever without an end. Therefore if we stop after four or five square roots, we get an approximation for 2. (Try it and see.) The approximation improves as we include more and more square-root nests but never equals the exact value until the infinite nests are summed (in principle, of course, you really can't sum them in practice). This is a simple example but of course you can have slightly more complicated infinite nested radicals.

Ramanujan's nested radical problem

The famous mathematician **Srinivasa Ramanujan** posed one such problem more than hundred years ago. He wanted to know the value of this beautiful nested expression:

$$? = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \cdots}}}}}}\ .$$

In fact the answer is very simple, it is simply equal to 3. Why is that? Let us start with the answer. We have,

$$3 = \sqrt{9} = \sqrt{1 + 8}\ .$$

Writing 8 as, $8 = 2 \times 4$, and further using $4 = \sqrt{16}$, we can write 8 as, $8 = 2\sqrt{16}$. Substituting in the above equation, we have,

$$3 = \sqrt{1 + 2\sqrt{16}}\,.$$

You may already have noticed that you can keep repeating this trick: write $16 = 1 + 15 = 1 + (3 \times 5) = 1 + (3 \times \sqrt{25})$, etc., to get, sequentially,

$$\begin{aligned}
3 &= \sqrt{1 + 2\sqrt{1 + 3\sqrt{25}}}\,, \\
3 &= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{36}}}}\,, \\
3 &= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \cdots}}}}}}\,,
\end{aligned}$$

and so we have the proof!

Infinite nested radicals for larger integers?

Now we have infinite nested square-root expression for 2,3. Obviously we may ask the next question, what about expressions for 4,5,..., or in general any positive integer n? To find the answer, we just need to follow in the foot-steps of Ramanujan's proof for 3. Let us do this now—just square the Ramanujan expression for 3. We have,

$$9 = 1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \cdots}}}}}\,.$$

Take the first number, 1, to the left hand side to get $(9 - 1) = 8$ and then divide the entire equation by 2. We have,

$$4 = \sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + 7\sqrt{1 + \cdots}}}}}}\,.$$

There you have it for 4. Now things become simple, square the expression for 4 and we get,

$$16 = 1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + 7\sqrt{1 + \cdots}}}}}\,.$$

Simplifying again in the same way by taking 1 to the left hand side and dividing by 3 this time, we have,

$$5 = \sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + 7\sqrt{1 + 8\sqrt{1 + \cdots}}}}}} \, ,$$

and we may continue endlessly for all positive integers so that the general expression for $n \geq 3$ is given by

$$n = \sqrt{1 + (n-1)\sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{1 + (n+3)\sqrt{1 + \cdots}}}}}} \, .$$

The golden nested radical

Here is another infinite nested square-root that looks nice and simple but the answer is in fact a famous number:

$$a = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}} \, ,$$

which involves only the number 1. To find the value of a, we use the same trick. Just square the expression to get,

$$a^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}} \, ,$$

or simply $a^2 = 1 + a$, whose solution is straight-forward (try it!):

$$a = \frac{1 + \sqrt{5}}{2} \, .$$

This is the famous **Golden Ratio** that was written about in the Sep-Oct, 2013, issue of JM! If you have such an old copy, you can find out that golden mean may also be written as a continued fraction involving only the number 1.

You can try to construct such examples on your own and have fun with numbers! •