How Polya found sums of square roots

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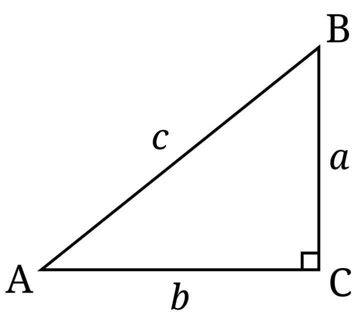
Dear Basheer, please redo this title in a nice font. I have left 2.5 cm space at the top of the article for this. Hope it's enough. Thanks, Indu

In geometry we first come across *square roots*. We learn that for a right-angled triangle ABC with sides BC, CA, AB of length a, b, c, AB of length c being the *hypotenuse*, the longest side which faces the right angle at C, we have Pythagoras's equation, $a^2 + b^2 = c^2$.



So, if length a is 3 and length b is 4, length c can be calculated. $a^2 + b^2$ is $3^2 + 4^2$, which is $9 + 16$, that is, 25. Since we know c^2 is 25, c must be its square root $\sqrt{25}$, which is 5.

Things may not be that simple. If for the same kind of triangle, we have a is 1 and b is 2, what is c? This time $a^2 + b^2$ is $1^2 + 2^2$, which is $1 + 4$, that is, 5. So c must be its square root $\sqrt{5}$. We cannot simplify further. Of course a calculator can show you what this number is—somewhere between 2 and 3.

Why? Because $2^2 = 4$ and $3^2 = 9$. If $c^2 = 5$ then c must be somewhere between 2 and 3, and perhaps a little closer to 2.

So $5 = c^2 = c \times c > 4 = 2 \times 2$. Geometrically, a square with sides of length 2 has area 4, so a square with area 5 must have sides little longer than 2.

So we have a geometrical demonstration of a property, not as famous as Pythagoras's equation: For two positive numbers, call them c and d, if $c^2 > d^2$, then $c > d$. Notice that there is nothing here about geometry, this is just a property of numbers. We will name this the **Square Root Property** (SRP in short).

For example, consider two positive whole numbers; call them a and b. If $a > b$, then $\sqrt{a} > \sqrt{b}$. A mathematician would say that $a > b$ is *sufficient* to show $\sqrt{a} > \sqrt{b}$. Since $5 > 4$ and $9 > 5$, these two conditions are sufficient to show $\sqrt{5} > 2$ and $3 > \sqrt{5}$, that is, $\sqrt{5}$ is between 2 and 3.

Mathematicians always like to extend what they know. So it was not long before a mathematician asked this question: if you have *four* positive whole numbers a, b, c, and d, what are the conditions sufficient to show that $\sqrt{a} + \sqrt{b} > \sqrt{c} + \sqrt{d}$?

A little thought will tell you that if $a > c$ and $b > d$, then this is sufficient. If you like, think about this geometrically, but like before, we are only talking about properties of numbers. So $a > d$ and $b > c$, this is also sufficient. You can try coming up with more sufficient conditions, or turning the question around a little bit.

But there are hard questions here. What if $a < c$ and $b > d$? For example, is $\sqrt{2} + \sqrt{7} > \sqrt{3} + \sqrt{5}$? That is hard to tell without a calculator.

Here is a demonstration that this is really so. The proof was given by the great Hungarian mathematician George Polya in one of his books. It uses another property of numbers, which we will call the **Square Equation**:

$$(a + b)^2 = a^2 + b^2 + 2 \times a \times b \ .$$

If you want to see this geometrically, take a square of side $a + b$, and inside it mark out squares of side a and side b at two opposite corners, and show that the remaining two rectangles both have area $a \times b$.

Let us get back to numbers and Polya's problem. You agree, he says, that $224 > 9$? Then you must also agree that $\sqrt{224} > \sqrt{9}$. That is because of our Square Root Property, SRP.

Now he does some simplification. The left hand side is $4 \times \sqrt{14}$ because $224 = 16 \times 14$. So you agree that

$$4 \times \sqrt{14} > 3 \ .$$

(Is it easy to see this directly, without using any sufficient condition?)

Now Polya does another trick. He adds a number on both sides, to get

$$57 + \left(4 \times \sqrt{14}\right) > 57 + (3) \ \ ,$$

or written another way,

$$1 + 56 + 4 \times \sqrt{14} > 60 \ .$$

Do you agree?

Now he uses this as a sufficient condition for SRP, to convince us that

$$\sqrt{1 + 56 + 4 \times \sqrt{14}} > \sqrt{60} \ .$$

Here is where the Square Equation comes in, because

$$1 + 56 + 4 \times \sqrt{14} = (1 + 2 \times \sqrt{14})^2 \ .$$

Do you see this? So

$$1 + 2 \times \sqrt{14} > 2 \times \sqrt{15} \ ,$$

where the right hand side is simplified a little bit.

George Pólya, 1887–1985

Polya was a Hungarian mathematician who worked for some time in Switzerland before moving to the United States, where he lived for 40 years till his death at the age of 97.

He worked on a number of areas of mathematics, but he is best known for his series of books on *"How to solve it, Mathematics and Plausible Reasoning"*. In these books, Polya discussed problem solving for both mathematical and non-mathematical problems. He also included advice for students on how to learn and teach mathematics.

The book has been translated into over 17 languages. Apart from telling the student how to make a plan to solve a problem (and carry it out), Polya stated a very important principle:

"Much can be gained by taking the time to reflect and look back at what you have done, what worked, and what didn't. Doing this will enable you to predict what strategy to use to solve future problems."

Okay so far? Now just look at what you have shown. Since the values of $\sqrt{14}$ and $\sqrt{15}$ are difficult to get without a calculator, it would be hard for you to believe this. Polya has shown you how to convince yourself. He will now do this magic one more time. First he adds 8 on both sides to get,

$$8 + 1 + 2 \times \sqrt{14} > 8 + 2 \times \sqrt{15} \ ,$$

which can be rewritten as

$$2 + 7 + 2 \times \sqrt{14} > 3 + 5 + 2 \times \sqrt{15} \ .$$

Agreed? So using the SRP, we have,

$$\sqrt{2 + 7 + 2 \times \sqrt{14}} > \sqrt{3 + 5 + 2 \times \sqrt{15}} \ .$$

Now look closely at this equation. You can see that the Square Equation can be used both on the left hand side and the right hand side. Doing this, we see that the two sides can be rewritten as

$$\sqrt{2} + \sqrt{7} > \sqrt{3} + \sqrt{5} \ .$$

If you find this sort of problem solving amusing and interesting, you can try to take this a little further. What Polya has done is actually shown you a *method* to answer the four-number question. But this method will not work always.

Try to find four whole numbers a, b, c, d, with $a < c$ and $b > d$ where the method breaks down.

–Compiled from several sources