what are the conditions sufficient to show that $\sqrt{a} + \sqrt{b} > \sqrt{c} + \sqrt{d}$?

A little thought will tell you that if $a > c$ and $b > d$, then this is sufficient. If you like, think about this geometrically, but like before, we are only talking about properties of numbers. So $a > d$ and $b > c$, this is also sufficient. You can try coming up with more sufficient conditions, or turning the question around a little bit.

But there are hard questions here. What if $a < c$ and $b > d$? For example, is $\sqrt{2} + \sqrt{7} > \sqrt{3} + \sqrt{5}$? That is hard to tell without a calculator.

Here is a demonstration that this is really so. The proof was given by the great Hungarian mathematician George Polya in one of his books. It uses another property of numbers, which we will call the **Square Equation**:

$$(a + b)^2 = a^2 + b^2 + 2 \times a \times b \ .$$

If you want to see this geometrically, take a square of side $a + b$, and inside it mark out squares of side a and side b at two opposite corners, and show that the remaining two rectangles both have area $a \times b$.

Let us get back to numbers and Polya's problem. You agree, he says, that $224 > 9$? Then you must also agree that $\sqrt{224} > \sqrt{9}$. That is because of our Square Root Property, SRP.

Now he does some simplification. The left hand side is $4 \times \sqrt{14}$ because $224 = 16 \times 14$. So you agree that

$$4 \times \sqrt{14} > 3 \ .$$

(Is it easy to see this directly, without using any sufficient condition?)

Now Polya does another trick. He adds a number on both sides, to get

$$57 + \left(4 \times \sqrt{14}\right) > 57 + (3) \ ,$$

or written another way,

$$1 + 56 + 4 \times \sqrt{14} > 60 \ .$$

Do you agree?

Now he uses this as a sufficient condition for SRP, to convince us that

$$\sqrt{1 + 56 + 4 \times \sqrt{14}} > \sqrt{60} \ .$$

Here is where the Square Equation comes in, because

$$1 + 56 + 4 \times \sqrt{14} = (1 + 2 \times \sqrt{14})^2 \ .$$