Never ending sequences

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I. Can you add up all the terms in a never ending sequence? For example, what is the value of

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

How can you find a definite value for an answer when it keeps on continuing? It is not always the case that a *sequence* like this can add to a definite value, but this one does. Consider the sequence:

$$\begin{aligned} 1 &= 1\,; \\ 1 + \frac{1}{2} &= \frac{3}{2}\,; \\ 1 + \frac{1}{2} + \frac{1}{4} &= \frac{7}{4}\,; \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{15}{8}\,; \\ \cdots & \quad \cdots \end{aligned}$$

This sequence of numbers $(1, 3/2, 7/4, 15/8, ...)$ converges to a “limit”. It is this limit which we call the “value” of the infinite sum.

How do we find this value?

If we assume it exists and just want to find what it is, let’s call it S. Now

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

So, if we multiply it by 1/2, we get

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Now, if we subtract the second equation from the first, the 1/2, 1/4, 1/8, etc. all cancel, and we get $S - (1/2)S = 1$. This means $S/2 = 1$ and so the sum of all the numbers, $S = 2$.

This sum can be easily visualised as in Figure 1.

II. Here is another example. What is the value of

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

Look at Figure 2, and you know the answer immediately. Now try and give a proof of this as we did above for the first example.

III. A slightly more difficult exercise now: look at the sequence of pictures in Figure 3 and write down both the infinite sum and its value! The answer is on page 32. Bonus: In using this technique, we have assumed that the infinite sum exists, then found the value. But we can also use it to tell whether the sum exists or not. Can you figure out how ?