Witnessing Nonnegative Rank

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30 September 2024

[0] Based on "Pavel Hrubeš, Hard Submatrices for Non-Negative Rank and Communication Complexity, CCC 2024".

Outline

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- $\text{rank}(M) \leq$ inn. dim. of any factorization.
- Proof:

$$M = XY \implies \text{rank}(M) \leq \min(\text{rank}(X), \text{rank}(Y)) = \text{inn. dim.}$$

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- How much close to the nnr can we get if $\mathcal{O}(r^c)$ columns are allowed?

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Approx. ratio of (M, r^3) *is* $\mathcal{O}(\log n)$.

This means there is a submatrix with r^3 columns and nnr $\Omega(\frac{r}{\log n})$.

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- Thus, M'' is a square witness with

$$\frac{r}{\mathrm{nnr}(M'')} = \mathcal{O}((\log n)(\log m)).$$

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Observe $\binom{[n]}{k}$ has both properties.

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For $k = r^3$, we expect $|\mathcal{G}_k| \leq 2^k$.

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- S is nonnegative iff $P_0 \subseteq P_1$

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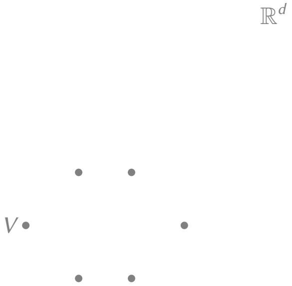

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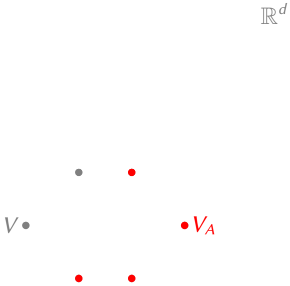

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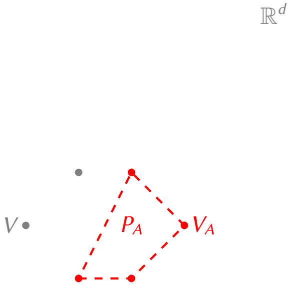

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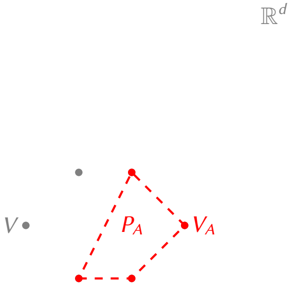

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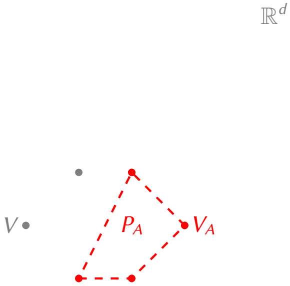

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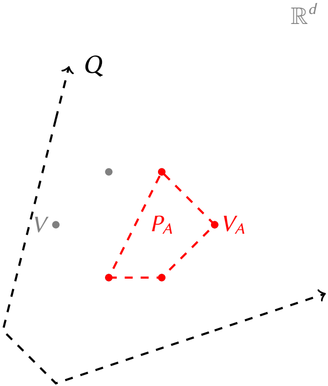

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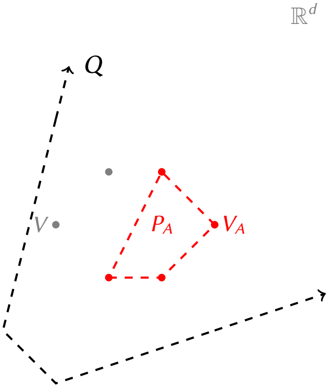

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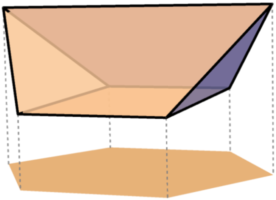

Source: Kwan, Sauermann, Zhao 2022

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For a polytope $P := P_0 = P_1$, $xc(P) = \mathrm{nnr}(S)$ (Yannakakis 1991).

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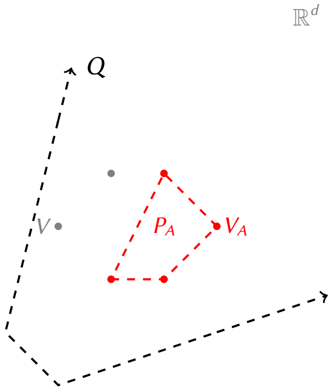

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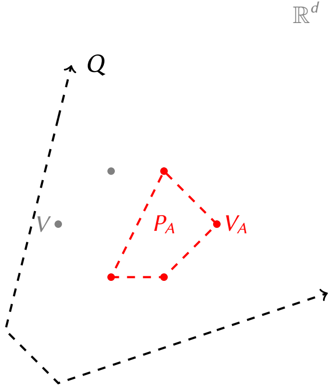

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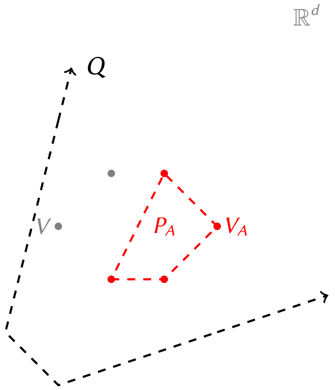

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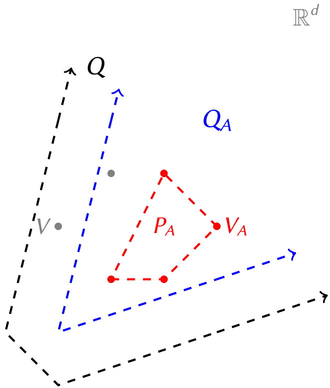

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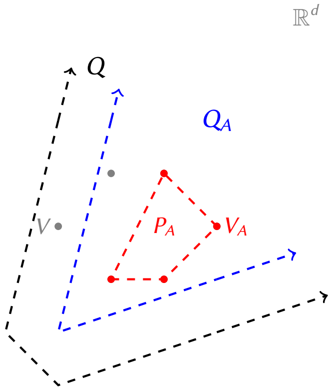

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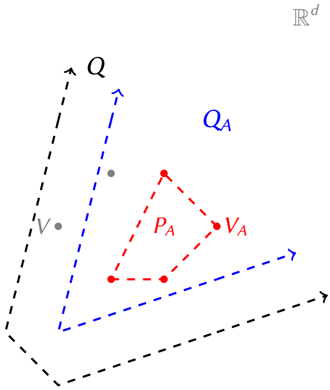

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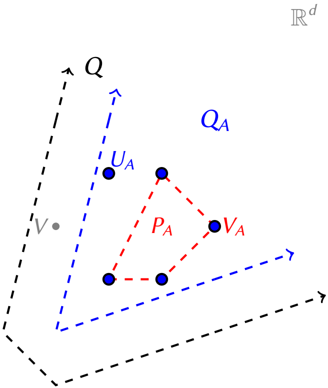

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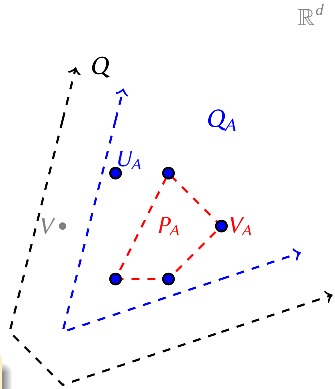

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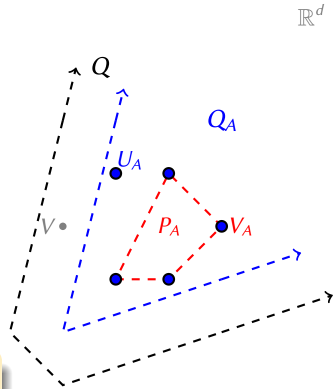

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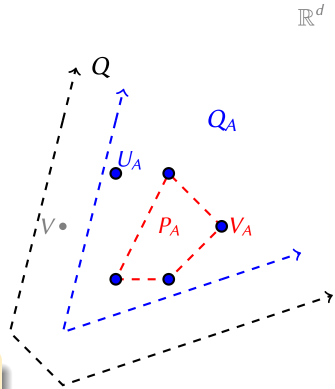

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Thus we have a cover for V

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- A polyhedron $P \subseteq \mathbb{R}^d$ is a *separating polyhedron* for U if

$$P \cap V = U.$$

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- For each $x \in V$,

$$x \in U \text{ iff } \exists_{y \in \mathbb{R}^s} \quad Cx + Dy = b, y \geq 0$$

Step 6: Separating polyhedron for a witness

- $P_A \subseteq Q$ with Slack matrix M_A.
- There is a polyhedron Q_A with

$$\mathrm{xc}(Q_A) \le \mathrm{nnr}(M_A).$$

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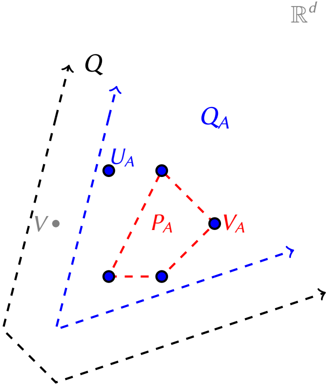

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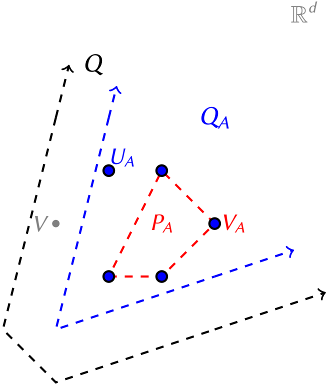

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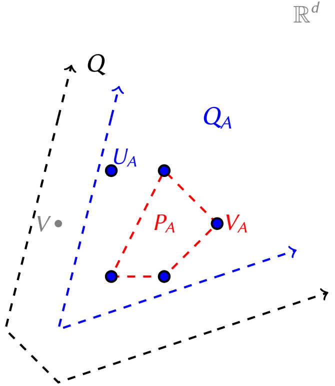

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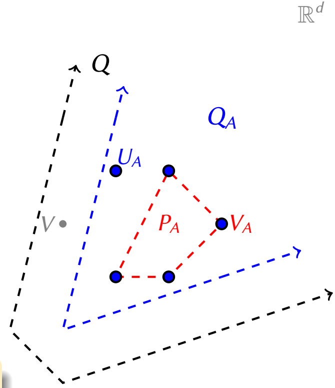

Since $\operatorname{nnr}(M_A) \leq s_k$ for all $A \in \binom{[n]}{k}$,

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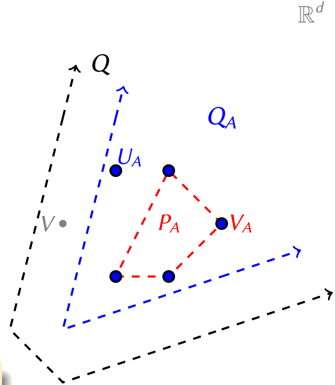

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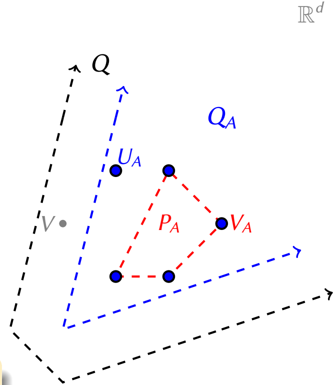

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Theorem. $|\mathcal{H}_s| = 2^{\mathcal{O}((s+d)^3 \log n)}$

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- Approx. ratio of (M, r^3) is $\mathcal{O}(\log n)$.
- There exists a submatrix with at most r^3 cols and nnr $\Omega(\frac{r}{\log n})$.

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Thank you! Any questions?

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- It is enough to focus of matrices of order 2^k with boolean rank $2^{\Omega(\sqrt{k})}$.