Towards the Real τ-Conjecture

MSc Thesis

Hitesh Wankhede

The Institute of Mathematical Sciences

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Outline

1. Real τ-Conjecture and History ($\leq$ 2011)
2. Wronskian Approach (2014)
3. Complex Variant (2013)

Polynomial Evaluation

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- $1 + \alpha(3 + \alpha(5 + 7\alpha)) \rightarrow$ 3 multiplications and 3 additions

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$$\sum_{i=0}^{n} a_i x^i = a_0 + x(a_1 + \cdots + x\,(a_{n-2} + x(a_{n-1} + xa_n))$$

at most n multiplications and n additions suffice.

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- Evaluate x^n at α using only $\times$? (Dellac 1894, Scholz 1937)

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- compute an integer using just addition starting from '1' $\rightarrow$ *addition chain* (Scholz 1937)

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- Borodin & Cook (1976), Risler (1985), Shub & Smale (1995)

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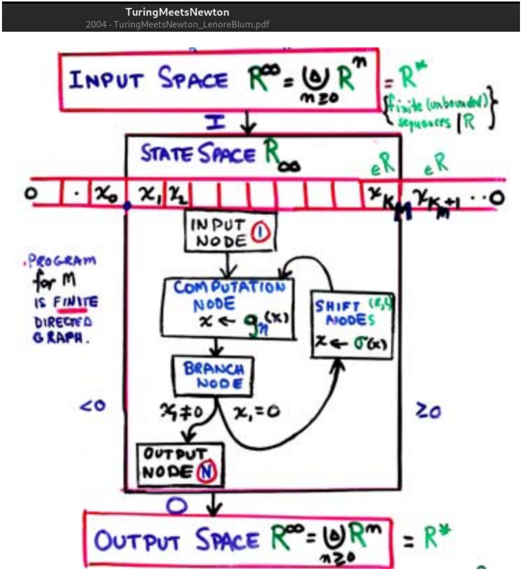
Theorem: τ-Conjecture implies $P_{\mathbb{C}} \neq NP_{\mathbb{C}}$ in BSS model over $\mathbb{C}$

BSS model over a ring R

For input $x \in \mathbb{C}^n$, $\text{size}(x) := n$



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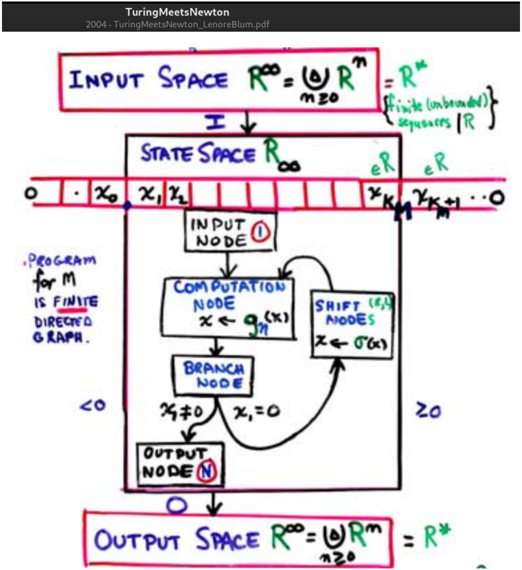
$L \subseteq \mathbb{C}^\infty$ is in $P_{\mathbb{C}}$ if there is a machine over $\mathbb{C}$ computing the characteristic function $\mathbb{1}_L$ and on all valid instances x, the computation length is at most $\text{poly}(\text{size}(x))$.



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- (Bürgisser 2009)

  τ-Conjecture implies $\text{PERM}_n \notin \text{VP}^0$

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- $(f_n) \in \mathrm{VP}^0$ if (f_n) is computable by such an SLP (φ_n) with $\tau(f_n)$ as well as intermediate coeffs/degree polynomially bounded in n. (Malod 2003)

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- $\mathrm{PERM}_n := \mathrm{permanent}([x_{i,j}]_{1 \leq i,j \leq n})$
- VP is defined over some field $\mathbb{F}$ (Valiant 1979).

If the goal is $\mathrm{PERM}_n \notin \mathrm{VP}^0$, what is the most relaxed variant of the τ-conjecture that leads to this conclusion?

Studying real or complex zeros might be more approachable than integer zeros.

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- (SPS τ-conjecture) For $f_{i,j} \in \mathbb{Z}[x]$ and k-sparse

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  where s controls size of exponents and coefficients of $f_{i,j}$

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- (Tavenas 2014) $\text{poly}(p2^q k)$ suffices & PERM $\notin$ VP follows

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- Let $f := \sum_{i=1}^{p} a_i f_i^{\alpha_i}$ be such that each $f_i \in \mathbb{R}[x]$ is k-sparse, $a_i \in \mathbb{R}$, and $\alpha_i \leq m$. Then, $N_f^{\circ}\mathbb{R} \leq \text{poly}(p2^m k)$.

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$$\text{perm}\begin{bmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \dots & x_n \end{bmatrix} = (-1)^n \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i=1}^{n} \sum_{j \in S} x_j$$

$$n! \prod_{i=1}^{n} x_i = \sum_{S \subseteq [n]} (-1)^{|S|+n} \left(\sum_{j \in S} x_j \right)^n$$

How do we count real zeros?

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Proof by induction:

- Assume $f(x) = a_n x^{\alpha_k} + \cdots + a_1 x^{\alpha_2} + a_0$ has r positive zeros
- By induction, $f'(x)$ has at most $k-2$ positive zeros.
- By Rolle's theorem, $f'(x)$ has at least $r-1$ positive zeros
- $r - 1 \leq k - 2 \implies r \leq k - 1$

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- By Rolle's theorem, $f'(x)$ has at least $r-1$ positive zeros
- $r-1 \leq k-2 \implies r \leq k-1$

Corollary

at most $2k-1$ distinct real zeros.

Upper Bounds

(Real τ-Conjecture for Powers) For $f_i \in \mathbb{R}[x]$ and k-sparse, $a_i \in \mathbb{R}$, and $\alpha_i \leq m$

$$f := \sum_{i=1}^{p} a_i f_i^{\alpha_i} \qquad \Longrightarrow \quad N_f^{\circ}\mathbb{R} \leq \mathrm{poly}(p2^m k).$$

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- $N_f^\circ \mathbb{R} \leq k^{\mathcal{O}(p^2)}$ using Wronskian approach.
- Conj. holds if any of p or m or k is bounded by a constant.

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Let $f = \phi_1 + \phi_2$. Then by Rolle's theorem

$$N_f^\circ \mathbb{R} \leq N_{f'}^\circ \mathbb{R} + 1$$

But f' is again sum of two polynomials!

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Let $f = \phi_1 + \phi_2$ *(e.g.* $\phi_i = f_i^{\alpha_i}$*). Then*

$$N_f^\circ \mathbb{R} \leq 1 + N_{\phi_1}^\circ \mathbb{R} + N_{W(\phi_1,\phi_2)}^\circ \mathbb{R},$$

$$\textit{where } W(\phi_1, \phi_2) = \det \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} = \phi_1 \phi_2' - \phi_1' \phi_2.$$

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Idea:

- Prove instead $N_f^\circ \mathbb{R} - (1 + N_{\phi_1}^\circ \mathbb{R}) \leq N_{W(\phi_1,\phi_2)}^\circ \mathbb{R}$

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- Prove instead $N_f^{\circ}\mathbb{R} - (1 + N_{\phi_1}^{\circ}\mathbb{R}) \leq N_{W(\phi_1,\phi_2)}^{\circ}\mathbb{R} = N_{W(\phi_1,f)}^{\circ}\mathbb{R}$

Wronskian approach (for $p = 2$)

Koiran, Portier, Tavenas (2015), motivated by Voorhoeve and van der Poorten (1975)

Theorem

Let $f = \phi_1 + \phi_2$ *(e.g.* $\phi_i = f_i^{\alpha_i}$*). Then*

$$N_f^\circ \mathbb{R} \leq 1 + N_{\phi_1}^\circ \mathbb{R} + N_{W(\phi_1,\phi_2)}^\circ \mathbb{R},$$

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- Use $W(\phi_1, f) = \phi_1^2 \left(\frac{f}{\phi_1}\right)'$ on the set $\mathbb{R} \backslash Z(\phi_1)$

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$$\begin{aligned} N_f^\circ \mathbb{R} &\leq 1 + N_{\phi_1}^\circ \mathbb{R} + N_{W(\phi_1,\phi_2)}^\circ \mathbb{R} \\ &\leq 1 + 2k - 1 + N_{W(f_1^{\alpha_1}, f_2^{\alpha_2})}^\circ \mathbb{R} \end{aligned}$$

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Finally,

$$N_f^\circ \mathbb{R} \leq 10k - 3$$

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For $\sum_{i=1}^{p} f_i^{\alpha_i}$, the bound $k^{\mathcal{O}(p^2)}$ can be shown similarly.

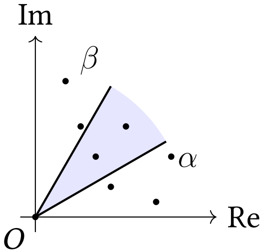
What’s so special about real zeros and real polynomials? Why count only distinct zeros?

Distribution of Zeros

Theorem (Hayman 1972)

$f \in \mathbb{C}[x]$ *be* k*-sparse and of degree* n *with* $f(0) \neq 0$

$$\left| N_f S(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} n \right| \leq k - 1$$



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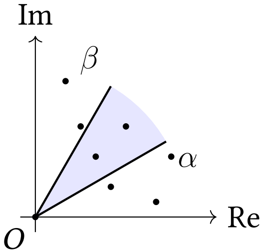

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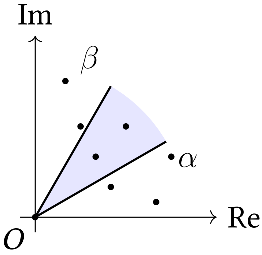

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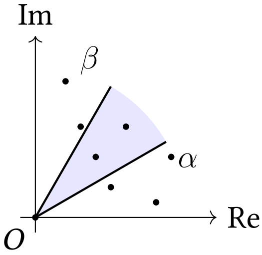

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Corollary: *Can also count zeros along any ray from the origin.*

Complex τ-Conjecture by Hrubeš 2013

TFAE:

- Let $f := \sum_{i=1}^{p} \prod_{j=1}^{q} f_{i,j}$ be such that each $f_{i,j} \in \mathbb{R}[x]$ is k-sparse. Then $N_f^{\circ}\mathbb{R} \leq \text{poly}(pqk)$
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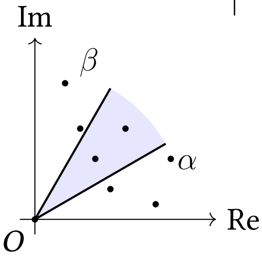

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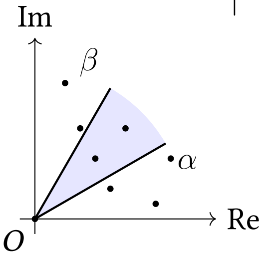

Proof idea: (Generalized Hayman)

1. Discrepancy $\le \#$ distinct zeros of $\Re(f)$ on the boundary of the sector
2. $\Re(f)$ has small representation if f does

How to Falsify Real τ-Conjecture?

Conjecture: *Let* $f := \sum_{i=1}^{p} \prod_{j=1}^{q} f_{i,j}$ *be of degree n with* $f(0) \neq 0$ *and each* $f_{i,j} \in \mathbb{C}[x]$ *be k-sparse. Then*

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- Candidate counterexample: $(x+1)^n$.
- Choose $\beta = \pi + \epsilon$ and $\alpha = \pi - \epsilon$
- Conjecture $\implies$ $pqk = \Omega(n^c)$ for some $c > 0$
- Can $(x+1)^n$ be expressed by a small complex SPS representation?

Summary

- Integer / Real / Complex τ-Conjecture are hard problems
- Hrubeš: Study distribution of zeros
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Possible directions:

- Does $gh + t$ have $\mathcal{O}(k)$ real zeros? (Chattopadhyay)
- Bivariate (Koiran, Portier, Tavenas, Thomassé 2015)
- Random (Briquel, Bürgisser 2020)
- SOS, SOC, …(Dutta 2021)

Thank you for you attention! Any questions?