Combinatorics and Zeros of Polynomials: Introduction to Chromatic Polynomial

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Introduction[1]

Gian-Carlo Rota was asked in an interview for Science magazine [7] in 1985,

> "SHARP: Gian-Carlo, tell us your contribution to combinatorics.
>
> ROTA: The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems of location of the zeros of certain polynomials and in giving these zeros a combinatorial interpretation..."

This answer of Rota is also something I am fascinated by and so I thought I'll tell you about this connection between counting and root-finding[2]. The combinatorial problem that we'll discuss is the four colour theorem, and the associated polynomial whose zeros we'll look at is the chromatic polynomial. It is also the 50th anniversary of the proof of four colour conjecture, and on this ocasion Wilson (2026) [8] wrote a survey article that you might want to check.

We define chromatic polynomial as a colouring-counting function.

Definition 1. *Given a graph $G = (V, E)$ and a positive integer t, we define $P_G(t)$ to be the number of proper t-colourings of vertices of G, i.e.,*

$$P_G(t) = \#\{\kappa : V \to [t] \mid \kappa \text{ is proper}\},$$

where $[t] := \{1, \dots, t\}$, and a proper t-colouring is a map $\kappa : V \to [t]$ such that every vertex is assigned some colour and $\kappa(u) \neq \kappa(v)$ for all edges $uv \in E$.

We shall prove in this note that the function $P_G(t)$ is indeed a polynomial. An immediate observation is that a graph can be coloured using t colours if and only if $P_G(t) > 0$. Also, if $P_G(t) > 0$ then $P_G(t+1) > 0$, which is true because it is not necessary to use all of the $t+1$ colors in a $(t+1)$-colouring (i.e. a colouring function need not be surjective). The *chromatic number*, which is simply the least number of colours required to properly colour a graph can also be alternatively defined as

$$\chi(G) = \min\{t \in \mathbb{N} \mid P_G(t) > 0\}.$$

[1] https://www.imsc.res.in/~hiteshw/

[2] I found out that there is convention to say 'zeros' of a polynomial and 'roots' of a polynomial equation.

As a result we have $P_G(t) = 0$ for each integer $t < \chi(G)$.

Let us compute this function for a few integers.

- $P_G(0) = 0$ because we cannot colour a graph with zero colours.
- $P_G(1) = 1$ if G is edgeless, otherwise $P_G(1) = 0$.
- $P_G(2) = 2^{\text{\#of components}}$ if G is bipartite, otherwise $P_G(2) = 0$.
- Even deciding whether $P_G(3) > 0$ is NP-hard (even for planar graphs), so computing $P_G(3)$ is even harder ($\#P$-complete). Also true for integers $t \geq 3$.
- $P_G(4) > 0$ if G is planar (by four colour theorem).

On Four Colour Theorem

Francis Guthrie in 1852 conjectured that every planar graph can be coloured using at most four colours, equivalently $\chi(G) \leq 4$ and also $P_G(4) > 0$. An example of a planar graph is not 3-colourable is K_4. In 1912, Birkhoff introduced the chromatic polynomial / counting function $P_G(t)$ to attack four colour conjecture. Counting the number of colourings may seem harder than showing existence of a colouring, but the hope was that by considering a general problem we may get some intuition about the hard problem.

There were a series of incorrect attempts at the four colour conjecture until 1976 when Appel and Haken (with assistance from Koch) announced a computer-assistance proof; it involved proving the claim for around 1900 configurations. Later Robsertson, Sanders, Seymour and Thomas in 1996 reduced the number of configurations to around 600. Meanwhile around 1890 Heawood had proved five colour theorem (which has a half-a-page proof using induction in West's graph theory book).

Now observe that the four colour theorem can be stated in terms of zeros of the chromatic polynomial.

Theorem 2. *If G is a planar graph, then $P_G(t)$ is zero-free (i.e. has not zeros in) in $[4, \infty) \cap \mathbb{Z}$.*

Although we do not have a proof of this theorem that we can easily understand, we could consider a slightly general version, where perhaps analytic or algebraic techniques could help.

Conjecture 3 (Birkhoff-Lewis Conjecture (1946))**.** *If G is a planar graph, then $P_G(t) > 0$ for all real numbers $t \geq 4$, i.e. $P_G(t)$ has no zeros in the real interval $[4, \infty)$.*

Birkhoff and Lewis were able to show that $[5, \infty)$ is a zero-free interval. A proof for the interval $[4, 5)$ is yet to be found. This will also imply an alternate proof for the four colour theorem, which we hope can be understood without a computer.[3]

[3] **Why is four colour theorem Interesting?**

- Because such concrete bound is not available in general. For complete graphs, we have an upper bound as large as $\chi(K_n) = n$. Brooks' theorem (1941) does give us the simple bound $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree. In the case of stars, which are planar, we have $\chi(K_{1,n-1}) = 2$ but Brooks' bound of $n - 1$ is very weak comparatively.
- For planar graphs, deciding "$P_G(k) > 0$?" is easy for $k = \mathbb{Z} \backslash \{3\}$ due to FCT but NP-Complete for $k = 3$.

With this motivation to study chromatic polynomial in mind, in this lecture we shall discuss the following questions:

1. Why is $P_G(t)$ a polynomial? Are all zeros of this polynomial integers or reals?
2. What can we say about the coefficients of $P_G(t)$?
3. What is the complexity of computing/evaluating $P_G(t)$? Do location of zeros give us any insights?

Simple examples where $P_G(t)$ is a polynomial

Let us see some examples.

Example 1. For the empty graph $\overline{K_n}$, we have

$$P_{\overline{K_n}}(t) = t^n.$$

Example 2. For the complete graph K_n, we have

$$P_{K_n}(t) = t(t-1)\dots(t-(n-1)) = \frac{t!}{(t-n)!}.$$

Notice that for each $t = 0, 1, \dots, n-1$ we have $P_G(t) = 0$. This is because a complete graph cannot be coloured using less than n colours.

Example 3. For the path graph P_n, we have

$$P_{P_n}(t) = t(t-1)^{n-1}.$$

Example 4. For a tree on n vertices T, the idea to fix a vertex as root and view the tree as a layered graph where the kth layer is the set of vertices that are distance k from the root. Since there are no cycles and not cross-edges between the branches, the argument is similar to that of paths. Then we have

$$P_T(t) = t(t-1)^{n-1}.$$

Observe that the chromatic polynomial of tree depends only on the number of vertices are not the structure, i.e. any two trees on the same number of vertices has the same chromatic polynomial.

Example 5. These were easy examples. Let us consider a cycle C_4 on four vertices. Let the vertices be a, b, c, d and edges be ab, bc, cd, ad, respectively. If we start colouring with a, then it can be coloured using t colours. Next, b has $t-1$ choices and c also has $t-1$ choices. Now we have a problem! Because d is adjacent to both a and c and we do not know if a and c have the same colour or not and depending on this there are $t-1$ or $t-2$ choices. Let κ denote a colouring; we are prompted to consider following two cases:

Case 1. $\kappa(a) = \kappa(c)$. In this case, if we start colouring with a, then there are t choices for it, and automatically that choice is also fixed for c. Next, b has $t-1$ choices and d also has $t-1$ choices. This contributes $t(t-1)^2$ to $P_{C_4}(t)$.

Case 2. $\kappa(a) \neq \kappa(c)$. Again, let us start colouring with a which has t choices of colours. Next, b has $t-1$ choices and c has (due to the condition imposed) has $t-2$ choices. Since a and c get different colour, d can use any of the other $t-2$ colours. Thus this case contributes $t(t-1)(t-2)^2$ to $P_{C_4}(t)$.

Finally, we have

$$\begin{aligned} P_{C_4}(t) &= t(t-1)^2 + t(t-1)(t-2)^2 \\ &= t(t-1)\left(t-1+(t-2)^2\right) \\ &= t(t-1)(t^2-3t+3). \end{aligned}$$

This also demonstrates that chromatic polynomial can have complex zeros.

Proof that $P_G(t)$ is a polynomial

Let us revisit the two cases we considered in the case of a cycle once agin. The two cases, in its generality, can be formulated as follows. Let G be a graph and $e = uv$ be a non-edge. Then

$$P_G(t) = P_G(t \mid \kappa(u) \neq \kappa(v)) + P_G(t \mid \kappa(u) = \kappa(v)), \tag{1}$$

where the first term on the RHS corresponds to the number of t-colourings where u and v get different colours.

What does it mean for the graph to have a colouring where endpoints of a non-edge get different colours? The effect is the same as assuming that $e = uv$ is an actual edge in the graph. Thus

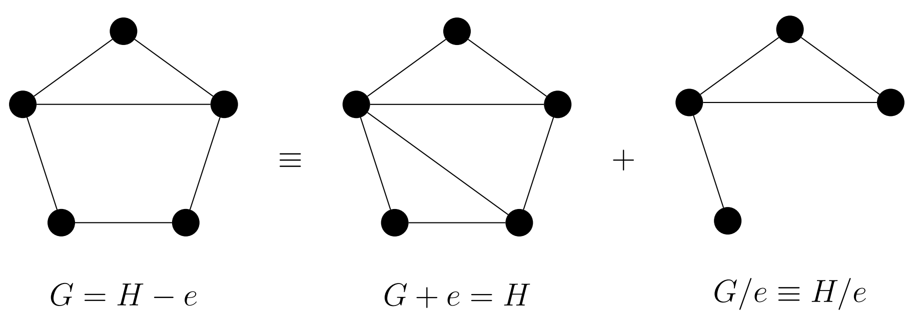
$$P_G(t \mid \kappa(u) \neq \kappa(v)) = P_{G+e}(t),$$

where $G + e$ is a graph obtained from G by simply adding the edge e.

For the other case, when the endpoints must get the same colour, the effect is the same as identifying the vertex u with v, i.e. *contracting* the non-edge $e = uv$. In the process, it may form loops or multi-edges. But they are irrelevant in counting colourings and therefore can be ignored. Thus

$$P_G(t \mid \kappa(u) = \kappa(v)) = P_{G/e}(t),$$

where G/e is a graph obtained from G by contracting the non-edge e.



For simplicity, let us consider a graph H with an edge $e = uv$. We define *deletion* operation as simply removing the edge e from the graph and denote the resulting graph by $H-e$. (Equivalently, in the notation used previously $H = G + e$.) Note that deletion does not reduce the number of vertices. Next, we define *contraction* operation of an edge $e = uv$ as identifying the vertex u

with v, and ignoring loops and replacing each multiple edge with a single edge; we denote the resulting graph by H/e. Note that contraction reduces the number of vertices by one. Thus Equation 1 becomes

$$P_{H-e}(t) = P_H(t) + P_{H/e}(t).$$

This is the fundamental identity for chromatic polynomial.

Lemma 4 (Deletion-Contraction Lemma[4]). *If $H = (V, E)$ is a graph and $e \in E$ is an edge, then*

$$P_H(t) = P_{H-e}(t) - P_{H/e}(t).$$

Theorem 5. *The function $P_G(t)$ is a polynomial of degree n.*

Proof. is by induction on the number of edges m. If $m = 0$, then we know that $P_G(t) = t^n$. Assume that the claim holds for $< m$, then by deletion-contraction identity, we get two graphs G/e and $G - e$, both have one edge less than G, and so the claim holds by induction. Since $P_{G-e}(t)$ is a polynomial of degree n, the function $P_G(t)$ is a polynomial of degree n. □

A naive algorithm. We may use deletion-contraction lemma to compute the chromatic polynomial recursively since on the RHS we have two graphs that have one less vertex and one less edge respectively. It is used by Mathematica as well. Both deletion and contraction operations reduce the number of edges by at least one. The idea will be to use this identity repeated until we reach a set of empty graphs for which we know that the chromatic polynomial is simply a power of t. A recurrence for the running time to compute chromatic polynomial of a graph on n vertices and m edges can be as follows

$$R(n, m) \leq R(n - 1, m - 1) + R(n, m - 1) + cm$$

with $R(n, 0) = 0$.

What can we say about the coefficients of $P_G(t)$?

Since a graph cannot be coloured using zero colours, we must have

Corollary 6. *The constant coefficient in $P_G(t)$ is zero, i.e, t always divides $P_G(t)$.*

From repeated use of $P_G(t) = P_{G-e}(t) - P_{G/e}(t)$, we notice that on the RHS there is always exactly one term that corresponds to a graph on n vertices; also it appears with a positive sign. This implies

Corollary 7. *The coefficient of t^n in $P_G(t)$ is 1, i.e. $P_G(t)$ is monic.*

Next, let us write $P_G(t) = \sum_{j=1}^{n} a_i t^i$, $P_{G-e}(t) = \sum_{j=1}^{n} b_i t^i$ and $P_{G/e}(t) = \sum_{j=1}^{n-1} c_i t^i$. Corollary 7, as well as, the Deletion-Contraction Lemma gives us

- $a_n = b_n = 1$,

[4] We use the word 'Deletion' first because $P_{H-e}(t)$ appears as the first term on the RHS.

- $c_{n-1} = 1$,
- $a_i = b_i - c_i$ for $i = 1, 2, \ldots, n-1$.

In particular, $b_{n-1} = a_{n-1} + 1$. Because we argued that there is exactly one term at the end with degree n which must necessarily originate from the sequence of deletion of edges from the original graph. This final term is t^n and its second coefficient (of t^{n-1}) is zero. Because there are m many sequence of deletions, we must have $a_{n-1} = -m$. Thus

Corollary 8. *The coefficient of t^{n-1} in $P_G(t)$ is $-m$ where m is the number of edges in G.*

So we saw the first coefficient is 1 and the second is $-m$. The coefficient sequence in fact alternates in sign.

Lemma 9. *The coefficients of $P_G(t)$ alternate in signs.*

Proof. is by induction on the number of edges m. For $m = 0$, the claim holds trivially since $P_G(t) = t^n$. Suppose that the claim holds for $< m$ edges, then we can assume coefficients in $P_{G-e}(t)$ and $P_{G/e}(t)$ alternate. Thus using same the notation of a_i's, b_i's and c_i's we see that whenener $b_i \geq 0$ we have $c_i \leq 0$, and whenever $b_i \leq 0$ we have $c_i \geq 0$.

Since $a_n = b_n = 1$ and $a_i = b_i - c_i$ for $i = 1, \ldots, n-1$, we notice that the sign of a_i is the same as the sign of b_i. Now b_i's are the coefficients of $P_{G-e}(t)$, which by induction, has coefficients altenating in sign. Thus the claim also holds for G. □

As a result of this lemma, we may express the chromatic polynomial as

$$P_G(t) = \sum_{i=1}^{n} (-1)^{n-i} a_i t^i \qquad \text{where } a_i \geq 0.$$

Then the zeros of the polynomial $P_G(-t) = (-1)^n \sum_{i=1}^{n} a_i t^i$ cannot be positive reals. Equivalently,

Theorem 10. *The chromatic polynomial $P_G(t)$ does not have zeros in $(-\infty, 0)$.*

Other intervals that have been shown to be zero free are $(0, 1)$ and $(1, \frac{32}{27}]$ (by Woodall 1977, and Jackson 1993 respectively). For planar graphs, Birkhoff and Lewis showed zero-free ness in $[5, \infty)$, and the question for the interval $[4, 5)$ is open and is equivalent to the Birkhoff-Lewis conjecture.

Breakthrough of June Huh

As alternation is signs in immediate, Read (1968) conjectured that the sequence of coefficients in absolute value is unimodal, i.e.,

$$1 = a_n \leq a_{n-1} \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_1 \geq a_0 = 0.$$

Later Hoggar (1974) conjectured that they are in fact log-concave, i.e.,

$$a_{k-1} a_{k+1} \leq a_k^2 \qquad \text{for each } k = 1, \ldots, n-1.$$

These conjectures remained opened for forty years until June Huh finally proved them in 2012 using algebraic geometry.[5]

Note that due to sign-alternations $\sum_{i=0}^{n}(-1)^{n-i}a_i z^i$ is log-concave if and only if $\sum_{i=0}^{n} a_i z^i$ is log-concave.

Why is unimodality or log-concavity interesting? First, let us observe

Proposition 11 (EXERCISE)**.** *Log-concavity implies unimodality*

Proof. Consider a log-concave sequence; we can write

$$\frac{a_0}{a_1} \leq \cdots \leq \frac{a_k}{a_{k+1}} \leq \cdots \leq \frac{a_{n-1}}{a_n}.$$

If the ratio $a_k/a_{k+1} \leq 1$ for some index k then each of the preceding ration stays at most 1. Then we have

$$a_0 \leq a_1 \leq \cdots \leq a_k \leq a_{k+1}$$

Also if $1 \leq a_{k+1}/a_{k+2}$ then each of the subsequent ratio is also at least 1. Then we have

$$a_{k+1} \geq a_{k+2} \geq \cdots \geq a_n.$$

□

Next,

Lemma 12 (Newton 1707, EXERCISE)**.** *If a polynomial with real coefficients has only real zeros, then it is log-concave.*

Although chromatic polynomial is not real-rooted, the closest property to it that we can observe is that its coefficients in absolute value are log-concave. It in some sense says that zeros are structured. The log-concavity property is at least helpful in finding a radius of a disk that contain all the zeros due to a result of Ostrowski.

Theorem 13 (Ostrowski 1940)**.** *Let* $f(z) = \sum_{i=0}^{n} a_i z^i$ *be a log-concave polynomial and* $\rho = \max_{f(\alpha_i)=0} |\alpha_i|$ *is the radius of the disk that contains all its zeros. Then*

$$\rho \leq 2\frac{|a_{n-1}|}{|a_n|}.$$

We have seen that for the chromatic polynomial $|a_n| = 1$ and $|a_{n-1}| = m$ (the number of edges). Thus we get

Lemma 14. *Let* G *be a graph with maximum degree* Δ*. Then all (complex) zeros of the chromatic polynomial* $P_G(t)$ *lie inside the disk* $D(0, n\Delta)$*, i.e, the region* $\{z \mid |z| > n\Delta\}$ *is zero-free.*

[5]He was a PhD student at that time, and was also awarded Fields medal in 2022, one of the supporting reasons for it being 'bringing ideas from algebraic geometry closer to combinatorics'.
Check out the quanta article: https://www.quantamagazine.org/june-huh-high-school-dropout-wins-the-fields-medal-20220705/

The current best improvement of this result is due to Bencs, Regts (2026) [3] who showed that $P_G(t)$ is zero free in $\{z \mid |z| > 4.25\Delta\}$[6] . Two and half decades ago the constant in the bound was around 8 due to Sokal.

- Recall Brooks' theorem says that $\chi(G) \leq \Delta(G)+1$. Therefore the discrete set $\{\Delta+1, \Delta+2, \ldots, \infty\}$ is zero-free. Whether the continuous interval (Δ, ∞) is also zero-free and in general the half-plane $\{z \mid \Re(z) > \Delta\}$ is also zero-free is an open problem.

Conjecture 15 (Sokal[7] 2002)**.** *Let G be a graph with maximum degree Δ. Then if t_0 is a zero of the chromatic polynomial $P_G(t)$ then $\Re(t_0) \leq \Delta$, i.e. the region $\{z \mid \Re(z) > \Delta\}$ is zero-free.*

The reason we want to study the location of zeros is that its has implications about the computational complexity of the polynomial.

Computational Complexity

The complexity classes P and NP are useful to study decision problems such as "Is $P_G(t) > 0$?". Such problem compute boolean functions of the type $f : \{0,1\}^n \to \{0,1\}$ where the image represents either YES or NO answer. On the other hand, the problem of computing $P_G(t)$ is modeled by boolean functions of the type $f : \{0,1\}^n \to \{0,1\}^m$. To study such problems, the relevant analogues complexity classes are FP and #P. Computing $P_G(3)$ is a #P-complete problem.

Next, we list some results on computing $P_G(t)$ exactly.

- #P-hard to compute $P_G(z)$ exactly for any algebraic $z \in \mathbb{C}\setminus\{0,1,2\}$ (Jaegar-Vertigan-Welsh 1990). Same is true even for planar graphs (Jaegar 2005).
- #P-hard to compute $P_G(z)$ exactly for algebraic $z \in \mathbb{R}\setminus\{0,1,2\}$.
- For **non-integers** $z > \frac{32}{27}$, even determining the sign of $P_G(z)$ is #P-hard. (Goldberg, Jerrum 2014).

If computing the polynomial exactly is generally hard, what can we say about computing it approximately?

- #P-hard to approximate for any non-real algebraic number z such that $|z-1| > 1$ or $\Re(z) > 3/2$. Also holds for planar graphs. (Fencs-Huijben-Regts 2022).
- Hardness of approximating for real z for planar graphs is open.
- Recall that the problem "Is $P_G(3) > 0$?" is NP-complete. On the other hand, Brooks' theorem trivializes the question "Is $P_G(t) > 0$?" when $t \geq \Delta+1$. Does this mean that computing $P_G(t)$ approximately when $t \geq \Delta+1$ is easier?

[6] I erroneously mentioned in class that proving this requires complex dynamics. Complex dynamics was used in proving some results about the density of the zeros.

[7] Just yesterday I heard three different talks online on chromatic polynomial at SokalFest, on the occasion of Sokal's 70th Birthday. Link: https://www.homepages.ucl.ac.uk/∼ucahrha/sokal.html

For **BOUNDED DEGREE GRAPHS**, there is Taylor interpolation method developed by Barvinok [1] to approximate polynomial evaluations efficiently in the zero-free regions. It implies

- an efficient approximation algorithm to compute $P_G(t)$ when $t \in \{z \mid |z| > 4.25\Delta\}$.
- If Sokal's conjecture is true, then this implies efficient approximation algorithm also for $t = \Delta + 1$. But we do not know this yet.
- Liu, Sinclair, Srivastava [5] and Bencs, Berrekal and Regts [2] provide efficient approximation algorithms for $P_G(t)$ when $t \in (2\Delta, \infty] \cap \mathbb{Z}$ and $t \in [(2-\eta)\Delta, \infty) \cap \mathbb{Z}$ respectively. They do this by considering the Tutte polynomial which generalizes chromatic polynomial.

Connection to Statistical Physics

Chromatic polynomial is known as the partition function of the anti-ferromagnetic t-state Potts model. Other graph polynomials such as independence polynomial, matching polynomial, Tutte polynomial also have connections to models in statistical physics. Statistical physicists are usually interested in computing $\ln Z$ where Z is some partition function; but it can be evaluated only in regions where Z is zero-free. Also zeros correspond to a state where phase transition can occur, e.g. liquid turning into gas, or metal suddenly starting to show magnetic properties. This is another reason we are interested in investigating zero-free regions of a graph polynomial.

Takeway

Zero-free regions imply efficient approximation algorithms for hard counting problems. If you wish to read more about this algorithmic connection check out the book by Barvinok (2016) [1]. Parts of this note are based on the exposition by Read (1968) [6]. There is a whole book just on the chromatic polynomial by Dong, Koh and Teo (2005) [4].

References

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Appendix

Theorem 16 (Bencs Regts [3], using method of Barvinok)**.** *For bounded degree graph G with maximum degree Δ such that the chromatic polynomial $P_G(z)$ is zero free in the disk of radius R (for $R > 0$) around origin, and given $\epsilon > 0$ and complex z such that $|z| < R$ there is an algorithm that computes $p(z)$ for some complex $|s| < 1$ such that*

$$p(z) = e^{\epsilon s} P_G(z) \lessapprox (1 + \epsilon s) P_G(z)$$

i.e.,

$$-\epsilon \leq \frac{P_G(z) - p(z)}{P_G(z)} \leq \epsilon$$

in time $\mathsf{poly}(n/\epsilon)$.

We may utilize this theorem through a small trick. Define $Q_G(s) := s^n P_G(1/s)$, a degree $n-1$ polynomial. Then $Q_G(0) = 1$ is the leading coefficient of $P_G(t)$. Also if $0, \alpha_2, \ldots, \alpha_n$ are the zeros of $P_G(t)$ then $1/\alpha_2, \ldots, 1/\alpha_n$ are the zeros of $Q_G(s)$. Thus

$$|\alpha_i| \leq 4.25\Delta \iff \frac{1}{|\alpha_i|} \geq \frac{1}{4.25\Delta},$$

and the open disk $D(0, 1/(4.25\Delta))$ around the origin is zero-free for $Q_G(s)$. Thus given a point t_0, we first compute $Q_G(1/t_0)$ and then use $P_G(t) = t^n Q_G(1/t)$ to recover $P_G(t_0)$.