TOPOLOGICAL (CHERN-SIMONS) QUANTUM MECHANICS IN REDUCED FORM
(Project Report)

Project Student:Himadri Barman
Dept. of Physics,University of Pune
Ganeshkhind,Pune 411007.
and Project Guide:Biswajit Chakraborty
S. N. Bose National Centre for Basic Sciences
JD Block, Sector III, Salt Lake City, Kolkata -700 098, India

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Abstract:

Here we look up the similarity between the Chern-Simons gauge field theory and the general quantum mechanical model. We also have a view on the zero-mass approach of this model.

Introduction:

The Chern-simons gauge theory is one of the special models needed to describe the topological phenomena in odd dimensional space-time. This play a vital role for 3-D dynamics in high temperature Quantum Hall Effect and String Theory in cosmology. This paper discusses the connection of the (2+1) dimensional Chern-Simons theory with the 1-dimensional space-time phenomena (i.e. our common quantum mechanics). It also shows that this relation holds even in the reduced (zero-mass) theory.

1 A Comparative study:

Consider the case of Landau Level Problem where a particle of m and charge e executes motion under crossed electromagnetic field.Its equation of motion can be written as:

$$L = \frac{m}{2}\dot{\mathbf{q}}^2 + \frac{e}{c}\dot{\mathbf{q}}.\mathbf{A}(\mathbf{q}) - e\mathbf{V}(\mathbf{q}) \qquad (\mathbf{1})$$

Here magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$ i.e. $B = \partial_i A^j - \partial_j A^i$ and electric field, $\mathbf{E} = -\nabla\mathbf{V}$ i.e. $E^i = -\partial_i V$

Now consider the gauge field $\mathbf{A}$ to be rotationally symmetric in (2+1) dimensional space time[i.e.$A^i(\mathbf{q}) = \epsilon^{ij}A(q)$] and so we take $\mathbf{A}^i(\mathbf{q}) = -\frac{1}{2}\epsilon^{ij}q^jB$ as we one can choose for uniform magnetic field $\mathbf{B}, \mathbf{A} = -\frac{1}{2}\mathbf{q}\times\mathbf{B}$

Also take $\mathbf{V}(\mathbf{q}) = k\mathbf{q^2/2}$ i.e. harmonic potential

Therefore $L = \frac{m}{2}\dot{q}^i\dot{q}^i - \frac{B}{2}\frac{e}{c}\dot{q}^i - \frac{k}{2}e\dot{q}^i\dot{q}^i$

$= \frac{m}{2}\dot{q}^i\dot{q}^i - \frac{B}{2}\dot{q}^i - \frac{k}{2}e\dot{q}^i\dot{q}^i$ [Choosing e=c=1]

$= \frac{m}{2}\dot{\mathbf{q}}^{\mathbf{2}} - \frac{B}{2}\dot{\mathbf{q}}\times\mathbf{q} - \frac{k}{2}\mathbf{q^2}$

i.e. $L = \frac{m}{2}\dot{\mathbf{q}}^{\mathbf{2}} + \frac{B}{2}\mathbf{q}\times\dot{\mathbf{q}} - \frac{k}{2}\mathbf{q^2}$ **(2)**

Now Lagrangian density in Weyl ($A^0 = 0$) gauge:

$\mathcal{L} = \frac{1}{2}\dot{A}^2 + \frac{\mu}{2}\dot{\mathbf{A}}^{\mathbf{2}} - \frac{1}{2}(\nabla\times\mathbf{A})^{\mathbf{2}}$ **(3)**

Setting $\mathbf{A} \to \sqrt{\frac{\kappa}{\mu}}$,

$\mathcal{L} = \frac{1}{2}\frac{\kappa}{\mu}\dot{\mathbf{A}}^{\mathbf{2}} + \frac{\kappa}{2}\dot{\mathbf{A}}\times\mathbf{A} - \frac{1}{2}\frac{\kappa}{\mu}(\nabla\times\mathbf{A})^{\mathbf{2}}$

Therefore $\mathcal{L}_{CS} = \frac{\kappa}{2}\dot{\mathbf{A}}\times\dot{\mathbf{A}}$ as $\mu\to\infty$ (4) where "CS" stands for pure Chern-Simons term which occur on a reduced phase-space

For $m \to 0$, Lagrangian in (2) can be written in reduced form:

$L_0 = \frac{B}{2}\mathbf{q}\times\dot{\mathbf{q}} - \frac{k}{2}\mathbf{q^2}$ **(5)**

If $k \to 0$ also,

$L = \frac{B}{2}\mathbf{q}\times\dot{\mathbf{q}}$ which is similar to eq.(4)

2 Hamiltonian formulations:

Now we are going to formulate the Hamiltonian from the above Lagrangian expressions:

From (2)

$L = \frac{m}{2}(q^i)^2 + \frac{B}{2}\epsilon^{ij}q^i\dot{q}^j$

$= \frac{m}{2}(q^i)^2 - \frac{B}{2}\epsilon^{ij}q^j\dot{q}^i$

Therefore canonically conjugate momentum,

$p^i = \frac{\partial L}{\partial\dot{q}^i} = m\dot{q}^i - \frac{B}{2}\epsilon^{ij}q^j$ (6)

gives the Hamiltonian,

$H = p^i\dot{q}^i - L$

$= p^i\dot{q}^i - \frac{m}{2}\dot{q}^i\dot{q}^i - \frac{B}{2}\epsilon^{ij}q^i\dot{q}^j + \frac{k}{2}q^iq^i$

$= p^i\dot{q}^i - \frac{1}{2m}(m\dot{q}^i)(m\dot{q}^i) + \frac{B}{2}\epsilon^{ij}q^j\dot{q}^i + \frac{k}{2}q^iq^i$

$= (p^i + \frac{B}{2}\epsilon^{ij}q^j)\dot{q}^i - \frac{1}{2m}(m\dot{q}^i)(m\dot{q}^i) + \frac{k}{2}q^iq^i$

$= \frac{1}{m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(m\dot{q}^i) - \frac{1}{2m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(p^i + \frac{B}{2}\epsilon^{ik}q^k) + \frac{k}{2}q^iq^i$

$= \frac{1}{m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(p^i + \frac{B}{2}\epsilon^{ik}q^k) - \frac{1}{2m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(p^i + \frac{B}{2}\epsilon^{ik}q^k) + \frac{k}{2}q^iq^i$

$= \frac{1}{2m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(p^i + \frac{B}{2}\epsilon^{ik}q^k) + \frac{k}{2}q^iq^i$ (7)

Since the Hamiltonian is obtained by Legendre transform of the Lagrangian the following conventional commutation relations hold:

$[q^i, q^j] = 0, \quad [p^i, p^j], \quad [q^i, p^i] = \delta^{ij}$ (8)

Now, $\dot{p}^i = i[H, p^i]$

$= i\frac{1}{2m}[(p^l + \frac{B}{2}\epsilon^{lj}q^j)(p^l + \frac{B}{2}\epsilon^{lk}q^k), p^i] + i\frac{k}{2}[q^lq^l, p^i]$

$= i\frac{1}{2m}(p^l + \frac{B}{2}\epsilon^{lj}q^j)[(p^l + \frac{B}{2}\epsilon^{lk}q^k), p^i] + i\frac{1}{2m}[(p^l + \frac{B}{2}\epsilon^{lj}q^j), p^i](p^l + \frac{B}{2}\epsilon^{lk}q^k)$

$+ikq^l[q^l, p^i]$

$= i\frac{1}{2m}(p^l + \frac{B}{2}\epsilon^{lj}q^j)\frac{B}{2}\epsilon^{lk}\delta^{ki} + i\frac{1}{2m}(p^l + \frac{B}{2}\epsilon^{lk}q^k)\frac{B}{2}\epsilon^{lj}\delta^{ji}$

$= -\frac{1}{2m}(p^l + \frac{B}{2}\epsilon^{lj}q^j)\frac{B}{2}\epsilon^{li} - \frac{1}{2m}(p^l + \frac{B}{2}\epsilon^{lj}q^j)\frac{B}{2}\epsilon^{li} - kq^i$

$= \frac{B}{4m}\epsilon^{il}(p^l + \frac{B}{2}\epsilon^{lj}q^j) + \frac{B}{4m}\epsilon^{il}(p^l + \frac{B}{2}\epsilon^{lk}q^k) - kq^i$

$= \frac{B}{2m}\epsilon^{ij}(p^j + \frac{B}{2}\epsilon^{jk}q^k) - kq^i$ $(9a)$

and $\dot{q}^i = i[H, q^i] = i\frac{1}{2m}[(p^l + \frac{B}{2}\epsilon^{lj}q^j)(p^l + \frac{B}{2}\epsilon^{lk}q^k), q^i] + i\frac{k}{2}[q^lq^l, q^i]$

$= i\frac{1}{m}(p^l + \frac{B}{2}\epsilon^{lj}q^j)[p^l, q^i]$

$= i\frac{1}{m}(p^l + \frac{B}{2}\epsilon^{lj}q^j)(-i\delta^{li})$

$= i\frac{1}{m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)$ $(9b)$

Therefore $\dot{p}^i = \frac{B}{2m}\epsilon^{ij}(p^j + \frac{B}{2}\epsilon^{jk}q^k) - kq^i$

(9a & 9b)

and $\dot{q}^i = i\frac{1}{m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)$

To find out the solutions take

$z = q^1 + iq^2$

and $p = p^1 + ip^2$

From (9b), $\dot{q}^1 = \frac{1}{m}(p^1 + \frac{B}{2}q^2)$& $\dot{q}^2 = \frac{1}{m}(p^2 - \frac{B}{2}q^1)$

Therefore $\dot{z} = \dot{q}^1 + i\dot{q}^2$

$= \frac{1}{m}(p^1 + ip^2) + \frac{B}{2m}(q^2 - iq^1)$

$= \frac{1}{m}(p^1 + ip^2) - i\frac{B}{2m}(q^1 - iq^2)$

i.e. $\dot{z} = \frac{1}{m} - \frac{iB}{2m}z$ $(10a)$

From (9a), $\dot{p}^1 = \frac{B}{2m}(p^2 - \frac{B}{2}q^1) - kq^1 = \frac{B}{2m}p^2 - (\frac{B^2}{4m} + k)q^1$

& $\dot{p}^2 = -\frac{B}{2m}(p^1 + \frac{B}{2}q^2) - kq^2 = -\frac{B}{2m}p^1 - (\frac{B^2}{4m} + k)q^2$

Therefore $\dot{p} = \dot{p}^1 + i\dot{p}^2$

$= \frac{B}{2m}(p^2 - ip^1) - (\frac{B^2}{4m} + k)(q^1 + iq^2)$

$= -\frac{iB}{2m}(p^1 + ip^2) - (\frac{B^2}{4m} + k)(q^1 + iq^2)$

$= -\frac{iB}{2m}(p^1 + ip^2) - m(\frac{B^2}{4m^2} + \frac{k}{m})(q^1 + iq^2)$

i.e. $\dot{p} = -\frac{iB}{2m}p - m\Omega^2 z \qquad (10b)$ where $\Omega^2 = (\frac{B^2}{4m^2} + \frac{k}{m})$

If we choose $z = e^{-\frac{iB}{2m}t}\{z(0)\cos\Omega t + \frac{p(0)}{m}\Omega \sin\Omega t\} \qquad (11a)$

and $p = e^{-\frac{iB}{2m}t}\{p(0)\cos\Omega t - m\Omega z(0)\sin\Omega t\} \qquad (11b)$

then

$\dot{z} = e^{-\frac{iB}{2m}t}\{z(0)\sin\Omega t + \frac{p(0)}{m}\Omega\cos\Omega t\} - \frac{iB}{2m}e^{-\frac{iB}{2m}t}\{z(0)\cos\Omega t + \frac{p(0)}{m}\Omega\sin\Omega t\}$

$\dot{z} = \frac{1}{m}p - \frac{iB}{2m}z$!satisfies (10a)

and $\dot{p} = e^{-\frac{iB}{2m}t}\{-\Omega p(0)\sin\Omega t - m\Omega^2 z(0)\cos\Omega t\} - \frac{iB}{2m}e^{-\frac{iB}{2m}t}\{z(0)\cos\Omega t + \frac{p(0)}{m}\Omega\sin\Omega t\}$

$= -m\Omega^2 e^{-\frac{iB}{2m}t}\{\frac{p(0)}{m\Omega}\sin\Omega t + z(0)\cos\Omega t\} - \frac{iB}{2m}e^{-\frac{iB}{2m}t}\{z(0)\cos\Omega t + \frac{p(0)}{m}\Omega\sin\Omega t\}$

$= -m\Omega^2 z - \frac{iB}{2m}p$!satisfies (10b)

Now the case where $m \to 0$ will be observed

Reduced Hamiltonian for $m \to 0$

$H_0 = \frac{k}{2}\mathbf{q^2} \qquad (\mathbf{12})$ [because $p^i + \frac{B}{2}\epsilon^{ij}q^j = 0$ as $m \to 0$ and so does not contribute in eq.(7)]

Also from (5)

$H_0 = p^i q^i - L_0 = p^i q^i - \frac{B}{2}\epsilon^{ij}q^i\dot{q}^j + \frac{k}{2}q^i q^i \quad$ [since $L_0 = \frac{B}{2}\epsilon^{ij}q^i\dot{q}^j - \frac{k}{2}q^i q^i$]

$= \frac{B}{2}\epsilon^{ji}q^j\dot{q}^j - \frac{B}{2}\epsilon^{ij}q^i\dot{q}j + \frac{k}{2}q^i q^i \quad$ [since $p^i = \frac{\partial L_0}{\partial \dot{q}^i} = \frac{B}{2}\epsilon^{ji}q^j$]

$= \frac{k}{2}q^i q^i \quad$ [since $\epsilon^{ji}q^j\dot{q}^j = \epsilon^{ij}q^i$]

Thus $H_0 = \frac{k}{2}\mathbf{q^2} \qquad (\mathbf{12})$

and $L_0 = \frac{B}{2}\epsilon^{ij}q^i\dot{q}^j - \frac{k}{2}q^i q^i$

$\quad = \frac{1}{2}q^i(\epsilon^{ij})\dot{q}^j - H_0$

Comparing with the symplectic form of Lagrangian

$L = \frac{1}{2}q^i\{q^i, q^j\}^{-1}\dot{q}^j - H$

we have,

$\{q^i, q^j\}^{-1} = \epsilon^{ij}B \Rightarrow \{q^i, q^j\} = 1/(\epsilon^{ij}B) = -\epsilon^{ij}/B$

$\Rightarrow [q^i, q^j] = i\{q^i, q^j\} = -i\epsilon^{ij}/B \qquad (13)$

Therefore, $\dot{q}^i = i[H_0, q^i] = i\frac{k}{2}[(q^j)^2, q^i]$

$= i\frac{k}{2}.2q^j[q^j, q^i]$

$= ikq^j(-\frac{i}{B}\epsilon^{ij})$

i.e. $\dot{q}^i = -\frac{k}{B}\epsilon^{ij}q^j \qquad (14)$

which means $\dot{q}^1 = -\frac{k}{B}q^2$ & $q^2 = \frac{k}{B}q1$

therefore $\dot{z} = \dot{q}^1 + iq^2$

$= -\frac{k}{B}q^2 + i\frac{k}{B}q^1$

$= i\frac{k}{B}(q^1 + iq^2)$

$= \frac{k}{B} z \qquad (15)$
which has the solution
$z = Ae^{i\frac{k}{B}t} = z(0)e^{i\frac{k}{B}t} \quad (16)$[since $A = z(0)$]
$(6) \Rightarrow p^i = m\dot{q}^i - \frac{B}{2}\epsilon^{ij}$
$(7) \Rightarrow H = \frac{1}{2m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(p^i + \frac{B}{2}\epsilon^{ik}q^k) + \frac{k}{2}q^iq^i$
As $m \to 0, \quad p^i \to -\frac{B}{2}\epsilon^{ij}q^j)$
i.e. $p^i + \frac{B}{2}\epsilon^{ij}q^j) \to 0$

Let $C^i = p^i + \frac{B}{2}\epsilon^{ij}q^j (= mq^i) \qquad (17)$
Therefore $[C^i, C^j] = [p^i + \frac{B}{2}\epsilon^{ik}q^k), p^j + \frac{B}{2}\epsilon^{jl}q^l)$
$= [p^i, p^j] + \frac{B}{2}\epsilon^{jl}[p^i, q^l] + \frac{B}{2}\epsilon^{ik}[q^k, p^j] + \frac{B^2}{4}\epsilon^{ik}\epsilon^{jl}[q^k, q^l]$
$= -i\frac{B}{2}\epsilon^{jl}\delta il + i\frac{B}{2}\epsilon^{ik}\delta kj$
$= -i\frac{B}{2}\epsilon^{ji} + i\frac{B}{2}\epsilon^{ij}$
i.e. $[C^i, C^j] = iB\epsilon^{ij} \neq 0 \qquad (18)$
So Dirac bracket commutation for operators O_1 and O_2 can be written as
$[O^1, O^2] = [O^1, O^2] - [O^1, C^i][C^i, C^j]^{-1}[c^j, O^2]$
$= [O^1, O^2] - [O^1, C^i]\frac{\epsilon^{ij}}{B}[C^j, O^2]$
since $[C^i, C^j]^{-1} = 1/(iB\epsilon^{ij}) = -i(-\epsilon^{ij}.1/B) = \frac{\epsilon^{ij}}{B}$
Considering $C^i \approx 0$ i.e. $p^i \approx -\frac{B}{2}\epsilon^{ij}q^j \quad (20)$
$L_0 = p^iq^i - H_0$
$\quad = -\frac{B}{2}\epsilon^{ij} - \frac{k}{2}q^iq^i$
$\quad = -\frac{B}{2}\dot{q} \times q - \frac{k}{2}q^iq^i$
i.e. $L_0 = \frac{B}{2}\mathbf{q} \times \dot{\mathbf{q}} - \frac{k}{2}\mathbf{q^2} \qquad (\mathbf{5})$
As $m \to 0$
$\Omega \sim \lim_{m\to 0} (\frac{B^2}{4m^2} + \frac{k}{m})^{1/2}$
$\quad \sim \lim_{m\to 0} \frac{B}{2m}(1 + \frac{k}{m}.\frac{4m^2}{B^2})^{1/2}$
$\quad \sim \lim_{m\to 0} \frac{B}{2m}(1 + \frac{2km}{B^2})^{1/2}$
$\quad \sim \lim_{m\to 0} \frac{B}{2m}(1 + \frac{1}{2}\frac{2km}{B^2} + \frac{2km}{B^2} + 0(\frac{2km}{B^2}^2))$
$\quad \sim \frac{B}{2m} + \frac{k}{B} \quad (21)$

3 Energy Spectrum for complete theory:

Now we shall see that both Hamiltonian and rotation generator posses simultaneous eigenstates which show an interesting picture in their energy spectra.

Agular momentum, $M = \mathbf{q} \times \mathbf{p} = \epsilon^{ij} q^i p^j \qquad (22)$

Therefore $[M, q^i] = [\epsilon^{lj} q^l p^j, q^i]$

$= \epsilon^{lj} q^l [p^j, q^i] + \epsilon^{lj} [q^l, q^i] p^j$

$= \epsilon^{lj} q^l (-i\delta^{ji})$

$= -i\epsilon^{li} q^l$

$= i\epsilon^{il} q^l$

i.e. $[M, q^i] = \epsilon^{ij} q^j \qquad (23a)$

And $[M, p^i] = [\epsilon^{lj} q^l p^j, p^i] \quad = \epsilon^{lj} [q^l, p^i] p^i$

$= \epsilon^{lj} \delta il p^j$

i.e. $[M, p^i] = i\epsilon^{ij} p^j \qquad (23b)$

indicating that M is the generator of rotation with the commutators in (8)

From (20),

$p^i = -\frac{B}{2} \epsilon^{ij} q^l p^j$

Therefore $M_0 = \epsilon^{ki} q^k p^i$

$= -\frac{B}{2} \epsilon^{ki} \epsilon^{ij} q^j q^k$

$= \frac{B}{2} \delta^{kj} q^j q^k$

$= \frac{B}{2} q^k q^k$

i.e. $M_0 = \frac{B}{2} \mathbf{q^2} \qquad (24)$

Also $i[M_0, q^i] = i\frac{B}{2} [q^k q^k, q^i]$

$= i\frac{B}{2} . 2q^k [q^k, q^i]$

$= -iB[q^i, q^k] q^k$

$= -iB(-iB)\epsilon^{ik}$ [from(14)]

$= -\epsilon^{ik} q^k$

$= -\epsilon^{ij} q^j \qquad (25)$

generates rotations with the commutator in (13)

Now, $[M, H] = \frac{1}{2m} [M, (p^i)^2] + \frac{1}{2} m\Omega^2 [M, (q^i)^2] - \frac{B}{2m} [M, M]$

$= \frac{1}{2m} \{[M, p^i] p^i + p^i [M, p^i]\} + \frac{1}{2} m\Omega^2 \{[M, q^i] q^i + q^i [M, q^i]\}$

$= \frac{1}{2m} (i\epsilon^{ij} p^j p^i + i\epsilon^{ij} p^i p^j) + \frac{1}{2} m\Omega^2 (i\epsilon^{ij} q^j q^i + i\epsilon^{ij} q^i q^j)$

$= 0 \qquad (26)$

i.e. M and H have simultaneous eigenkets represented as:

$M|N, n\rangle = n|N, n\rangle \qquad (27a)$

$H|N, n\rangle = E(N, n)|N, n\rangle \qquad (27b)$

$E(N, n) = \Omega(2N + |n| + 1) - \frac{B}{2m} n \qquad (27c)$

and the eigen function in coordinate representation :

$\langle q|N,n\rangle = (\frac{N!}{\pi(N+|n|)!})^{1/2}(m\Omega)^{(1+|n|)/2} r^{|n|} e^{in\theta} e^{-(m/2)\Omega r^2} L_N^{|n|}(m\Omega r^2) \qquad (27d)$

This is known as Fock-Darwin spectra[2,3]

Here $L_N^{|n|}$ is the associated Laguerre polynomial,satisfying the differential eq.

$w\frac{d^2}{dw^2}L_N^{|n|}(w) + (|n|+1-w)\frac{d}{dw}L_N^{|n|}(w) + NL_N^{|n|}(w) = 0 \qquad (27e)$

Here N is a non-negative integer and n is any integer.

4 Energy spectrum for reduced theory

Now we come back to our reduced theory ($m \to 0$) and we shall find out simultaneous eigenkets of M_0 and H_0 since it is obvious from (12) and (24) that they commute.

Choose creation and annihilation operators as defined below:

$a = \sqrt{\frac{B}{2}}(q^1 - iq^2) \qquad (28a)$

$a\dagger = \sqrt{\frac{B}{2}}(q^1 + iq^2) \qquad (28b)$

Using eq.(13) for $m \to 0$ we get,

$[a, a\dagger] = \frac{B}{2}[q^1 - iq^2, q^1 + iq^2]$

$= \frac{B}{2}\{[q^1, q^1] + i[q^1, q^2] - i[q^2, q^1] + [q^2, q^2]\}$

$= \frac{B}{2}.2i[q^1, q^2]$

$= iB(\frac{i}{B}\epsilon^{12})$

$= -i^2 = 1 \qquad (29)$

$\hat{n} = a\dagger a = \frac{B}{2}(q^1 + iq^2)(q^1 - iq^2)$

$= \frac{B}{2}(q^1)^2 + \frac{B}{2}(q^2)^2 - \frac{B}{2}(iq^1q^2 + iq^2q^1)$

$= \frac{B}{2}(q^1)^2 + \frac{B}{2}(q^2)^2 - \frac{B}{2}[q^1, q^2]$

$= \frac{B}{2}((q^1)^2 + (q^2)^2) - i\frac{B}{2}(-\frac{i}{B}\epsilon^{12})$

$= \frac{B}{2}(q^1)^2 + \frac{B}{2}(-\frac{2}{p^1})^2 - \frac{1}{2} \quad$ [since $p^1 = -\frac{B}{2}q^2$ for $m \to 0$]

$= \frac{2p^1}{B} + \frac{B}{2}(q^1)^2 - \frac{1}{2} \quad (30)$

$H_0 = \frac{k}{2}(q^1)^2 + \frac{k}{2}(q^2)^2$

$= \frac{k}{2}(q^1)^2 + \frac{k}{2}(-\frac{2}{B})(p^1)^2$

$= \frac{2k}{B^2}(p^1)^2 + \frac{k}{2}(q^1)^2$

$= \frac{k}{B}(\frac{2(p^1)^2}{B} + \frac{B}{2}(q^1)^2)$

$= \frac{k}{B}(\hat{n} + \frac{1}{2}) \quad$ [from (30)]

Therefore $H_0|n\rangle = \frac{k}{B}(n + \frac{1}{2})|n\rangle \qquad (31)$ where n=0,1,2,...etc.

and so $M_0|n\rangle = \frac{B}{2}\dot{\mathbf{q}}^2|n\rangle = \frac{B}{k}.\frac{k}{2}\dot{\mathbf{q}}^2|n\rangle$

$= \frac{B}{k} H_0 |n\rangle$
$= \frac{B}{k} . \frac{k}{B} (n + \frac{1}{2}) |n\rangle \qquad = (n + \frac{1}{2}) |n\rangle \quad (32)$

We can choose $q^1 \equiv \frac{\sqrt{k}}{B} x$& $q^2 \equiv i \frac{1}{\sqrt{k}} \frac{d}{dx}$ satisfying eq.(13)

i.e. $[q^1, q^2]\psi = (q^1 q^2 - q^2 q^1)\psi$

$= \frac{\sqrt{k}}{B} \frac{i}{\sqrt{k}} x \frac{d\psi}{dx} - \frac{\sqrt{k}}{B} \frac{i}{\sqrt{k}} (\frac{d}{dx} x\psi + x \frac{d\psi}{dx})$

$= -\frac{i}{B} \psi$

i.e. $[q^1, q^2] = -\frac{i}{B}$

Now recall our wellknown harmonic oscillator expressions:

$a = 1/\sqrt{2}(\sqrt{m\omega} x + i \frac{1}{\sqrt{m\omega}} p) \quad (33)$ where $-i \frac{d}{dx} \equiv p$

which gives wave function

$\psi(x) = (2^n n!)^{-1/2} (\frac{m\omega}{\pi})^{1/4} e^{-1/2 m\omega x^2} H_n(\sqrt{m\omega} x) \quad (34)$ [considering $\hbar = 1$]

Now in our case

$a = 1/\sqrt{2}(\sqrt{B} q^1 - i\sqrt{B} q^2)$

$= 1/\sqrt{2}(\sqrt{B} . \frac{\sqrt{k}}{B} + i \frac{\sqrt{B}}{\sqrt{k}} p)$ [Here $q^2 \equiv -\frac{1}{\sqrt{k}}(-i \frac{d}{dx}) = -\frac{i}{k} p$]

$= 1/\sqrt{2}(\sqrt{\frac{k}{B}} x + i \frac{1}{\sqrt{\frac{k}{B}}} p) \quad (35)$

Therefore $m\omega \leftrightarrow \frac{k}{B}$ [" $\leftrightarrow$ "implies "corresponds to"]

Therefore the wave function in our current problem will be

$u(x) = (2^n n!)^{-1/2} (\frac{k}{\pi B})^{1/4} e^{-\frac{k}{2B} x^2} H_n(\sqrt{\frac{k}{B}} x) \qquad (36)$

5 Review of The Complete Theory:

Now we return to our complete theory and we want to see that if it agrees in the limit $m \to 0$ with our previously obtained results for the reduced theory.

As $m \to 0$ eqs.(27c)gives

$E(N, n) \underset{m \to 0}{\sim} \Omega(2N + |n| + 1) - \frac{B}{2m} n$

$\sim (\frac{B}{2m} + \frac{k}{B})(2N + |n| + 1) - \frac{B}{2m} n$

$\sim \frac{B}{2m}(2N + |n| - n + 1) + \frac{k}{B}(2N + |n| + 1) \quad (37a)$

For $N = 0$ & $n \geq 0$(i.e. n is non-negative integer)

$E(N, n) \underset{m \to 0}{\sim} \frac{B}{2m} + \frac{k}{B}(n + 1) \qquad (37b)$

Now to reach eq. (30) [i.e.$E_0(n) = \frac{k}{B}(n + 1/2)$] we have to subtract an infinite term $\frac{B}{2m}$ and a finite term $\frac{k}{2B}$ from eq. (37b)and so there lies a discrepancy!

Also in eq.(32) an extra 1/2 factor comes the eigenvalue of M_0 compared to that of M in eq. (27a) i.e. discrepancy arises here too.

The wave functions in (27d) become,in the zero-mass limit,

$\langle \mathbf{q}|0,|n|\rangle \underset{m\to 0}{\longrightarrow} (\frac{1}{\pi|n|})^{1/2}[m(\frac{B}{2m}+\frac{k}{B}]^{(1+|n|)/2} r^{|n|} e^{in\theta} e^{-\frac{m}{2}(\frac{B}{2m}+\frac{k}{B})r^2} \times 1$

$= (\frac{1}{\pi|n|})^{1/2} \frac{B}{2}^{(1+|n|)/2} r^{|n|} e^{i|n|\theta} e^{-\frac{B}{4}r^2}$

$= (\frac{B}{2})^{|n|/2} (\frac{B}{2\pi})^{1/2} \frac{1}{\sqrt{|n|!}} r^{|n|} e^{i|n|\theta} e^{-\frac{B}{4}r^2}$

$= (\frac{B}{2\pi})^{1/2} \frac{1}{\sqrt{|n|!}} (\sqrt{\frac{B}{2}} re^{i\theta})^{|n|} e^{-\frac{B}{4}r^2}$

$= (\frac{B}{2\pi})^{1/2} \frac{1}{\sqrt{|n|!}} (\sqrt{\frac{B}{2}} re^{i\theta})^{|n|} e^{-\sqrt{\frac{B}{2}}re^{i\theta}\sqrt{\frac{B}{2}}re^{-i\theta}}$

$= (\frac{B}{2\pi})^{1/2} \langle\alpha||n|\rangle e^{-\alpha^*\alpha/2}$ (38)

Here α is defined by the state $\langle|\alpha$ as $\langle\alpha|a\dagger = \langle\alpha|\alpha = \langle\alpha|\sqrt{\frac{B}{2}}re^{i\theta}$ and following this Holomorphic representation[5] we have $\langle\alpha|n\rangle = \frac{\alpha^n}{\sqrt{n!}}]$ (as we already know that $|n\rangle = \frac{\dagger a^n}{\sqrt{n!}}|0\rangle$)

Again this result tells us that the complete wave functions do not approach those of the reduced theory since the former involves two variables α and α^*(or rather $q^1 and q^2$)but the reduce theory depends only one coordinate x. So there may be some anomaly in our representation.

6 New Representation of The Complete theory wave-function :

Hamiltonian in (7)

$H = \frac{1}{2m}(p^i + \frac{B}{2}\epsilon^{ij}q^j)(p^i + \frac{B}{2}\epsilon^{ik}q^k) + \frac{k}{2}q^iq^i$

$= \frac{1}{2m}(p^ip^i + \frac{B}{2}\epsilon^{ik}p^iq^k + \frac{B}{2}\epsilon^{ij}q^jp^i + \frac{B}{4}\epsilon^{ik}\epsilon^{ij}q^jq^k) + \frac{k}{2}q^iq^i$

$= \frac{1}{2m}p^ip^i + \frac{B}{4m}(\epsilon^{ik}p^iq^k + \epsilon^{ij}q^jp^i) + \frac{1}{2}(\frac{B^2}{4m}+k)q^iq^i$

$= \frac{1}{2m}p^ip^i + \frac{B}{4m}(\epsilon^{ik}p^iq^k + \epsilon^{ij}(i\delta ij + p^iq^j)) + \frac{m}{2}\Omega^2q^iq^i$

$= \frac{1}{2m}p^ip^i + \frac{B}{4m}(\epsilon^{ik}p^iq^k + \epsilon^{ij}p^iq^j) + \frac{m}{2}\Omega^2q^iq^i$

$= \frac{1}{2m}p^ip^i + \frac{B}{2m}\epsilon^{ij}p^iq^j + \frac{1}{2}m\Omega^2q^iq^i$ $(39a)$

$= \frac{1}{2m}\mathbf{p^2} + \frac{1}{2}m\Omega^2\mathbf{q^2} + \frac{B}{2m}\mathbf{p}\times\mathbf{q}$ **(39b)**

Now we choose canonical pairs $(p\pm, q\pm)$ such that

$p_\pm = (\frac{\omega_\pm}{2m\Omega})^{1/2}p^1 \pm (\frac{m\Omega\omega_\pm}{2})^{1/2}q^2$ $(40a)$

and $q_\pm = (\frac{m}{2\omega_\pm})^{1/2}q^1 \mp (\frac{1}{2m\Omega\omega_\pm})^{1/2}p^2$ $(40b)$

where $\omega_\pm = \Omega \pm \frac{B}{2m}$ $(40c)$

Now $\sqrt{\omega_-}p_+ + \sqrt{\omega_+}p_- = 2\sqrt{\frac{\omega_+\omega_-}{2m\Omega}}p^1 = \frac{1}{\sqrt{m\omega}}.\sqrt{2\omega_+\omega_-}p^1$ $(41a)$

and $\sqrt{\omega_-}p_+ - \sqrt{\omega_+}p_- = 2\sqrt{\frac{m\Omega\omega_+\omega_-}{2}}q^2 = \sqrt{m\omega}.\sqrt{2\omega_+\omega_-}q^2$ $(41b)$

Also $\frac{q_+}{\sqrt{\omega_-}} + \frac{q_-}{\sqrt{\omega_+}} = 2\sqrt{\frac{m\Omega}{2\omega_+\omega_-}}q^1 = \sqrt{m\omega}.\sqrt{\frac{2}{\omega_+\omega_-}}q^1$

and $\frac{q_-}{\sqrt{\omega_+}} - \frac{q_+}{\sqrt{\omega_-}} = 2\frac{1}{\sqrt{2m\Omega\omega_+\omega_-}}p^2 = \frac{1}{\sqrt{m\Omega}}.\sqrt{\frac{2}{\omega_+\omega_-}}p^2$

Thus the Hamiltonian expressed in co-ordinates and momenta,

$$H = \frac{1}{2m}((p^1)^2 + (p^2)^2) + \frac{1}{2}m\Omega^2((q^1)^2 + (q^2)^2) + \frac{B}{2m}(p^1q^2 - p^2q^1)$$
$$= \frac{1}{2m}\frac{m\Omega}{2\omega_+\omega_-}(\sqrt{\omega_-}p_+ + \sqrt{\omega_+}p_-)^2 + \frac{1}{2m}.m\Omega\frac{\omega_+\omega_-}{2}(\frac{q_-}{\sqrt{\omega_+}} - \frac{q_+}{\sqrt{\omega_-}})^2$$
$$+\frac{1}{2}m\Omega^2.\frac{1}{m\Omega}\frac{\omega_+\omega_-}{2}(\frac{q_+}{\sqrt{\omega_-}} + \frac{q_-}{\sqrt{\omega_+}})^2 + \frac{1}{2}m\Omega^2.\frac{1}{m\Omega}.\frac{1}{2\omega_+\omega_-}(\sqrt{\omega_-}p_+ - \sqrt{\omega_+}p_-)^2$$
$$+\frac{B}{2m}(p^1q^2 - p^2q^1)$$
$$= \frac{\Omega}{4\omega_+}p_+^2 + \frac{\Omega}{4\omega_-}p_-^2 + \frac{\Omega p_+p_-}{2\sqrt{\omega_+\omega_-}} + \frac{\Omega\omega_-}{4}q_-^2 + \frac{\Omega\omega_+}{4}q_+^2 - \frac{\Omega\sqrt{\omega_+\omega_-}}{2}q_+q_-$$
$$+\frac{\Omega\omega_+}{4}q_+^2 + \frac{\Omega\omega_-}{4}q_-^2 + \frac{\Omega\sqrt{\omega_+\omega_-}}{2}q_+q_- + \frac{\Omega}{4\omega_+}p_+^2 + \frac{\Omega}{4\omega_-}p_-^2 - \frac{\Omega p_+p_-}{2\sqrt{\omega_+\omega_-}}$$
$$+\frac{B}{2m}(p^1q^2 - p^2q^1)$$
$$= \frac{\Omega}{2\omega_+}p_+^2 + \frac{\Omega}{2\omega_-}p_-^2 + \frac{\Omega\omega_+}{4}q_+^2 + \frac{\Omega\omega_-}{4}q_-^2$$
$$+\frac{B}{2m}\sqrt{m\Omega}\frac{1}{\sqrt{2\omega_+\omega_-}}\frac{1}{\sqrt{m\Omega}}\frac{1}{\sqrt{2\omega_+\omega_-}}(\omega_-p_+^2 - \omega_+p_-^2)$$
$$+\frac{B}{2m}\sqrt{m\Omega}\sqrt{\frac{\omega_+\omega_-}{2}}\frac{1}{\sqrt{m\Omega}}\sqrt{\frac{\omega_+\omega_-}{2}}(\frac{q_-^2}{\omega_+} - \frac{q_+^2}{\omega_-})$$
$$= \frac{1}{2}(\Omega + \frac{B}{2m})\frac{p_+^2}{\omega_+} + \frac{1}{2}(\Omega - \frac{B}{2m})\frac{p_-^2}{\omega_-} + \frac{1}{2}(\Omega - \frac{B}{2m})\omega_-q_-^2 + \frac{1}{2}(\Omega + \frac{B}{2m})\omega_+q_+^2$$
$$= (\frac{1}{2}p_+^2 + \frac{1}{2}\omega_+q_+^2) + (\frac{1}{2}p_-^2 + \frac{1}{2}\omega_-q_-^2) \qquad \text{[usng eq.s]}$$

Therefore $H = H_+ + H_-$ (42) [considering $m = 1$]

Thus the wave functions of the complete problem can be presented in the "new coordinate" representation:

$\langle q \pm | N, n\rangle = u^+_{n_+}(q_+)u^+_{n_+}(q_+)$ (43a)

where $u_{n_\pm}$ is the harmonic oscillator wave functions(36) with frequencies $\omega_\pm$ and $n_\pm = N + \frac{|n| \mp n}{2}$ (43b)

satisfies (27)

i.e. $E(N, n) = \Omega(n_+ + 1/2 + n_- + 1/2) - \frac{B}{2m}n$ [Look in eq. (39b) the last term equals to $-\frac{B}{2m}M$ follwing eq.(22)]

Now as $m \to 0$, $\omega_+ \to B/m + k/B$ and $\omega_- \to k/B$.So only the minus signed oscilator contributes in eq.(43a)

Alternatively we can get the same wave function using the holomorphic representation of the new coordinates i.e. here we take

$a_\pm = \frac{1}{\sqrt{2}}(\sqrt{\omega_\pm}q + i\frac{p\pm}{sqrt\omega_\pm}$(44a)

and $\langle\alpha_\pm|\dagger\alpha_\pm = \langle\alpha|\alpha_\pm$(44b)

and so $\langle\alpha_\pm|\ N, n\rangle = \frac{{\alpha_+}^{n_+}}{\sqrt{n_+!}}\frac{{\alpha_-}^{n_-}}{\sqrt{n_-!}}$ (44c)

7 Chiral oscillator problem -finding similarity:

From the previous discussions we can easily find out a nice similarity with the chiral oscillator problem.

The Lagrangian for chiral oscillator can be written as:

$L_+ = \frac{1}{2}\epsilon^{ij}q^i\dot{q}^j - \frac{k}{2}q^iq^i$ (45)

which is similar to L_0 of eq.(5) upto a factor B in the first term. Now the canonically conjugate momentum: $p^j = \frac{\partial L_+}{\partial \dot{q}^j} = \frac{1}{2}\epsilon^{ij}q^i$ (46) So the Hamiltonian,

$$\begin{aligned} H_+ &= p^i q^i - L_+ \\ &= \frac{1}{2}\epsilon^{ji}q^j\dot{q}^i - (\frac{1}{2}\epsilon^{ij}q^i\dot{q}^j - \frac{k}{2}q^iq^i) \\ &= \frac{1}{2}\epsilon^{ij}q^i\dot{q}^j - (\frac{1}{2}\epsilon^{ij}q^i\dot{q}^j - \frac{k}{2}q^iq^i) \qquad (47) \end{aligned}$$

This Hamiltonian is identical with H_0 in eq.(12) Also it satisfies the noncommutative relation in eq.(13)[5]. Thus we can conclude that the Lagrangian and the Hamiltonian behave like a chiral oscillator in the reduced theory($m \to 0$).

Conclusion:

Thus we observed an quantum mechanical analog of the gauge field theory.We also saw the non-commutativity in the reduced phase-space.The shift in the zero-point eigen values agrees the non-commutivity between phase-space reduction and quantization.This anomaly can be removed by introduction of the operators in eqs.(40)

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