Microscopic Fermi liquid theory

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Almost all the metals behave in the same way at low temperature even though some of them, having narrow (localized) d or f orbitals (e.g. transition metal Cu and rare-earth compound $CeAl_3$) and hence the electronic interactions are not ignorable there. Lev Landau in 1956 came up with a phenomenological theory which predicts a renormalized version of Drudé metals. The phenomenological idea was that the low energy excited state due to interaction has a one-to-one correspondence to the non-interacting state and the former state can be achieved by adiabatically[1] switching on the interaction. This phenomenological model was later carried out by Abrikosov and Kalatnikov in the formalism of diagrammatic perturbation which is essentially known as the microscopic Fermi liquid theory. The non-interacting single particle propagator (Green's function) can be written as

$$\mathcal{G}(\mathbf{k},\omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} + \mu + i\eta} \tag{1}$$

where $\eta \to 0+$, μ is the chemical potential, and $\epsilon_{\mathbf{k}}$ is the non-interacting dispersion relation ($\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/(2m)$ for Fermi gases with particle mass m.) The interacting Green's function modify the above equation by an extra term $\Sigma(\omega)$, known as the self-energy:

$$G(\mathbf{k},\omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} + \mu - \Sigma(\mathbf{k},\omega) + i\eta} \tag{2}$$

$$\begin{aligned}\mathrm{Re}\Sigma(\mathbf{k},\omega) = {}& \mathrm{Re}\Sigma(\mathbf{k}=\mathbf{k}_F,\omega=\mu) - b(\mathbf{k}_F)(\omega-\mu) \\ & - \mathcal{O}((\omega-\mu))^2 + a(\mathbf{k}_F)(k-k_F) + \mathcal{O}((k-k_F)^2)\end{aligned} \tag{3}$$

[1]In a loose sense, it means *very slowly*.

$$\begin{aligned}\mathrm{Im}\Sigma(\mathbf{k},\omega) = -\Gamma(\omega-\mu))^2 + \mathcal{O}((\omega-\mu))^3 \\ \zeta(\mathbf{k}_F)(k-k_F)^2 + \mathcal{O}((k-k_F)^3)\end{aligned} \tag{4}$$

This gives rise to the renormalized Green's function,

$$G^R(\mathbf{k},\omega) \simeq \frac{Z_{\mathbf{k}_F}}{\omega - \epsilon^*(\mathbf{k}-\mathbf{k}_F) + i[Z_{\mathbf{k}_F}\{\Gamma(\omega-\mu)^2 - \zeta(\mathbf{k}_F)(\mathbf{k}-\mathbf{k}_F)^2\}]} \tag{5}$$

where

$$Z_{\mathbf{k}_F} = \frac{1}{1+b(\mathbf{k}_F)} \tag{6}$$

and

$$\epsilon^*(\mathbf{k}-\mathbf{k}_F) \simeq Z_{\mathbf{k}_F}[\epsilon_0 + a(\mathbf{k})(\mathbf{k}-\mathbf{k}_F)] \tag{7}$$

0.1 Luttinger theorem

In 1960, Luttinger showed that the for a Fermi liquid (FL), the volume of the Fermi surface (FS) remains the same though its shape may change due to interaction [**?**]. This is known as the *Luttinger theorem*, and the later work by Langer and Ambegaokar (1961) and by Langreth (1966) gave a generalized Friedel sum-rule for a FL :

$$\mathrm{Im}\int_{-\infty}^{\mu} d\omega\, G(\omega)\frac{\partial\Sigma(\omega)}{\partial\omega} = 0 \quad . \tag{8}$$