Quantum Field Theory for Many Body Systems:
2016-2017
Lecture 1: Introduction, Second Quantization

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1 Introduction

The philosophical case for studying many particle physics was first laid out by Philip W. Anderson in *More is Different*, Science, 177 393 (1972). Broadly speaking, we may take a reductionist approach or a constructionist approach to understanding nature. The reductionist approach is exemplified by high energy physics which breaks down matter into its constituent parts and tries to understand their interactions in terms of ‘fundamental’ laws. However, the whole is greater than the sum of its parts. In Anderson’s words,

*The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe.*

For example, we know that quantum particles are described by the Schrödinger equation. A large collection of such particles forms a ball, which as we know, is well described by Newton’s classical equations of motion. In principle, one must be able to start with the Schrödinger equation and deduce Newton’s laws. However, this is an impossible task due to issues of scale and complexity. Similarly, we can not begin with Newton’s laws to explain all of thermodynamics.

Moral of the story: every energy scale is ‘fundamental’ in its own right. At every energy scale, we need a new physics with its own building blocks and its own laws.

An important motivation for many body theory is that it helps us bridge scales. It allows us to begin with a ‘microscopic’ picture and to describe ‘macroscopic’ phenomena. Naïvely, the microscopic theory is quantum mechanical in nature while the macroscopic theory is classical. However, there are several dramatic exceptions to this, e.g., superconductivity is macroscopic but quantum mechanical, spins with easy axis anisotropy are microscopic but classical, etc. Seen from this perspective, many body theory interpolates between quantum mechanics and statistical mechanics.
2 Comparison with quantum field theory as applied to HEP:

Quantum field theory, as applied to many body physics, is essentially the same as that used in high energy physics. The strongest commonality is the mathematical structure. This is reflected in several common themes, e.g.,

- spontaneous symmetry breaking
- Higg’s mechanism (first proposed in superconductors by Anderson)
- topological invariants
- gauge theories
- renormalisation group.

However, there are two key differences. The first is the non-relativistic nature of many body physics. The systems that are typically studied in many body physics are well described by non-relativistic theories. For example, the velocity of electrons in a metal is nowhere near the speed of light. It is not necessary to invoke relativity in a theory for metals. An apt example is to compare a collision of protons in a particle accelerator with the scattering of electrons in a metal. In a typical course on quantum field theory, relativistic effects are presented as strong motivation for the need to invoke quantum field theory. Relativistic energy conservation allows the protons in an accelerator to disintegrate into several new particles – this effect can be captured within quantum field theory. With electrons in a metal, it more than suffices to treat the motion of electrons non-relativistically. Relativistic corrections to energy are negligible. However, we still need a field theoretic description, e.g., the electron-electron scattering may create photons (lattice vibrations). Note that there are several exceptions to this rule. For example, to describe the physics of a neutron star, we need a relativistic many body field theory. There are also many body analogues of relativity, the most prominent of which is graphene.

The second key difference vis-à-vis HEP is the nature of the vacuum. In many body systems, the vacuum (the background) already has many particles. Taking the example of electrons scattering in a metal, the electrons are scattering in the background of a Fermi sea. There are several other electrons that are occupying available states – note that these are real particles which have conserved quantum numbers (e.g., electronic charge) and not just virtual (short lived) particles. The outgoing electrons from the scattering process cannot go into these levels due to Pauli’s exclusion principle. This is markedly different from the case of protons in an accelerator which sees an ‘empty’ vacuum.

3 Typical systems of applicability

A key selling point of many body theory is its high testability. Its principles apply to a wide variety of systems with various microscopic interactions, many
of which are readily available for experimental study. The experimental systems also come with a large number of tuning parameters which help us explore various limits of our theories. Some experimentally accessible many body systems are

- 2d electron gap realised in semiconductor well geometries (Coulomb)
- electrons in metals (lattice-mediated)
- liquid He (van der Waal’s)
- ultracold atomic gases, e.g., K atoms at nanoKelvin temperatures (contact)
- quantum magnets (dipolar, exchange)
- possible superfluid core of neutron stars (strong interactions)

For each system, we have given the nature of the interactions in parentheses. Many body theory gives us a systematic way of approaching such a wide range of systems.

4 Second quantisation

We will deal with quantum states of many particles. A convenient book keeping tool to describe many body states is called second quantisation.

Let us first consider single particle states, e.g., in a quantum harmonic oscillator (QHO). The Hamiltonian is given by

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 x^2. \]  

(1)

From basic quantum mechanics, the eigenvalues are known to be \( E_n = (n + \frac{1}{2})\hbar\omega \). The eigenstates are also known. They are described by Hermite polynomials of \( x \).

The time evolution of an eigenstate is given by

\[ i \frac{\partial \psi_n}{\partial t} = \hat{H}\psi_n = \epsilon_n \psi_n \rightarrow \psi_n(x, t) = e^{-i\epsilon_n t}\psi_n(x). \]  

(2)

We have set \( \hbar = 1 \). An arbitrary state can be written as a combination of eigenstates, leading to

\[ \psi_{arb}(x, t) = \sum_n A_n e^{-i\epsilon_n t}\psi_n(x); \quad \sum_n |A_n|^2 = 1. \]  

(3)

The Schrödinger equation for a QHO is linear. That is, any linear combination of solutions (suitably normalised) is also a solution. Typically, interactions between particles introduces non-linearities. This necessitates a field theory description.
Consider two particles in a QHO. The Hamiltonian is
\[
\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{\hat{p}_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2.
\] (4)
As the Hamiltonian is of the form \(\hat{H}_1 + \hat{H}_2\), the solution is separable, i.e.,
\[
\Phi(x_1, x_2, t) = \phi_1(x_1, t)\phi_2(x_2, t).
\] (5)
Suppose the first particle is in the \(n = 3\) state of the QHO and the second particle is in the \(n = 5\) state. We have \(\Phi(x_1, x_2, t) = \psi_3(x_1, t)\psi_5(x_2, t)\). This is a valid solution of the Schrödinger equation, however the particles are distinguishable. For example, we may evaluate the energy of the first particle,
\[
\langle \hat{H}_1 \rangle = \int dx_1 dx_2 \psi_3^*(x_1, t)\psi_5^*(x_2, t)\hat{H}_1\psi_3(x_1, t)\psi_5(x_2, t) = \epsilon_3.
\] (6)
Similarly, we will find \(\langle \hat{H}_2 \rangle = \epsilon_5\). Thus, we can distinguish the two particles by making a measurement.

To enforce the principle of indistinguishability, we can write
\[
\Phi_\zeta(x_1, x_2, t) = \frac{1}{\sqrt{2}} [\psi_3(x_1, t)\psi_5(x_2, t) + \zeta\psi_5(x_1, t)\psi_3(x_2, t)],
\] (7)
where \(\zeta\) is the exchange parameter which takes values \(\pm 1\). We call \(\zeta = 1\) bosonic, it represents wavefunctions symmetric under exchange. We call \(\zeta = 1\) fermionic representing wavefunctions which are anti-symmetric under exchange.

Note: in dimensions \(D = 1\) and \(D \geq 3\), we only have two choices for \(\zeta\). Two dimensional systems are much more richer allowing for ‘anyons’. We will discuss this possibility in this course. In \(D = 1\), it is easy to see that we have only two options for \(\zeta\) since \(\hat{P}_{12}^2 = 1\) (the exchange operator squares to identity). This constrains \(\zeta\) to \(\pm 1\).

We highlight the following aspects from the above discussion. For particles in a one-dimensional QHO,

- \(\Phi(x_1, x_2, t)\) is a function defined in two spatial dimensions and one time dimension. It must obey the usual quantum mechanical constraint of normalisability, i.e., at any time \(t\), the modulus-squared of the wavefunction integrated over space should be unity. However, not all functions that satisfy this are allowed wavefunctions \(\rightarrow\) only symmetric/anti-symmetric functions are allowed.

- For \(N\) particles, we have symmetric/anti-symmetric functions in \(N\) dimensions.

- To deal with \(N\) particles, we must work in \(N\) dimensions while always ensuring symmetry/anti-symmetry.

- Second quantisation is a tool that allows us to work in one dimension and automatically keeps track of symmetry/anti-symmetry.
Before discussing second quantisation, we note that the above way of expressing $N$-particle wavefunctions as functions of $N$ variables is called ‘first quantisation’.

To apply second quantisation, we must first fix two aspects of the system of interest:

1. The particles are fixed to be bosonic or fermionic
2. A suitable basis for single-particle wavefunctions is decided – we will refer to these as ‘reference’ states

We will further assume that the reference states can be labelled by suitable quantum numbers and sorted in some order. For example, to describe $N$ electrons sitting in a QHO, we first fix the nature of the particles to be fermionic. We may choose the energy eigenstates of the QHO to be our reference states. The states are labelled by the energy eigenvalues.

We define the key notion of Fock space that underlies second quantisation.

**Definition**: $\mathcal{F} = \bigoplus N \mathcal{F}_N$.

Here, $\mathcal{F}_N$ is the set of all legitimate $N$-particle wavefunctions, i.e., functions of $N$ variables that are normalisable and (anti-)symmetric. The symbol $\bigoplus$ denotes a direct product over all $N$. The Fock space thus contains all allowed wavefunctions of $0, 1, \cdots$ particles.

To denote an arbitrary element of the Fock space, we will use the occupation number representation, which denotes a state by the number of particles present in each reference state.

**Definition**: A state in the occupation number representation is denoted as $|n_1, n_2, \cdots\rangle$. This represents a state with $n_1$ particles in reference state 1, $n_2$ particles in state 2 and so on.

The total number of particles in this state is $N_{\text{tot}} = \{n_1 + n_2 + \cdots\}$. It is understood that this wavefunction is appropriately symmetrised/anti-symmetrised. To show that this is always consistent, we explicitly demonstrate that we can construct such a $N$-dimensional wavefunction.

### 5 Wavefunctions for fermions

To translate from the occupation number representation to first quantised notation, we use a ‘Slater determinant’. A wavefunction for $M$ fermions is given by

$$|n_1, n_2, \cdots\rangle \rightarrow \Psi_F(x_1, \cdots, x_M) = \frac{1}{\sqrt{M!}} \text{Det} \begin{bmatrix}
\phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_M(x_1) \\
\phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_M(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_M) & \phi_2(x_M) & \cdots & \phi_M(x_M)
\end{bmatrix}.$$  

(8)

Here, $\text{Det}[]$ represents the matrix determinant. It can easily be seen that this determinant gives a function of $M$ variables $x_1, x_2, \cdots, x_M$. The total number
of particles is \( n_1 + n_2 + \cdots = M \). The wavefunctions \( \phi_1(x) \), \( \phi_2(x) \), etc. are reference single-particle wavefunctions. In the above expression, we have \( M \) such wavefunctions \( \phi_i(x) \). We choose \( n_1 \) of these wavefunctions to be the first reference wavefunction (corresponding to the first entry in \( |n_1, n_2, \cdots \rangle \)), \( n_2 \) of these to be the second reference wavefunction, etc.

Clearly, \( \Psi_F(x_1, \cdots, x_N) \) is a wavefunction for an \( M \) particle system. Interchanging any two particles simply corresponds to switching two rows of the matrix, which gives an overall minus sign in the determinant. Thus, \( \Psi(x_1, \cdots, x_N) \) is anti-symmetric under exchange of any particles.

**N. B.**: the Slater determinant automatically enforces the Pauli exclusion principle. If we have two particles in one reference state, we see that two columns of the matrix in \( \Psi_F \) become equal, making the determinant zero.

**Exercise:** Consider fermions in a QHO. Take a simple 3 particle wavefunction \( |1, 1, 1, 0, \cdots \rangle \) in which only the first three reference levels are occupied. Write this wavefunction in first quantised form. Evaluate the expectation value of the Hamiltonian \{\( H_1 + H_2 + H_3 \)\} in this wavefunction.

### 6 Wavefunctions for bosons

We can write an analogous form of the \( M \) particle wavefunction for bosons that is symmetric under exchange of any two particles.

\[
|n_1, n_2, \cdots \rangle \rightarrow \Psi_B(x_1, \cdots, x_N) = \frac{1}{\sqrt{M!}} \text{Per} \begin{bmatrix}
\phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_M(x_1) \\
\phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_M(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_M) & \phi_2(x_M) & \cdots & \phi_M(x_M)
\end{bmatrix}.
\]

The notation \( \text{Per}[] \) denotes a ‘permanent’ – this is identical to the determinant, except there are no minus signs in the expansion. For example,

\[
\text{Per} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ab + cd. \tag{10}
\]

This can easily be seen to symmetric under exchange of particles.

**Exercise:** Consider bosons in a QHO. Take the same simple 3 particle wavefunction \( |1, 1, 0, \cdots \rangle \) as above. Write this wavefunction in first quantised form. Evaluate the expectation value of the Hamiltonian \{\( H_1 + H_2 + H_3 \)\} in this wavefunction.

The Slater determinant and the Slater permanent give us a prescription to go from second-quantised notation (occupation number representation) to first quantised notation.

**Exercise:** Consider two particles in a QHO, one in the ground state and one in the excited state. TakeWhat is the average separation (distance squared) between them if they are (a) bosons, or (b) fermions?
7 Operators in Fock space

Let us first examine a two particle wavefunction of bosons and fermions in first quantised notation. We have \( |\Psi_{B/F}(x_1, x_2)\rangle = \frac{1}{\sqrt{2}} \{ \phi_1(x_1) \phi_2(x_2) + \zeta \phi_1(x_2) \phi_2(x_1) \} \).

To write this, we have designated one particular reference state as \( \phi_1(x) \) and another as \( \phi_2(x) \). Suppose we switch our designation by interchanging \( \phi_1(x) \) and \( \phi_2(x) \). This does not change the two-particle wavefunction in the bosonic case. However, in the fermionic case, the wavefunction picks up a minus sign. This is a general property of fermionic systems – the sign of the first quantised wavefunction depends on our labelling of the reference states. The occupation number representation in second quantised language also inherits from this defect, viz., the wavefunction of a given state depends on our initial choice of ordering in the set of reference states. This is quite a nuisance; before calculating any matrix element, we will have to verify the reference state set. We will now develop an operator formalism that frees us of this responsibility by consistently defining the occupation number states.

We define a creation operator in Fock space \( a_i^\dagger \) as follows
\[
a_i^\dagger |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1} \zeta_i s_i |n_1, n_2, \ldots, n_i + 1, \ldots\rangle,
\]
where \( (n_i + 1) \) is defined modulo 2 for fermions. This ensures that if we already have one particle in the reference state, then \( a_i^\dagger \) makes the wavefunction zero. As defined before, \( \zeta = \pm 1 \) for bosons/fermions. The quantity \( s_i = n_1 + n_2 + \ldots + n_{i-1} \) counts the number of particles in reference states that occur before the \( i^{th} \) state in our sorted list of reference states. The \( \zeta s_i \) factor is a book-keeping tool to keep track of the fermionic sign.

The operator \( a_i^\dagger \) increases the number of particles in the \( i^{th} \) reference state. N.B. : \( a_i \) and \( a_i^\dagger \) are NOT the usual creation and annihilation operators of the QHO algebra. The QHO operators connect different single particle states. Here, the operators connect different sectors in Fock space which differ in particle number.

We can now see how the fermionic/bosonic nature is reflected in the commutation relations of operators, it is easy to verify (try this as an exercise) that
\[
[a_i^\dagger, a_j^\dagger]_\zeta \equiv [a_i^\dagger, a_j^\dagger]_\zeta = a_i^\dagger a_j^\dagger - \zeta a_j^\dagger a_i^\dagger = 0.
\]
Thus, fermionic creation operators anti-commute while bosonic operators commute. The symmetrisation properties of the wavefunctions have now been absorbed into the creation operators.

For fermions, it is easy to see that \( (a_i^\dagger)^2 = 0 \). This is a consequence of a more general property by which the symmetric/anti-symmetric nature of particles reflects in the commutation properties of creation operators.

We can now define an arbitrary state in Fock space using creation operators,
\[
|n_1, n_2, \ldots\rangle = \left\{ \prod_i \frac{(a_i^\dagger)^{n_i}}{(n_i!)^{1/2}} \right\} |0, 0, \ldots\rangle
\]
Here, the state $|0,0,\ldots\rangle$ denotes the vacuum which has no particles. We see that this definition cures the ordering ambiguity. If the ordering of the reference states is changed, the creation operators change accordingly so that the state remains the same.

**Exercise:** Consider two fermionic particles in two states of a QHO; one in the ground state and the other in the first excited state. Take two labelling conventions: (i) ground state – a, excited state – b, and (ii) ground state – b, excited state – a. In each case, write the state in occupation number basis and in first quantised language. Now, in each case, represent the state using creation operators as above. By arguing that switching the basis should still give us the same state, show that the fermionic creation operators must anti-commute.

### 7.1 Properties of creation and annihilation operators

Let us now define an overlap of states in Fock space.

$$\langle m_1, m_2, \ldots | n_1, n_2, \ldots \rangle = \delta_{m_1,n_1} \delta_{m_2,n_2} \ldots$$

(14)

We assert this as a postulate. Note that the overlap matrix element between states with different particle numbers will always be zero with this definition. Using the definition of the creation operator, we can now assert that

$$\langle m_1, m_2, \ldots | a_i^\dagger | n_1, n_2, \ldots \rangle = (n_i + 1)^{1/2} \zeta_s^{m_i} \delta_{m_1,n_1} \delta_{m_2,n_2} \ldots \delta_{m_i,n_i+1} \ldots,$$

(15)

where $\zeta_s^{m_i} = n_1 + n_2 + \ldots + n_{i-1}$. Let us postulate that Hermitian conjugate of this equation is

$$\langle n_1, n_2, \ldots | a_i | m_1, m_2, \ldots \rangle = (m_i)^{1/2} \zeta_s^{m_i} \delta_{m_1,n_1} \delta_{m_2,n_2} \ldots \delta_{m_i,n_i+1} \ldots,$$

(16)

We note that $\zeta_s^{m_i} = n_1 + n_2 + \ldots + n_{i-1} = m_1 + m_2 + \ldots + m_{i-1} = \zeta_s^{m_i}$.

This defines the action of our annihilation operator $a_i$ as

$$a_i | n_1, n_2, \ldots, n_i, \ldots \rangle = \sqrt{n_i} \zeta_s^{m_i} | n_1, n_2, \ldots, n_i - 1, \ldots \rangle.$$  

(17)

If the initial state does not have any particles in the $i^{th}$ reference state, then $a_i$ makes the wavefunction zero. As with the creation operators, we can easily check that

$$[a_i, a_j] \zeta = [a_i, a_j] \zeta = a_i a_j - \zeta a_j a_i = 0.$$  

(18)

**Exercise:** Using the above definitions, show that $[a_i, a_j^\dagger] \zeta = \delta_{ij}$.

We have shown how to represent many particle states using creation and annihilation operators. In order to have a complete prescription, we must also be able to represent arbitrary operators using these creation and annihilation operators – we will show this next. Once we achieve this, we can perform any calculation (any matrix element) using creation and annihilation operators alone!