Duality in Communication Complexity

Alexander Sherstov
Microsoft Research

Communication complexity

Alice

Bob

[Yao 1979]

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$x \in X$

$y \in Y$

Objective: compute $F(x, y)$ with probability $> 2/3$.

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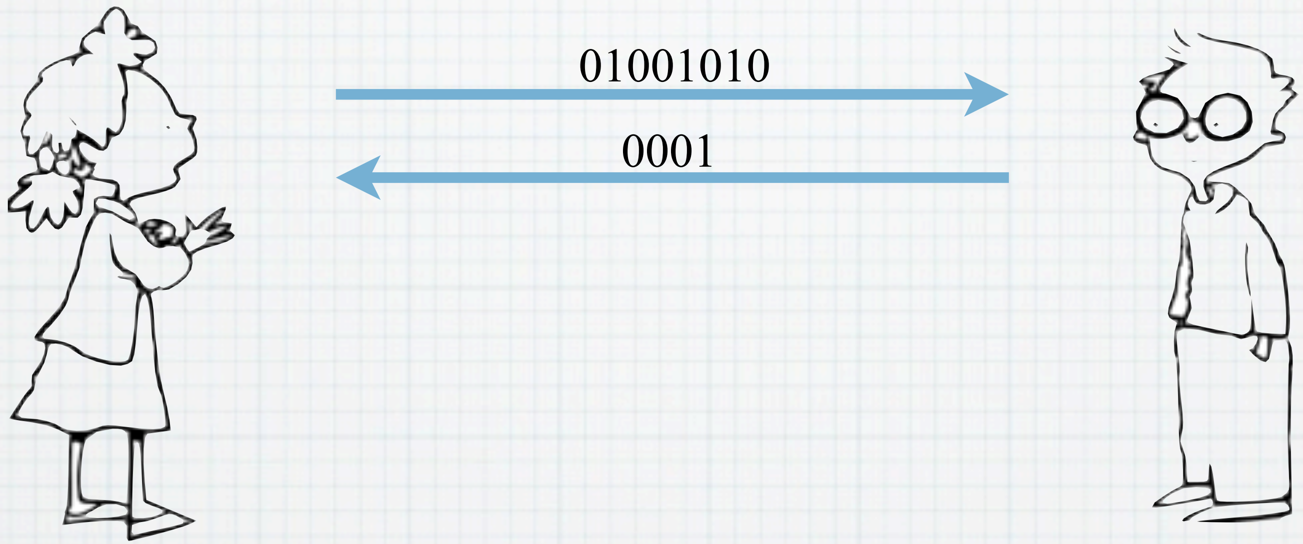

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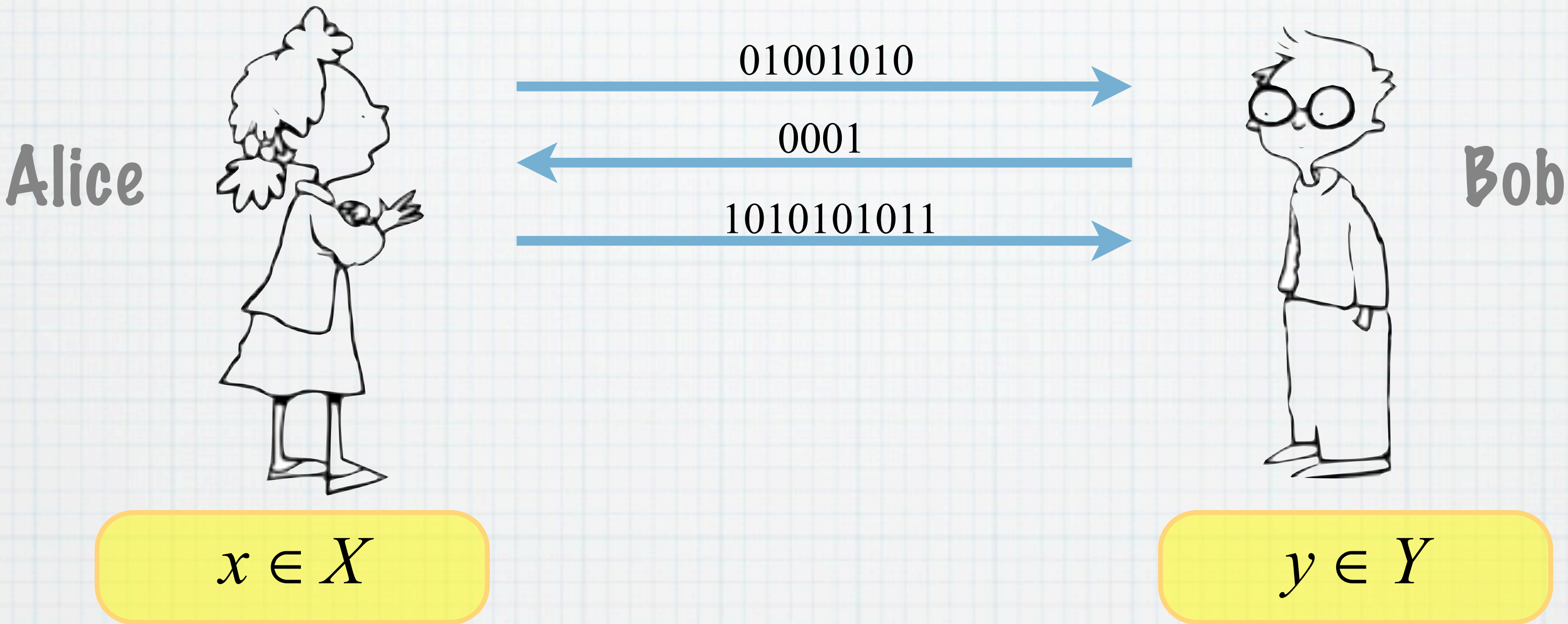

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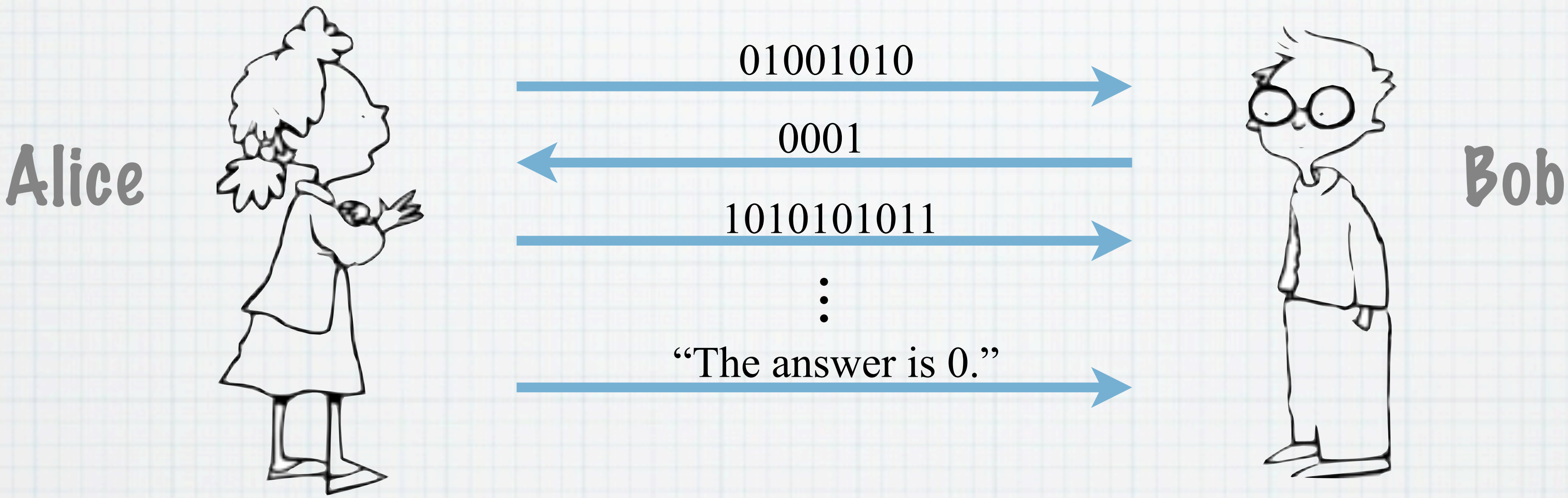

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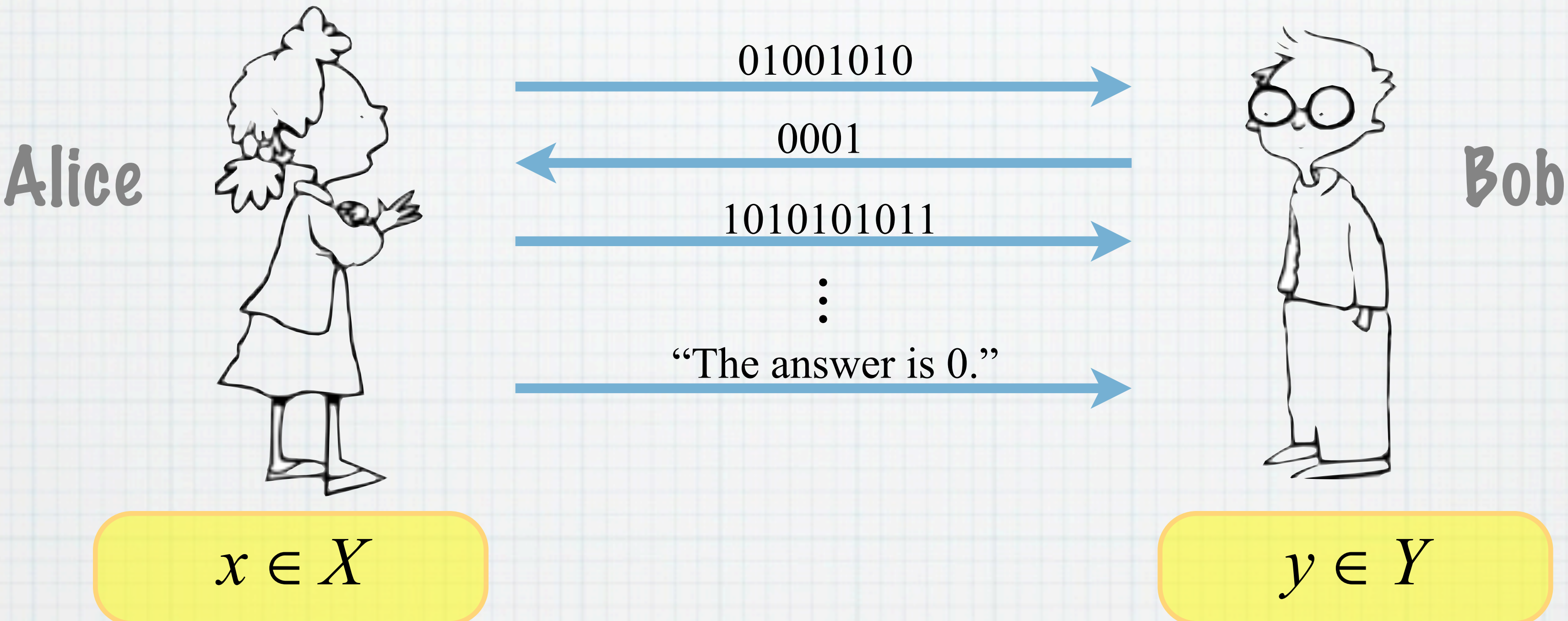

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Complexity measure: # bits exchanged.

[Yao 1979]

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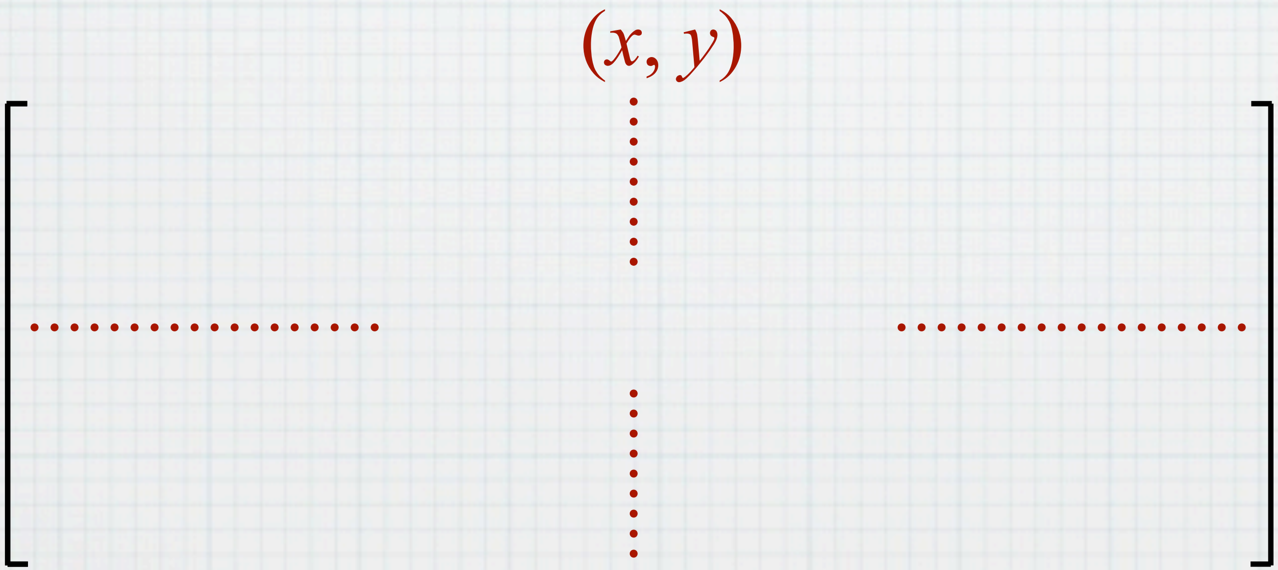

Π: deterministic protocol of cost $\le c$

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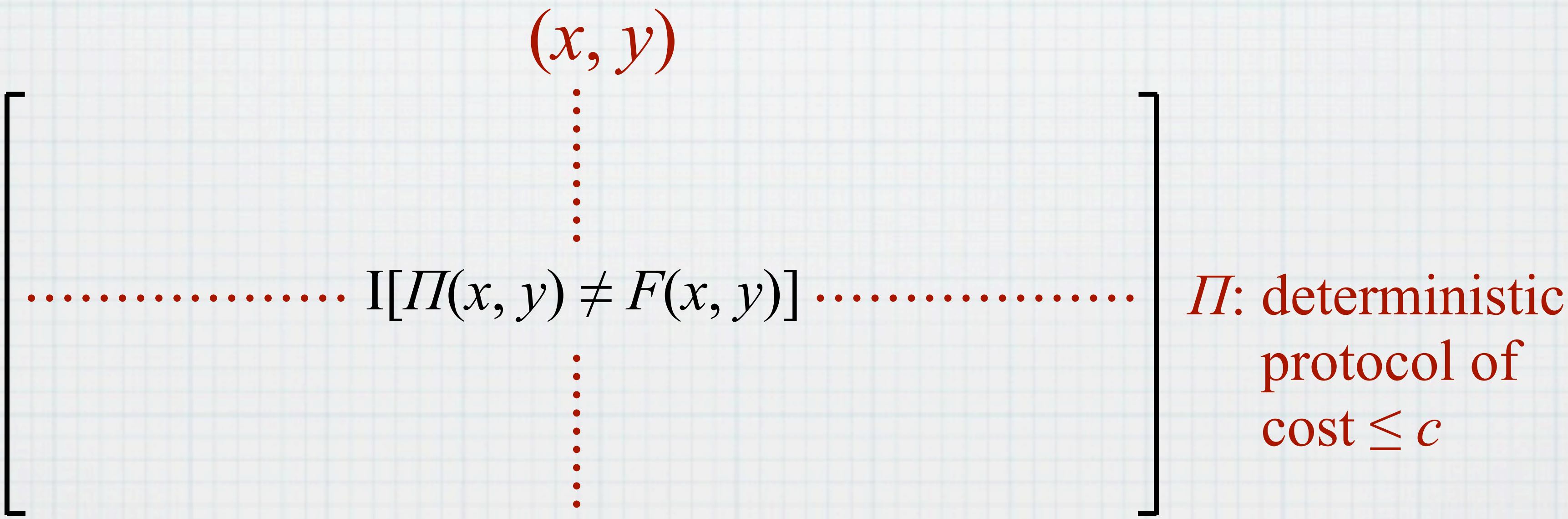

In this talk

N, norm on $\mathbf{R}^n$

$$N^*(x) = \max\{\langle x, y\rangle : N(y) \le 1\}.$$

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- relations among communication complexity measures

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- relations among communication complexity measures
- lower bounds
- ties to other areas

Quantum communication

[Yao 1993]

$F : X \times Y \rightarrow \{0,1\}$

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- Complex inner product spaces

$$\mathcal{A} = \text{span}\{\, |x, w'\rangle \;:\; x \in X, w' \in W'\}$$

$$\mathcal{B} = \text{span}\{\, |y, w''\rangle \;:\; y \in Y, w'' \in W''\}$$

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- State of protocol = unit vector in $\mathcal{A} \otimes \mathcal{C} \otimes \mathcal{B}$

- Cost-k protocol is $\{M_1, M_2, M_3, \ldots, M_k\}$, where

  $M_1, M_3, \ldots$ are unitary transformations in $\mathcal{A} \otimes \mathcal{C}$

  $M_2, M_4, \ldots$ are unitary transformations in $\mathcal{C} \otimes \mathcal{B}$

Quantum communication

- Start state:

$$\text{Initial}(x, y) = |x,0\rangle|0\rangle|y,0\rangle \qquad \text{(no prior entanglement)},$$

$$\text{Initial}(x, y) = \frac{1}{\sqrt{|E|}} \sum_{e \in E} |x,0,e\rangle|0\rangle|y,0,e\rangle \qquad \text{(prior entanglement)}$$

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- Protocol starts in state Initial(x, y) $\in \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{B}$ and ends in the state

$$\text{Final}(x, y) = (M_k \otimes I) \ldots (I \otimes M_4)(M_3 \otimes I)(I \otimes M_2)(M_1 \otimes I)\text{Initial}(x, y)$$

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- The output is 1 with probability $\| \text{Proj}_{\mathcal{A} \otimes |1\rangle \otimes \mathcal{B}} \ \text{Final}(x, y) \|^2$

Analytic reformulation

Theorem (Yao 1993; Kremer 1995; Razborov 2002). *Let* $\Pi = [\Pi_{xy}]$ *be the matrix of acceptance probabilities of a quantum protocol with cost* c*, with or without entanglement. Then*

$$\Pi = AB,$$

where

$$\| A \|_F \leq 2^{O(c)} \, |X|^{1/2},$$
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From this fact alone, a clean and elegant theory arises.

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Theorem (Yao 1993; Kremer 1995; Razborov 2002). *Let $\Pi = [\Pi_{xy}]$ be the matrix of acceptance probabilities of a quantum protocol with cost c, with or without entanglement. Then*

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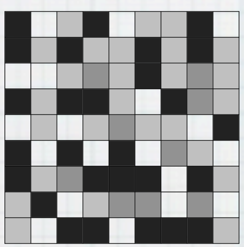
$\Pi =$ 

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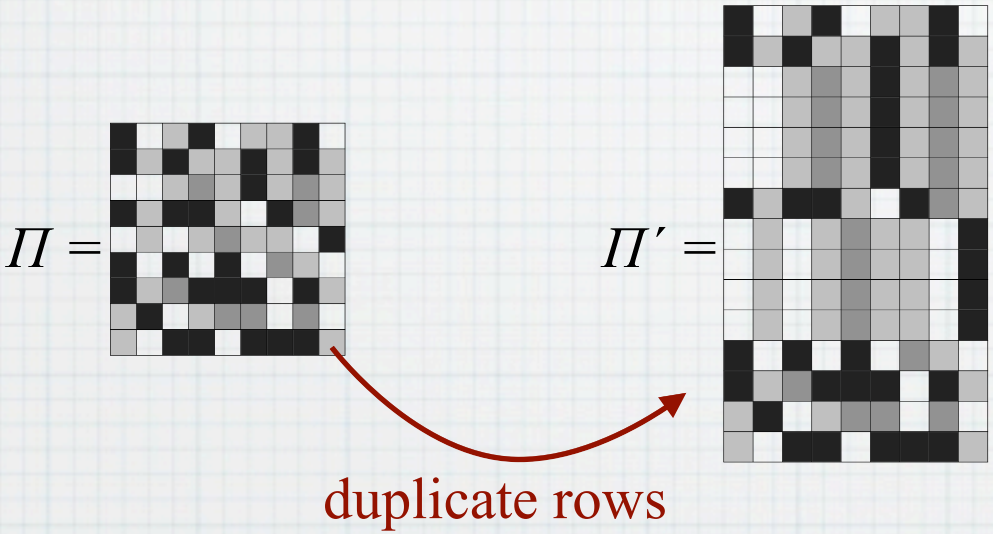

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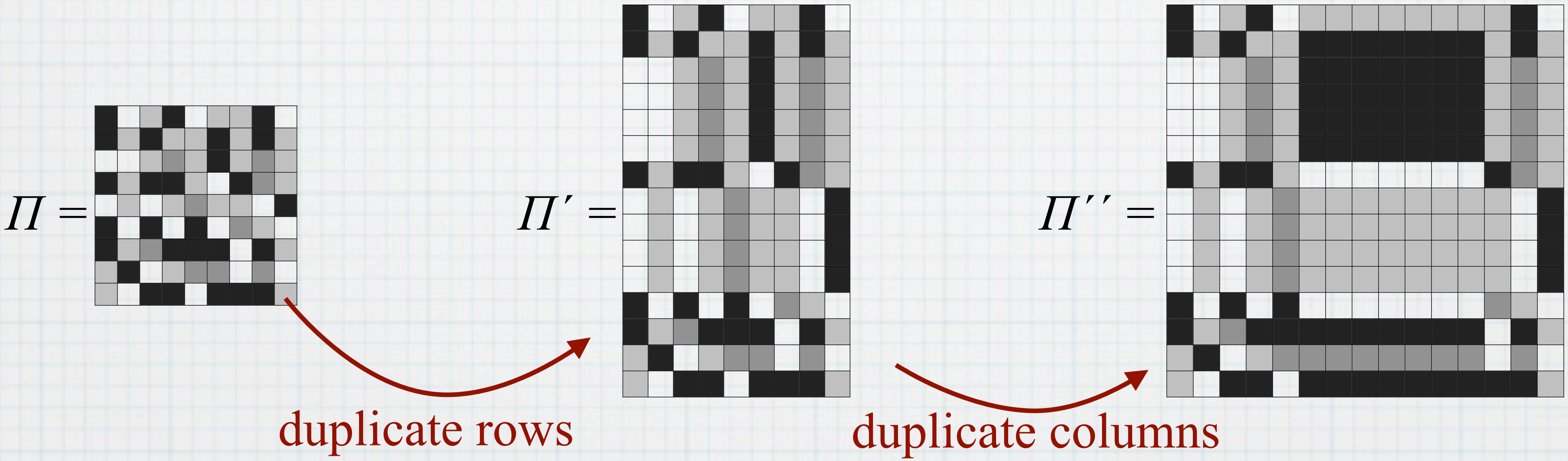

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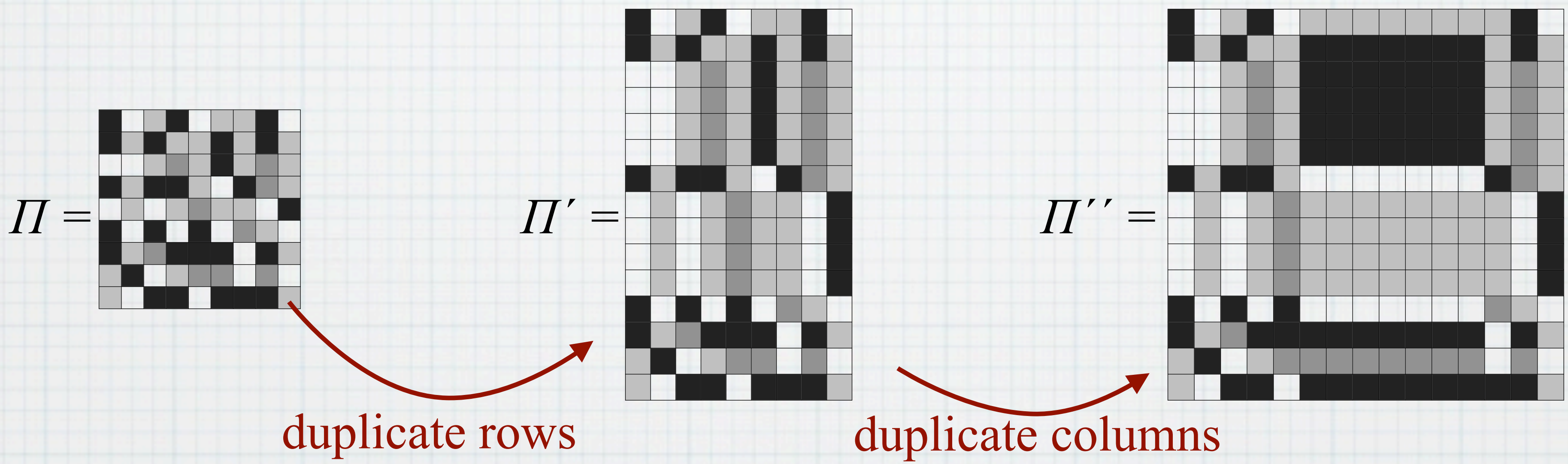

By Yao-Kremer-Razborov, we still have

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Analytic reformulation

This sketches:

Theorem (Linial and Shraibman 2007). *Let $\Pi = [\Pi_{xy}]$ be the matrix of acceptance probabilities of a quantum protocol with cost c, with or without entanglement. Then*

$$\max_{p,q} \| \operatorname{diag}(\ldots, p_x^{1/2}, \ldots)\, \Pi\, \operatorname{diag}(\ldots, q_y^{1/2}, \ldots) \|_{\Sigma} \le 2^{O(c)},$$

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called $\gamma_2(\Pi)$

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Quantum	conv$\{\pm R_1 \pm R_2 \pm R_3 \pm \ldots \pm R_{2^c}$: ~~nonoverlapping~~ rectangles$\}$

Discrepancy and generalized discrepancy

Definition (Razborov 2002). The *ε-approximate trace norm* of $F \in \{-1, 1\}^{X \times Y}$ is

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Generalized discrepancy method (Klauck 2001, Razborov 2002). For all distributions P and all sign matrices H,

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Definition (Razborov 2002). The *ε-approximate trace norm* of $F \in \{-1, 1\}^{X \times Y}$ is

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Notes (Linial and Shraibman 2007).

- GDM is most general criterion known for high quantum c.c.
- Strong duality ⇒ GDM is complete.

I. Inner product

Theorem (Kremer 1995; Cleve, van Dam, Nielsen, and Tapp 1998).

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II. Fourier coefficients

Theorem (Klauck 2001).

Let $f\colon \{0,1\}^n \to \{-1,+1\}$ *be given. Put* $F = [\,f(x \wedge y)]_{xy}$ *. Then for each* $S \subseteq \{1, 2, \ldots, n\}$,

$$Q^*_{1/3}(F) \geq \Omega\left(\frac{|S|}{\log(2/|\hat{f}(S)|)}\right).$$

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But

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II. Fourier coefficients

$$\widehat{\mathrm{MAJ}}(\{1,2,\ldots,n\}) = \Theta\left(\frac{1}{\sqrt{n}}\right)$$ gives :

Corollary (Klauck 2001).

Computing MAJ($x \wedge y$) *to accuracy* 1/3 *requires* $\Omega(n/\log n)$ *qubits.*

III. Hahn matrices

Theorem (Razborov 2002).

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- Tight
- Razborov handles $f(x \wedge y)$ for any symmetric f.

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$$\begin{bmatrix} \langle F, \mu_0\rangle \\ \langle F, \mu_1\rangle \\ \langle F, \mu_2\rangle \\ \vdots \\ \langle F, \mu_k\rangle \end{bmatrix} = \begin{bmatrix} \langle \Pi, \mu_0\rangle \\ \langle \Pi, \mu_1\rangle \\ \langle \Pi, \mu_2\rangle \\ \vdots \\ \langle \Pi, \mu_k\rangle \end{bmatrix} + \begin{bmatrix} \langle E, \mu_0\rangle \\ \langle E, \mu_1\rangle \\ \langle E, \mu_2\rangle \\ \vdots \\ \langle E, \mu_k\rangle \end{bmatrix}$$

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By Nisan-Szegedy (1992), $\log\{\|\Pi\|_\Sigma \binom{n}{n/4}^{-1}\} \ge \sqrt{n}$. $\square$

IV. Pattern matrices

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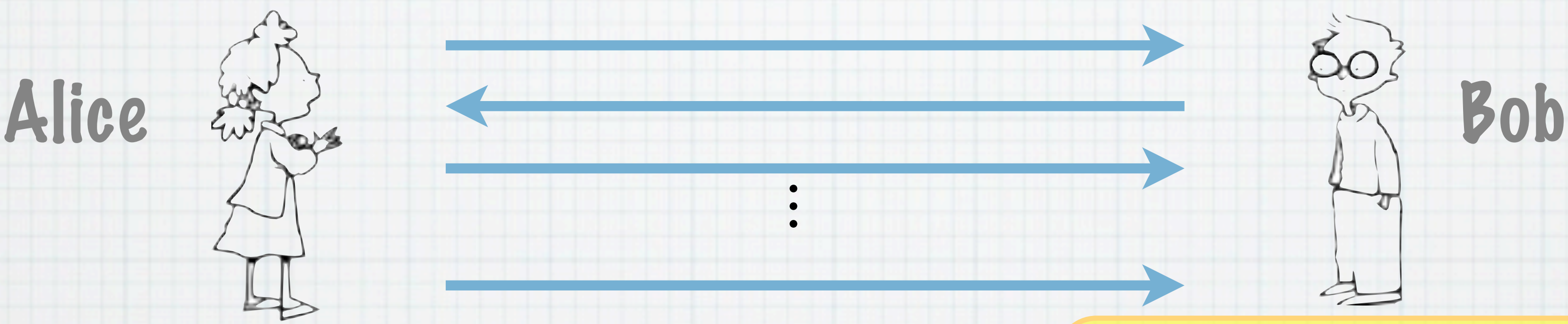

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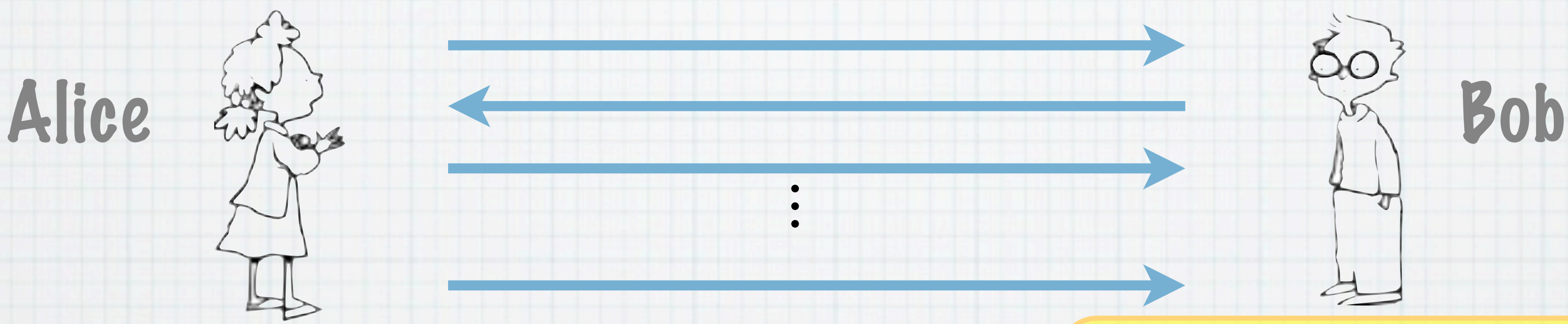

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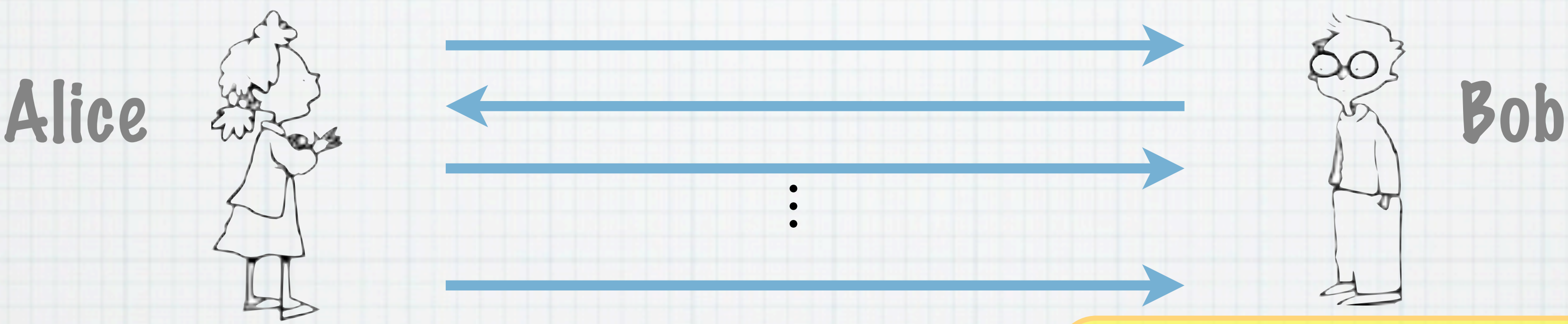

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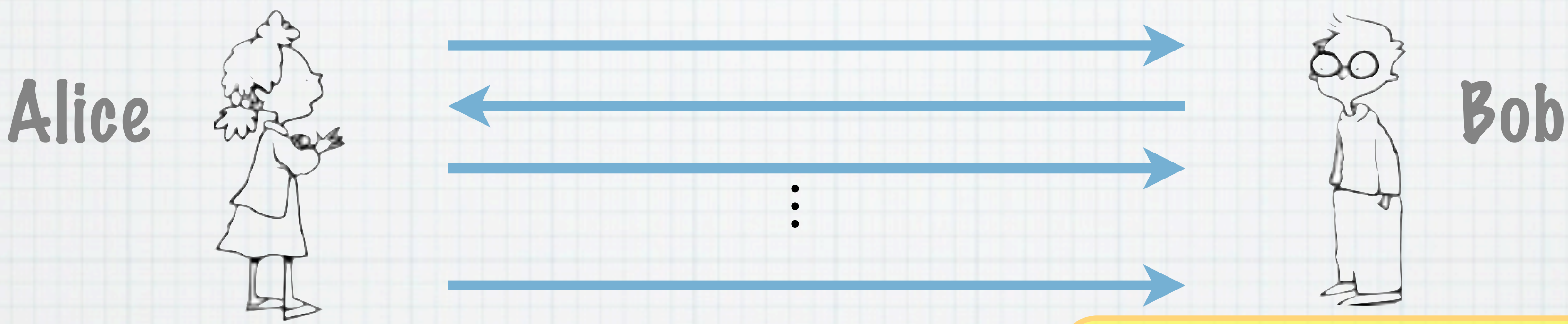

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least degree of a polynomial p with $|f - p|_\infty < 1/3$.

IV. Pattern matrices

$$\mathcal{V}(n,t) = \left\{1, 2, \ldots, \frac{n}{t}\right\} \times \left\{\frac{n}{t} + 1, \ldots, \frac{2n}{t}\right\} \times \cdots \times \left\{\frac{(t-1)n}{t} + 1, \ldots, n\right\}.$$

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Definition. Fix $\phi : \{0,1\}^t \to \mathbb{R}$. The (n,t,ϕ)-*pattern matrix* is

$$\Big[\ \phi(x|_V \oplus w)\ \Big]_{\substack{x\in\{0,1\}^n,\\ (V,w)\in\mathcal{V}(n,t)\times\{0,1\}^t}}.$$

IV. Pattern matrices

Theorem (S. 2008). *Fix* $\phi : \{0,1\}^t \to \mathbb{R}$. *Let* A *be the* (n,t,ϕ)-*pattern matrix. Then*

$$\|A\| = \sqrt{2^{n+t}\left(\frac{n}{t}\right)^t} \quad \max_{S\subseteq[t]} \left\{ |\hat{\phi}(S)| \left(\frac{t}{n}\right)^{|S|/2} \right\}.$$

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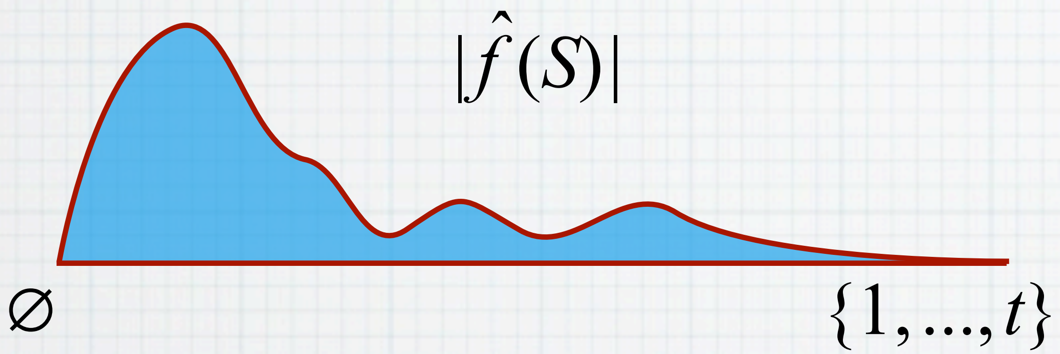
Low-order Fourier coeffs of ϕ **small**

$\Rightarrow \|A\|$ **small.**

IV. Pattern matrices

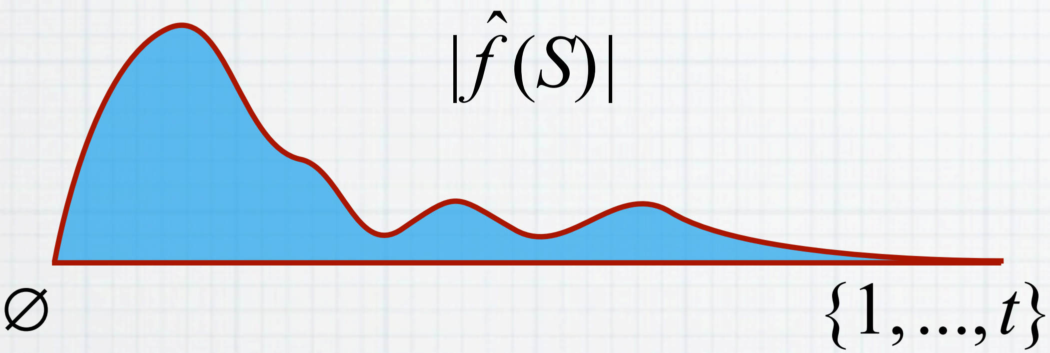


Original function

$f:\{0,1\}^t \to \{-1,1\}$

$\deg_{1/3}(f) = d$

IV. Pattern matrices



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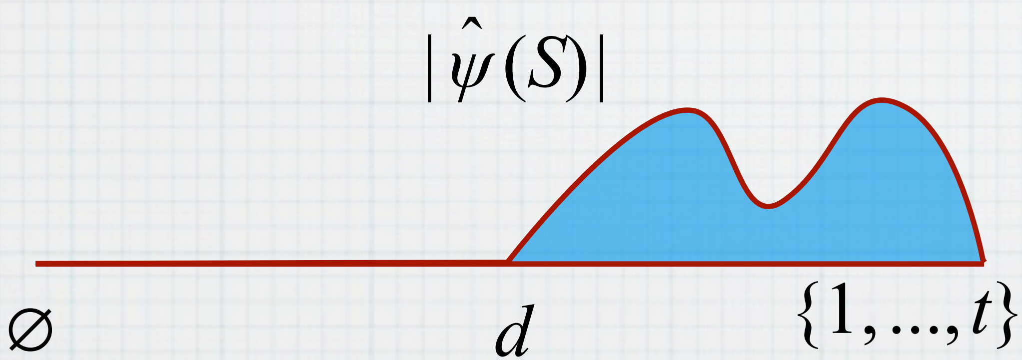

Modified function ψ:

$\langle f, \psi \rangle > \frac{1}{3} \| \psi \|_1$

IV. Pattern matrices

Theorem (S. 2008).

Fix $f\colon \{0,1\}^t \to \{-1,+1\}$, $d = \deg_{1/3}(f)$.

Let F be the (n, t, f)*-pattern matrix. Then*

$$Q^*_{1/3}(F) \geq \Omega\left(d \log \frac{n}{t}\right).$$

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Fix $\psi : \{0,1\}^t \to \mathbb{R}$ such that

$$\hat{\psi}(S) = 0 \qquad \text{for } |S| < d,$$

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$$\|K\| \leqslant \left(\frac{t}{n}\right)^{d/2}\left(2^{n+t}\left(\frac{n}{t}\right)^{t}\right)^{-1/2}$$

$$\|K\|_1 = 1$$

$$\langle K, F \rangle > \frac{1}{3}$$

IV. Pattern matrices

1

Fix $\psi : \{0,1\}^t \to \mathbb{R}$ such that

$$\hat{\psi}(S) = 0 \qquad \text{for } |S| < d,$$

$$\sum_{z \in \{0,1\}^t} |\psi(z)| = 1,$$

$$\langle \psi, f \rangle > \frac{1}{3}.$$

Let K be the $\left(n, \ t, \ 2^{-n}\binom{n}{t}^{-t}\psi\right)$-pattern matrix.

F is the $(n, \ t, \ f)$-pattern matrix.

2

3

$$\|K\| \leqslant \left(\frac{t}{n}\right)^{d/2}\left(2^{n+t}\left(\frac{n}{t}\right)^{t}\right)^{-1/2}$$

$$\|K\|_1 = 1$$

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GDM

$$Q^*_{1/5}(F) > \frac{1}{4}\, d \log\left(\frac{n}{t}\right) - 2.$$

□

IV. Pattern matrices

Follow-up work

- Multiparty disjointness function
  [Lee & Shraibman, 2008]
  [Chattopadhyay & Ada, 2008]
- $\mathrm{NP}^{cc} \not\subset \mathrm{BPP}^{cc}$
  [David & Pitassi, 2008]
- Explicit separation of NP^{cc} and BPP^{cc}
  [David, Pitassi & Viola, 2008]
- Constant-depth circuits
  [Beame & Huynh-Ngoc, 2008]
- Explicit separation of $\mathrm{NP}^{cc} \neq \mathrm{coNP}^{cc}$, $\mathrm{NP}^{cc} \neq \mathrm{coAM}^{cc}$
  [Gavinsky and S., 2009]

V. Block composition

[Shi and Zhu, 2008]

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$$F = f(g, g, \ldots, g)$$

any function $f : \{0,1\}^n \to \{0,1\}$

gadget $g : \{0,1\}^k \times \{0,1\}^k \to \{0,1\}$

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Theorem (Shi and Zhu, 2008). *Put* $d = \deg_{1/3}(f)$. *Then*

$Q^*_{1/3}(F) \geq \Omega(d)$ *for any gadget g with spectral discrepancy* $O(d/n)$.

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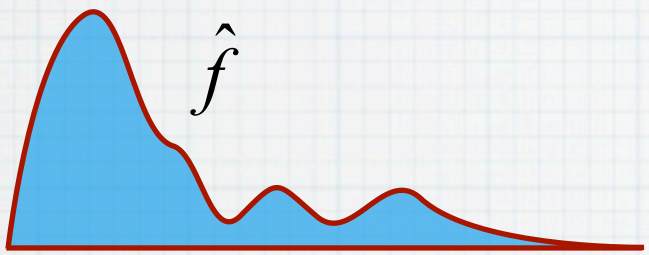
- Independent of [S. 2008]
- Broader class than pattern matrices (g = selector gadget)
- Bounds weaker than pattern matrix, e.g., $Q^*(\mathrm{DISJ}) = \Omega(n^{1/3})$.

V. Block composition



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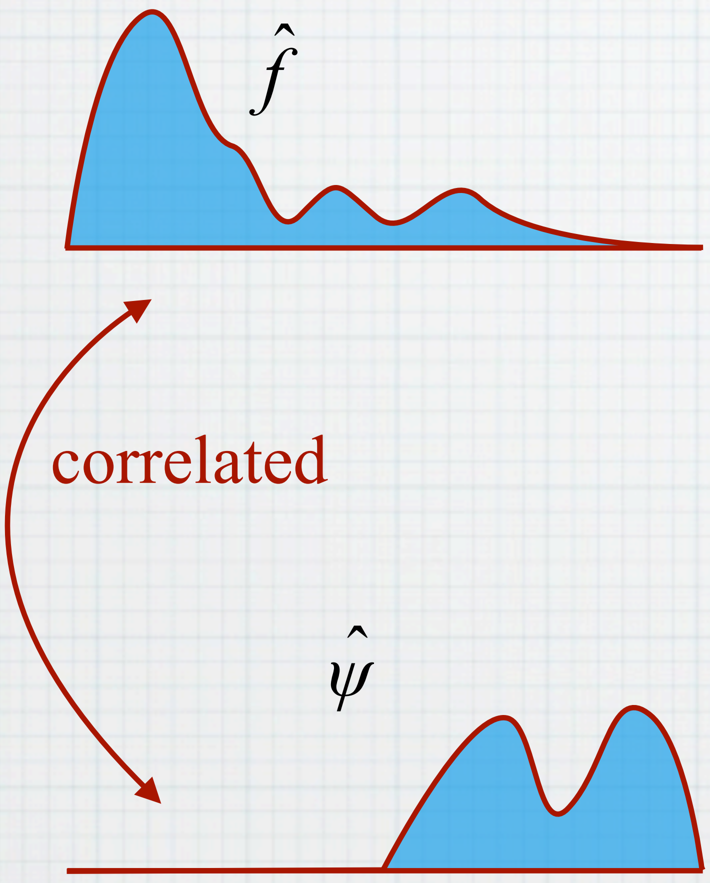

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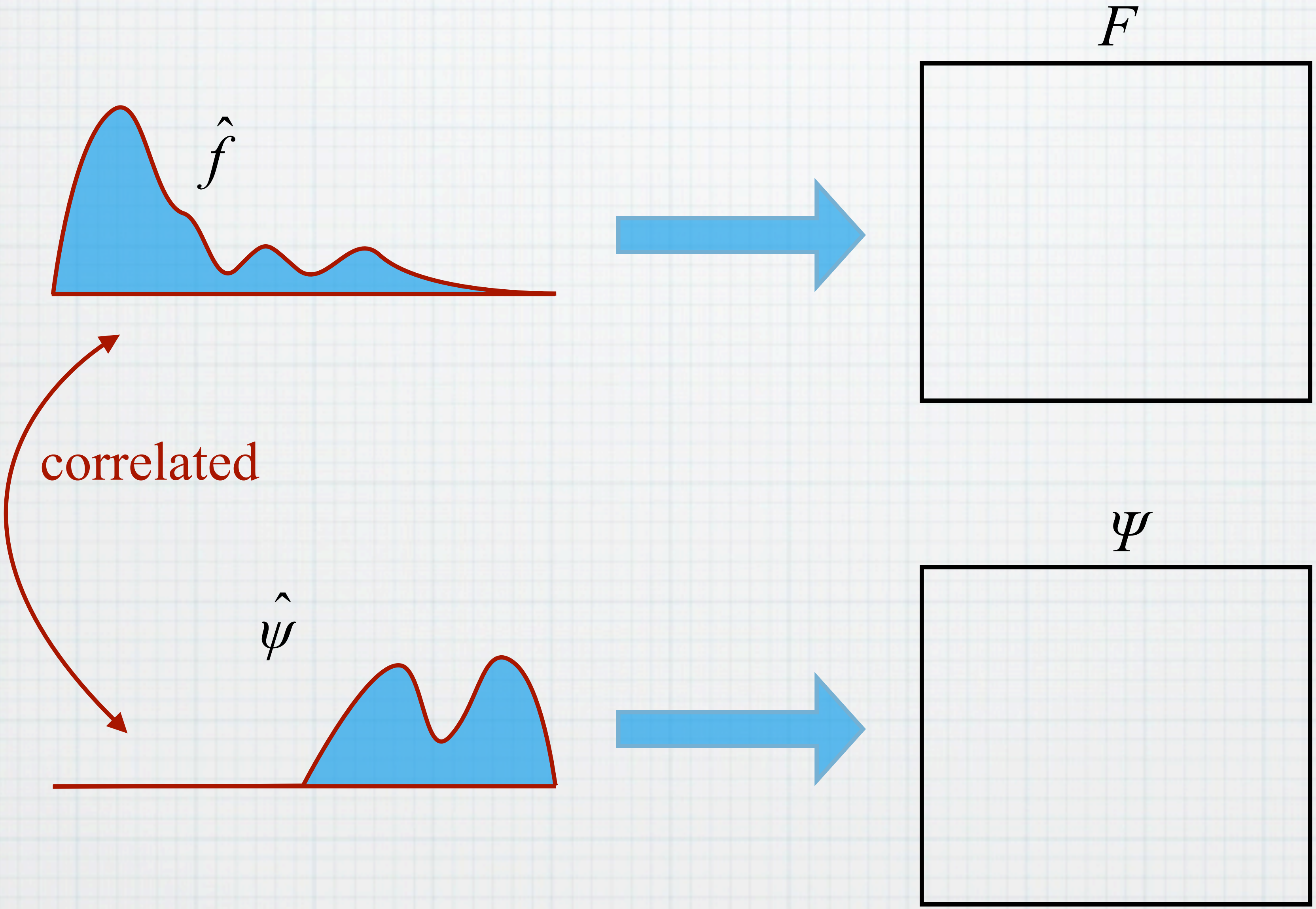

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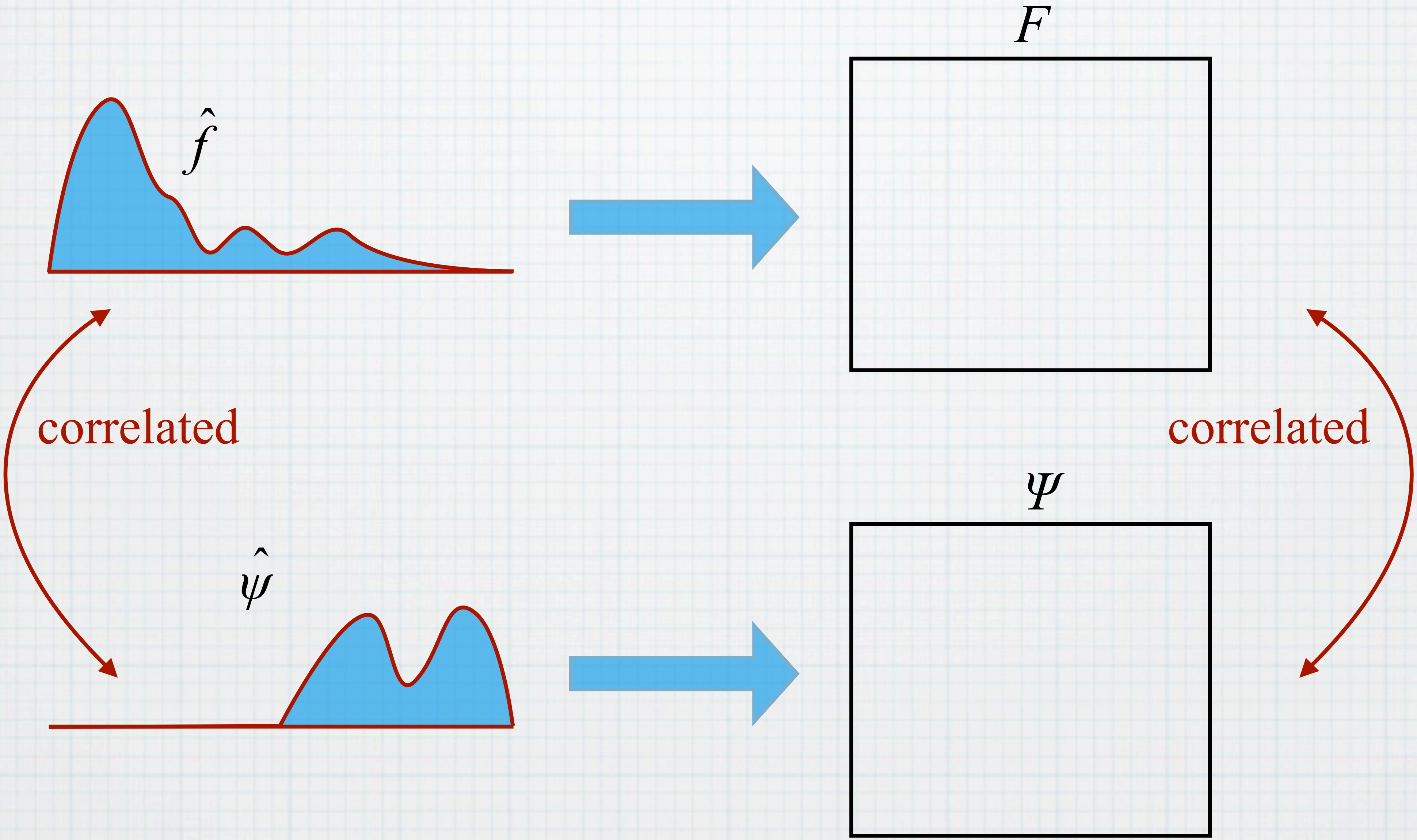

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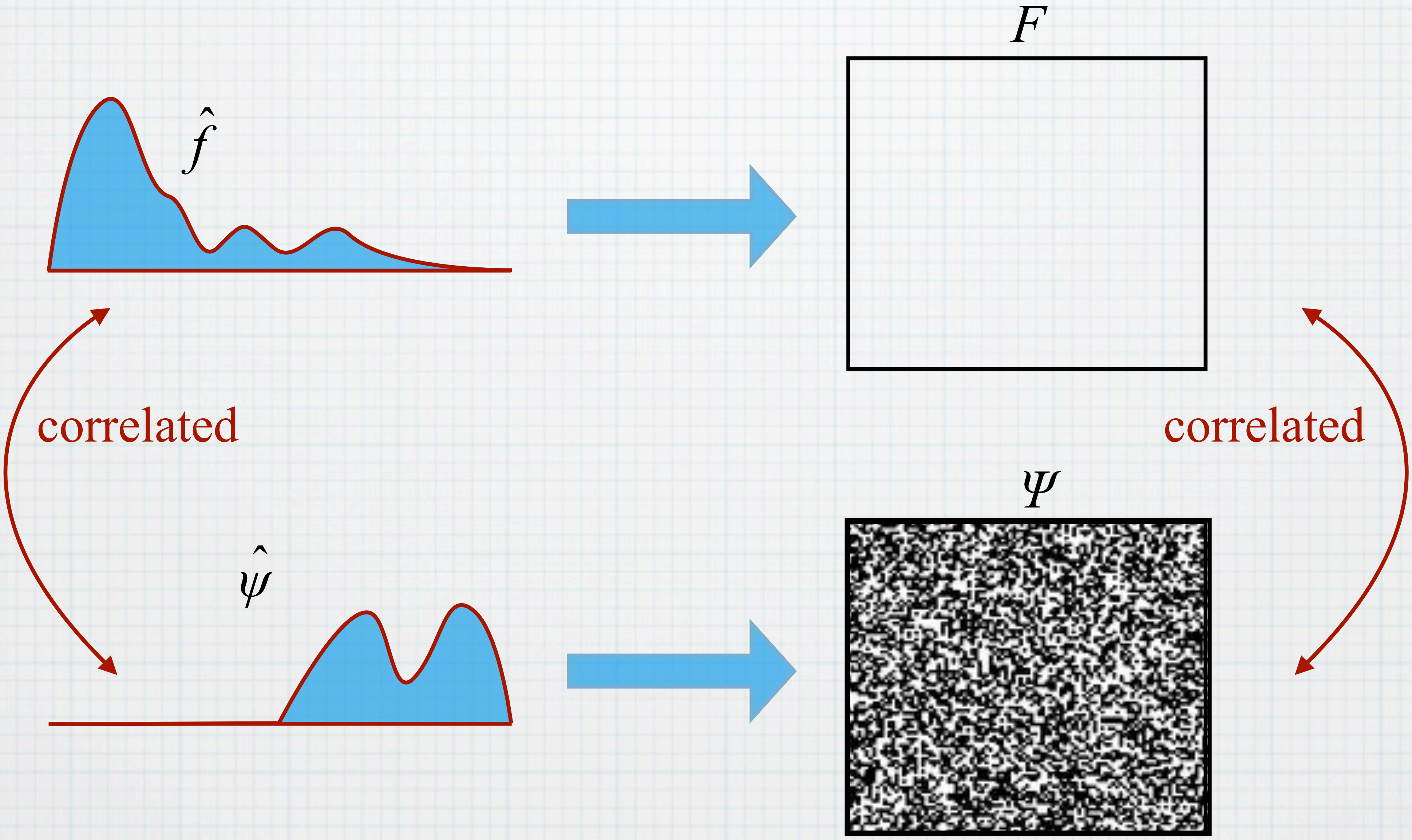

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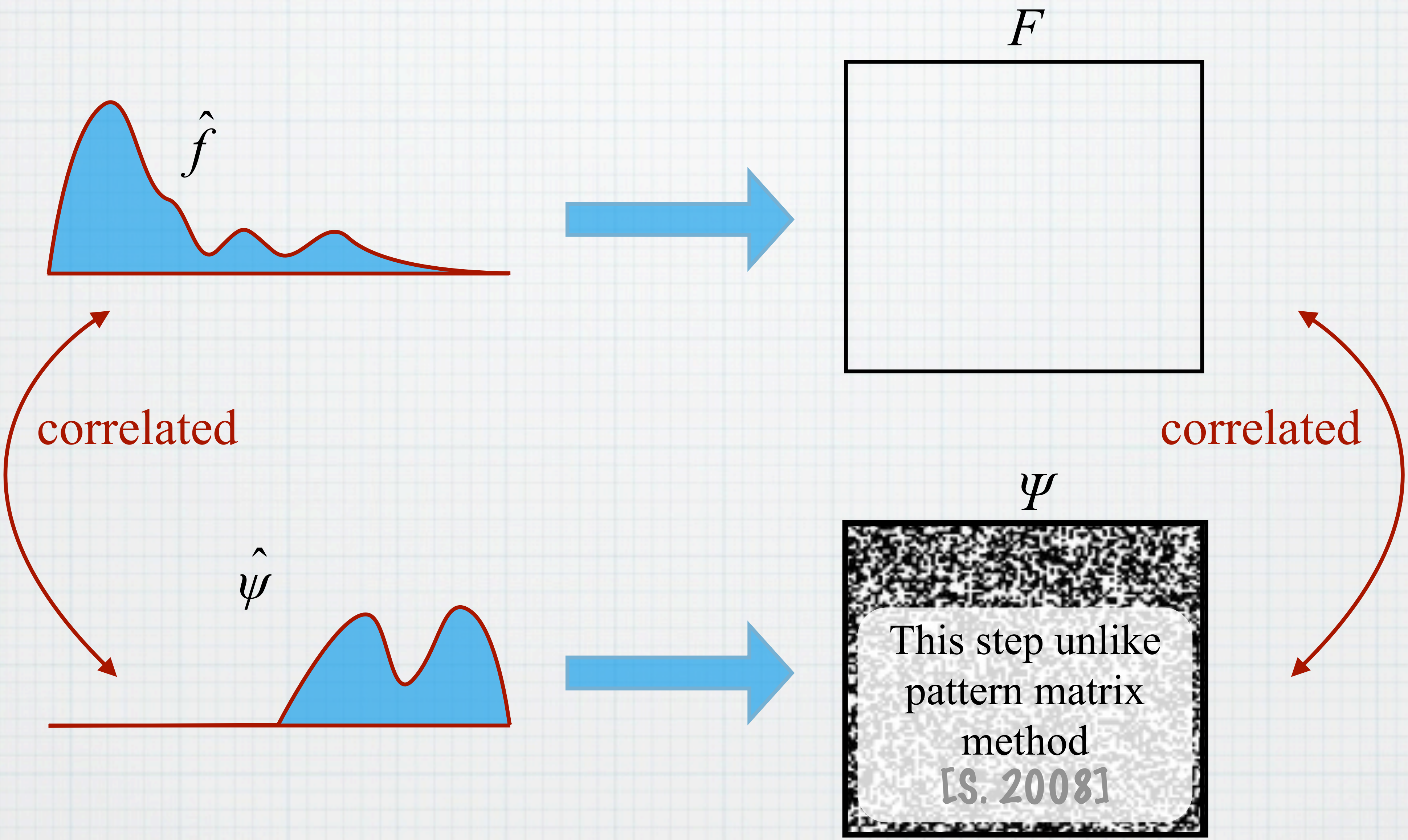

VI. Margin vs. discrepancy

[Linial and Shraibman 2007]

$F \in \{-1, 1\}^{X \times Y}$

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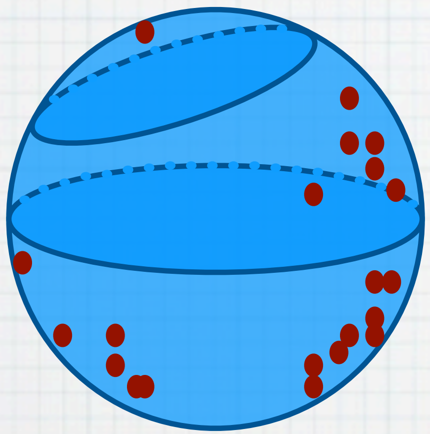

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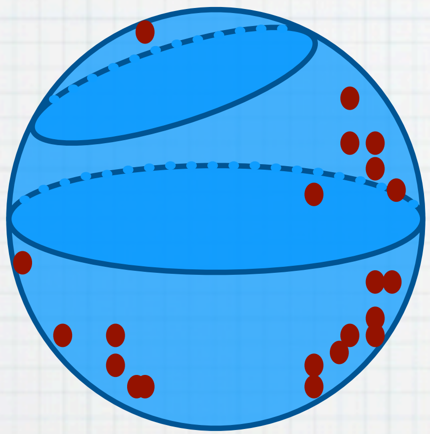

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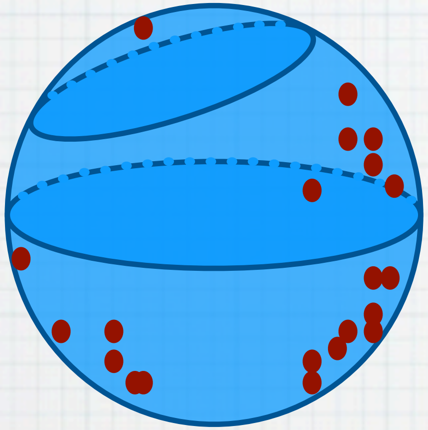

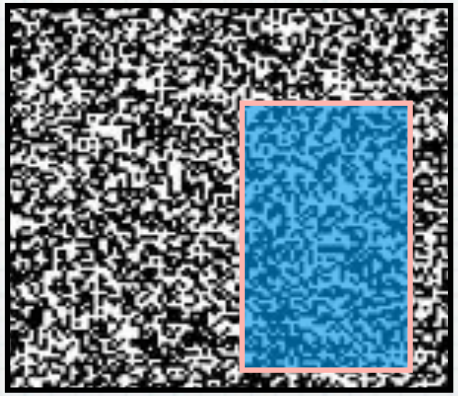

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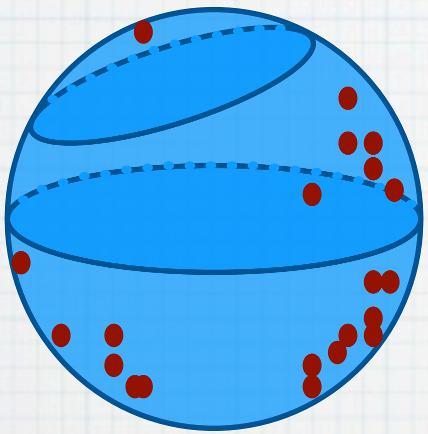

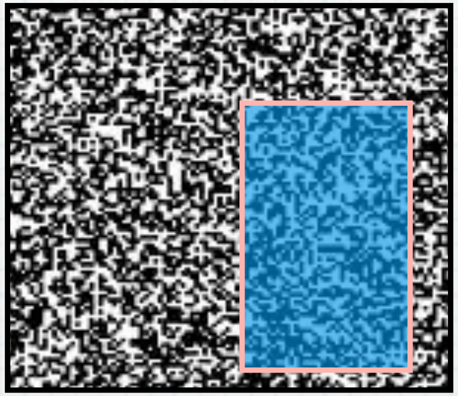

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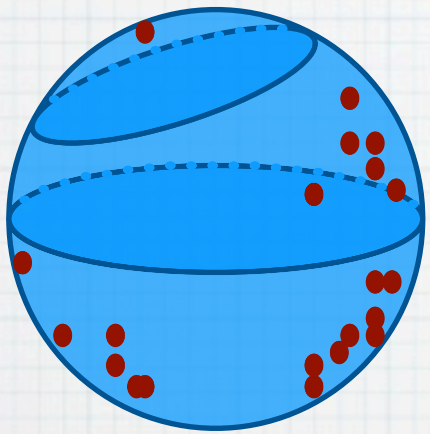

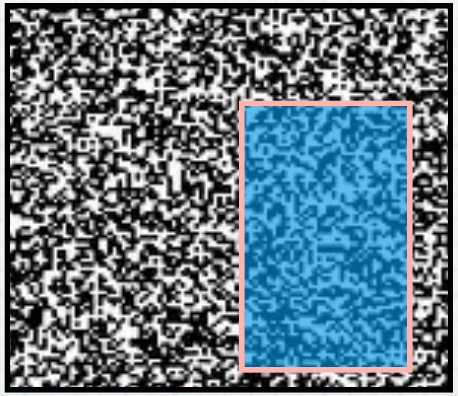

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$\approx$ Grothendieck's Ineq.

Part 2

Unbounded-Error Communication

Sign-rank defined

$A \in \{-1,+1\}^{m \times n}$

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Sign-rank of A is least rank of a *real* matrix with A's sign pattern.

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Definition.

$$\text{sign-rank}(A) = \min_B \left\{\text{rank}(B) : A_{ij} B_{ij} > 0 \quad \forall i, j\right\}$$

An example

$$A = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \end{bmatrix}_{n \times n}$$

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$$A = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \end{bmatrix}_{n \times n}$$

$$\text{rank}(A) = n$$

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$$\text{rank}(A) = n$$
$$\text{sign-rank}(A) = 2$$

Solution

$$B = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Solution

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ & & & & & & & & \\ \end{bmatrix}$$

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A trickier example

$$A = \begin{bmatrix} +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \end{bmatrix}_{n \times n}$$

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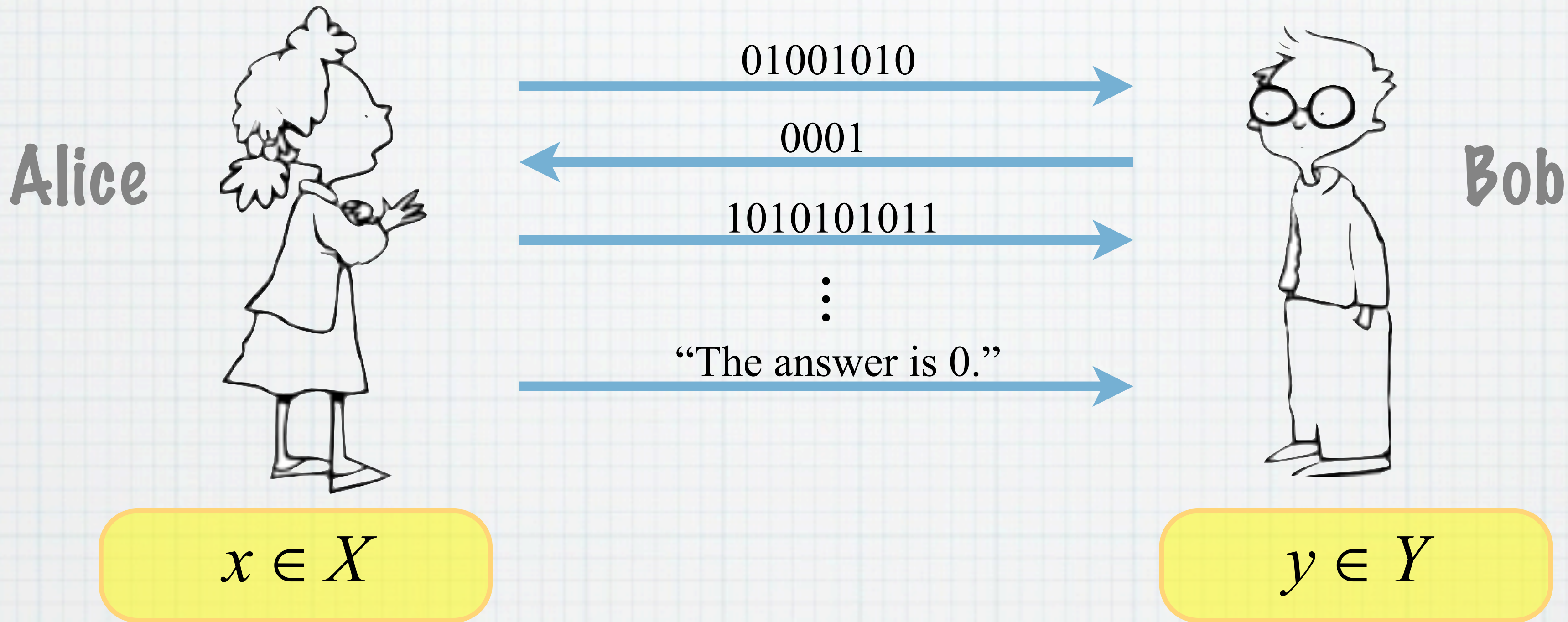
Unbounded-error model

[Paturi and Simon 1986]



Objective: compute $F(x, y)$ with probability >1/2.

Complexity measure $U(F)$ = # bits exchanged.

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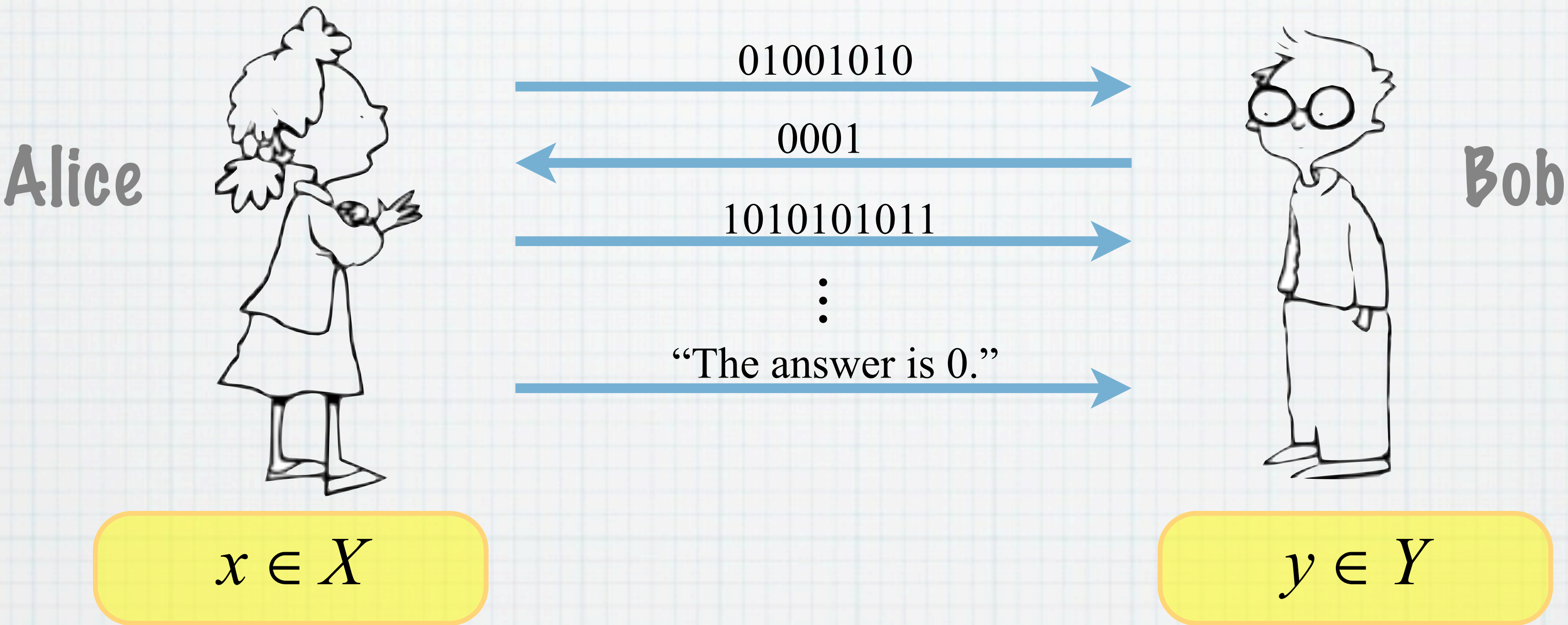

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[Razborov 2002]
[Kalyanasundaram and Schnitger 1992]
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- $\exists\, f: \{0,1\}^n \to \{-1,1\}$ such that

$$U(f) = O(\log n),$$
$$Q^*(f) = \Omega(n^{1/2}) \text{ for advantage } \exp(-n^{1/2})$$

[Buhrman, Vereshchagin, and de Wolf 2007]
[S. 2007]

Relation to sign-rank

Theorem (Paturi and Simon 1986).
Put $F = [\, f(x, y)\,]_{x,y}$. *Then*

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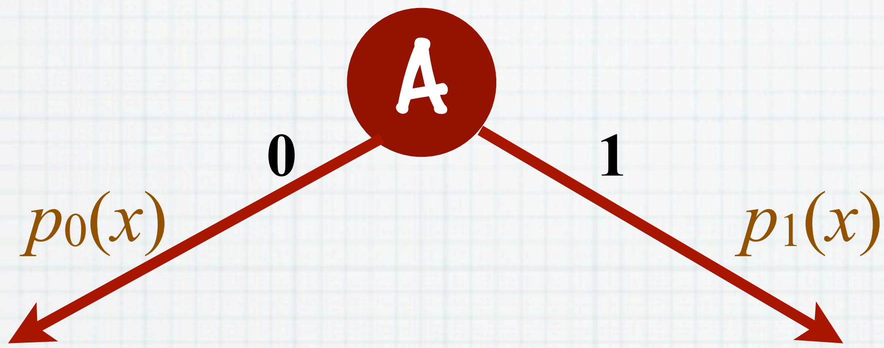
Will show: $U(f) \geq \log_2(\text{sign-rank } F)$

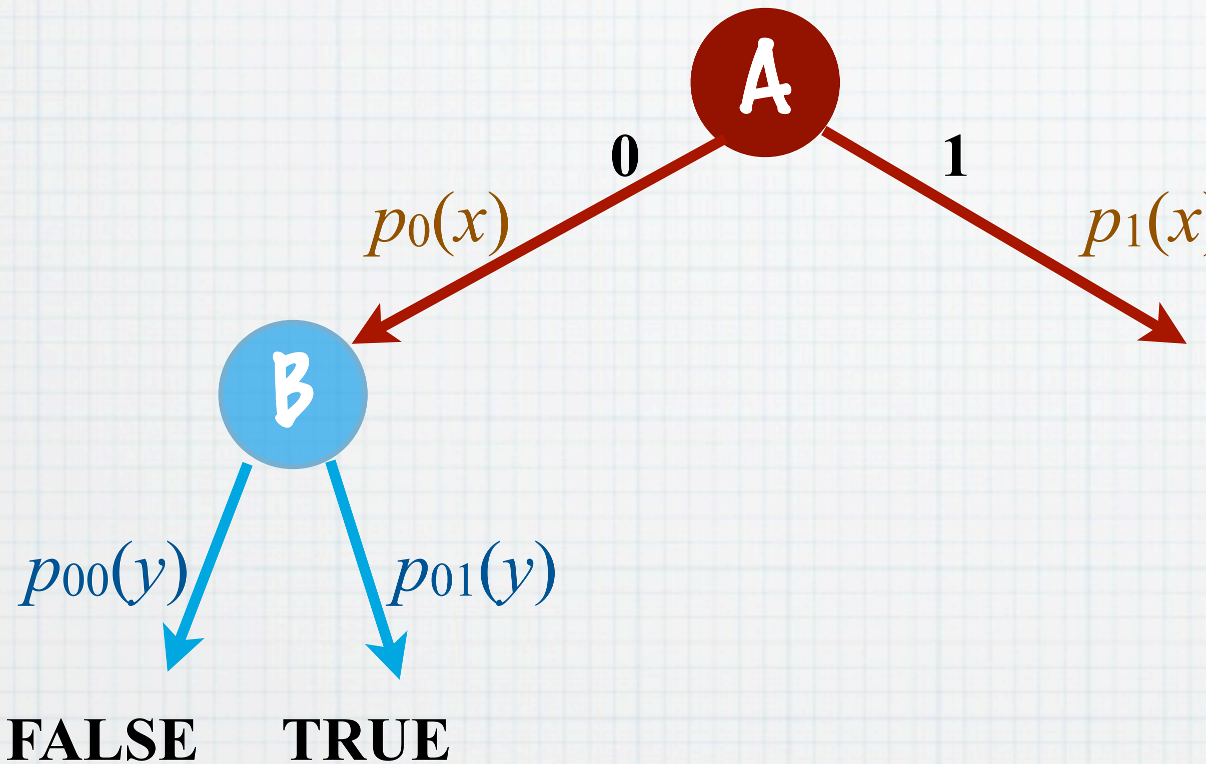
Proof

A

Proof



Proof



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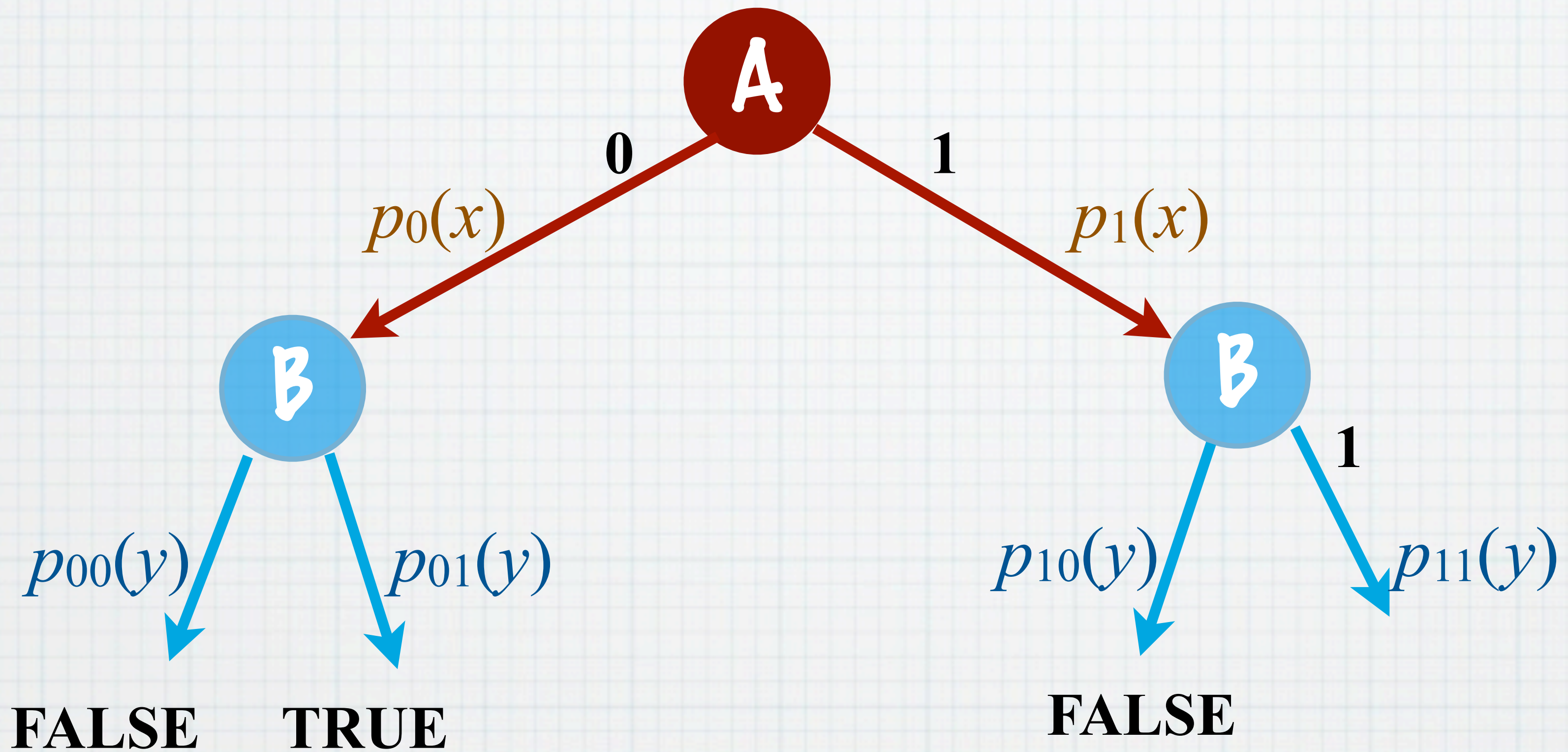

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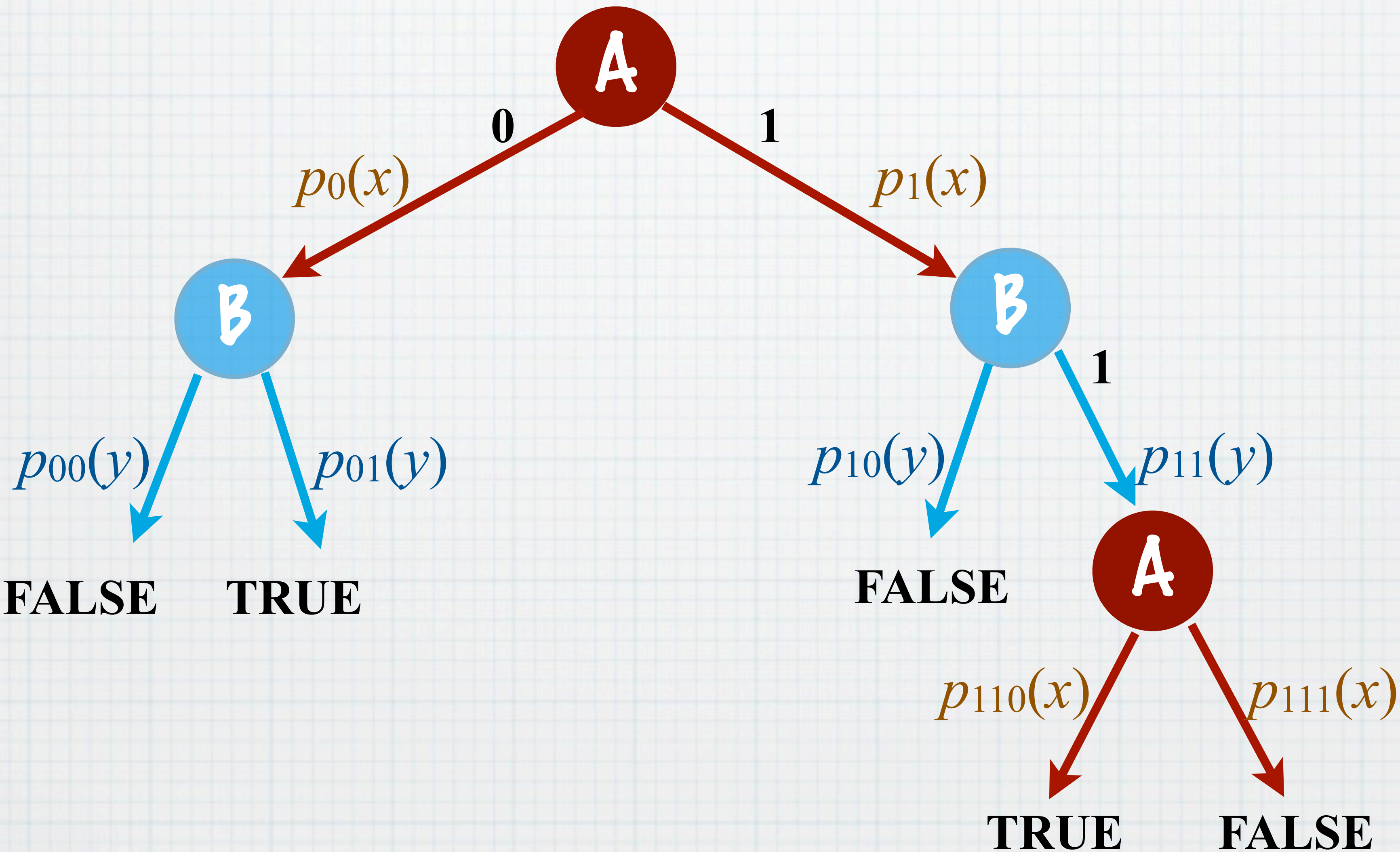
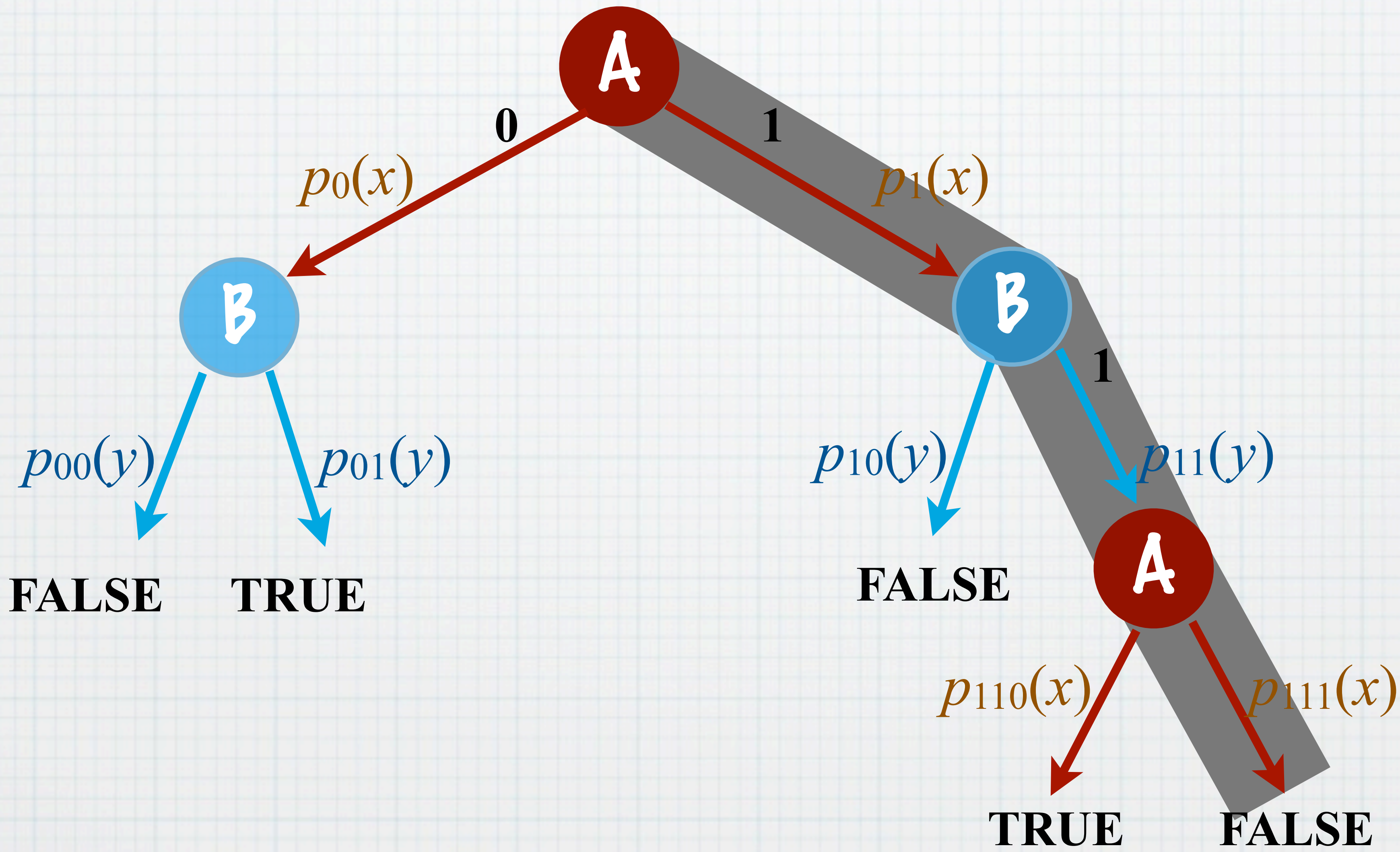


A
0
1
$p_0(x)$
$p_1(x)$
B
B
1
$p_{00}(y)$
$p_{01}(y)$
$p_{10}(y)$
$p_{11}(y)$
FALSE
TRUE
FALSE
A
$p_{110}(x)$
$p_{111}(x)$
TRUE
FALSE

Proof



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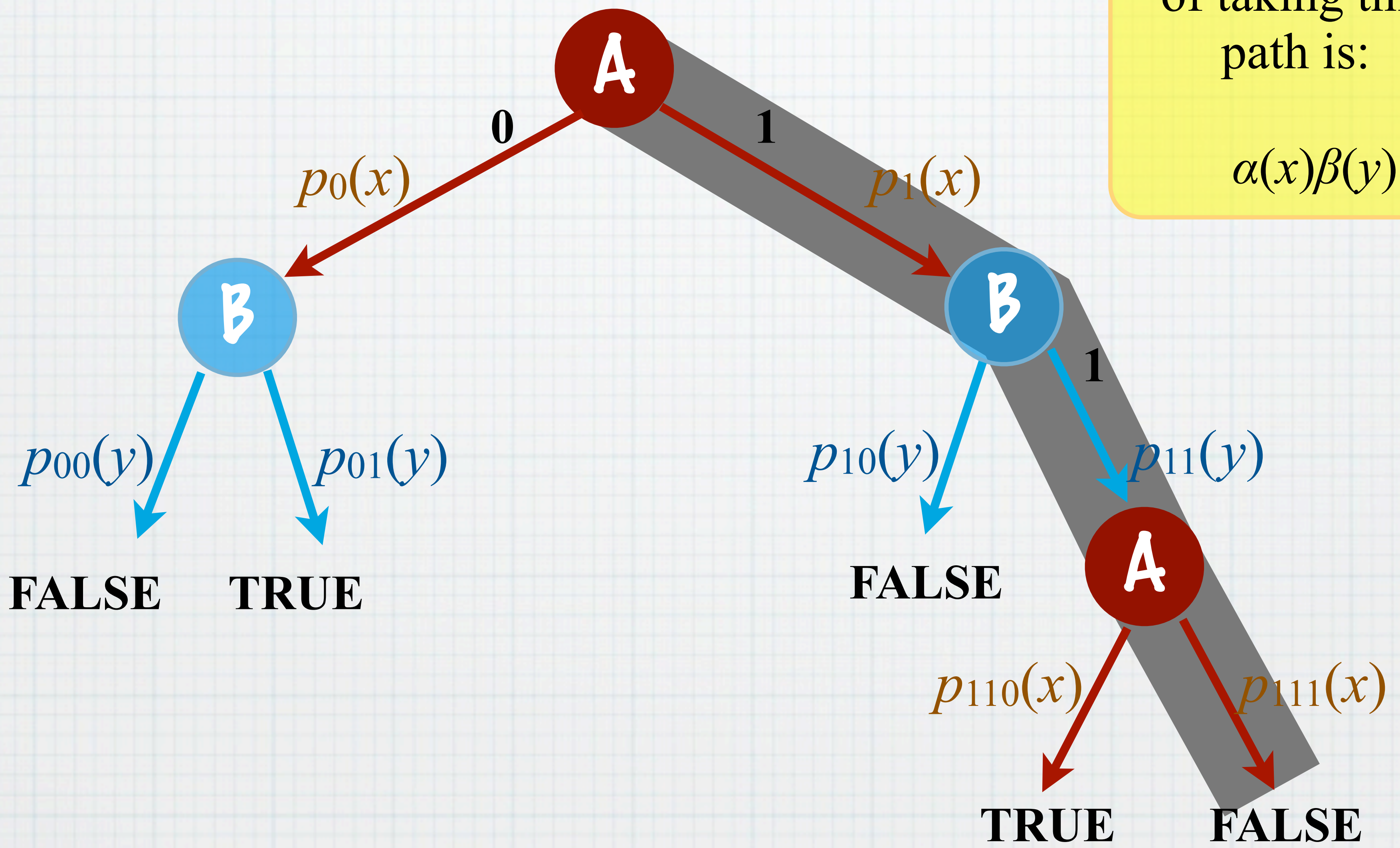
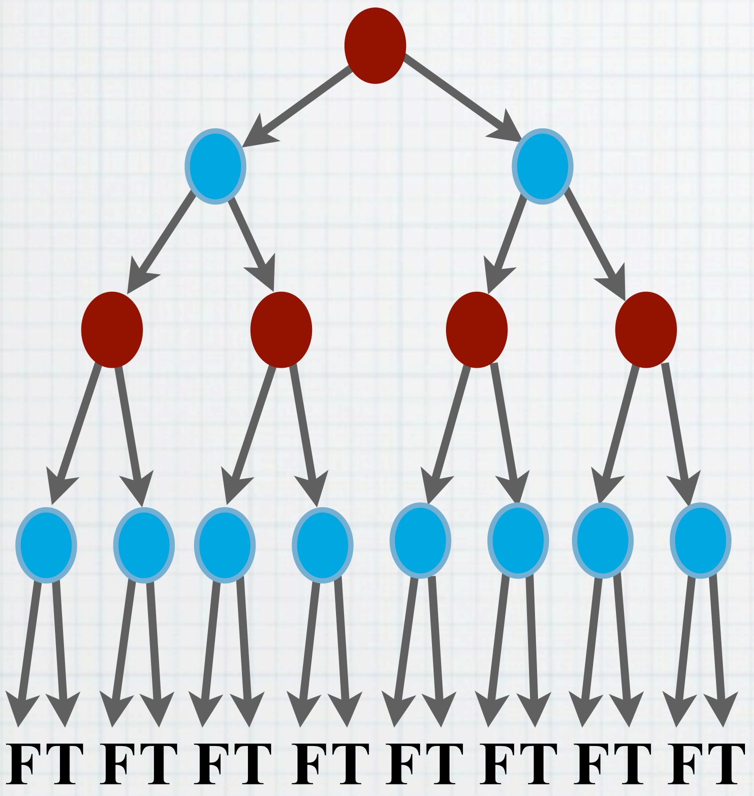


probability of taking this path is:

$\alpha(x)\beta(y)$

Proof



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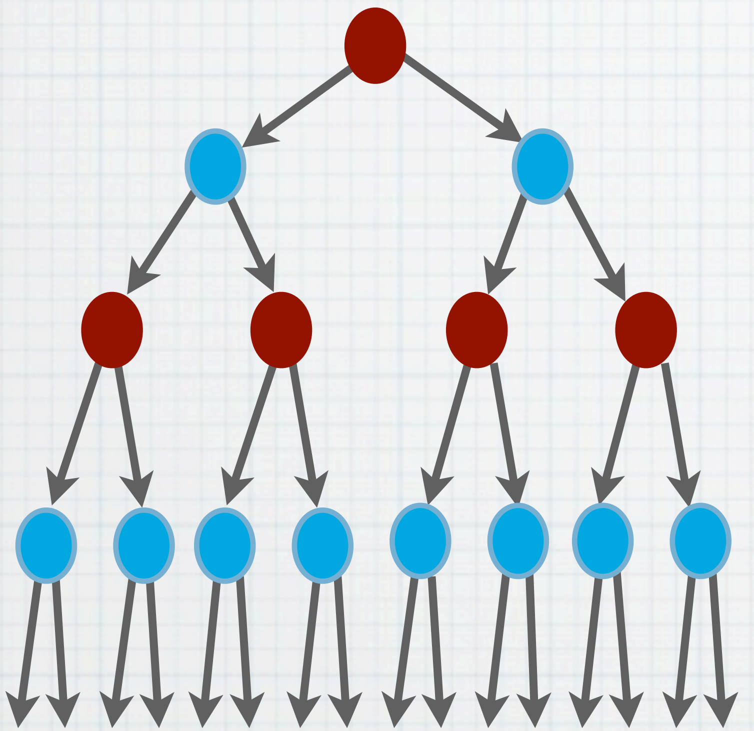
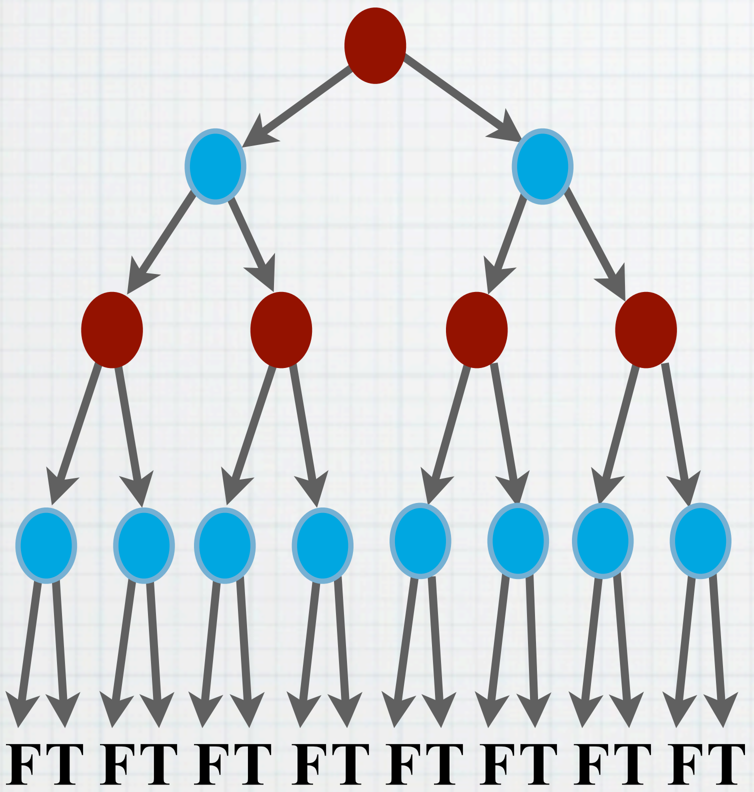


$$\mathbf{P}[P(x,y) = \text{TRUE}]$$

$$= \sum_{\substack{\text{accepting} \\ \text{paths } \pi}} \alpha_\pi(x)\beta_\pi(y)$$

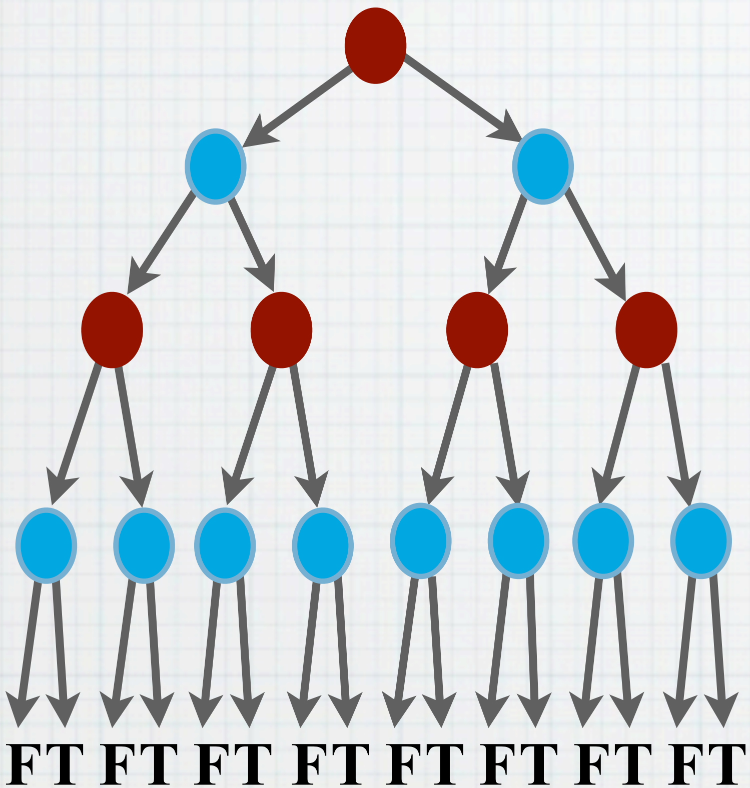
Proof



$$\mathbf{P}[P(x,y) = \text{TRUE}] = \sum_{\substack{\text{accepting} \\ \text{paths } \pi}} \alpha_\pi(x)\beta_\pi(y)$$

$$F = \text{sign}\left(\left[\sum_{\substack{\text{accepting} \\ \text{paths } \pi}} \alpha_\pi(x)\beta_\pi(y) - \frac{1}{2}\right]_{x,y}\right)$$

Proof



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$\text{rank} \leq 2^{U(f)}$

□

I. Counting arguments

First nontrivial result on sign-rank:

Theorem (Alon, Frankl & Rödl, 1985).
A random matrix in $\{-1,+1\}^{n \times n}$ *has sign-rank* $\Theta(n)$ *w.v.h.p.*

II. Forster's method

$A \in \{-1,+1\}^{N \times M}, \quad \text{sign-rank}(A) = r$

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$A \in \{-1,+1\}^{N\times M}$, sign-rank$(A) = r$

$\Rightarrow$ $\exists$ $u_1, u_2, \ldots, u_N \in \mathbf{R}^r$,

$v_1, v_2, \ldots, v_M \in \mathbf{R}^r$ such that

$$A_{ij} = \operatorname{sign} \langle u_i, v_j \rangle.$$

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Lemma (Forster). Can choose u_i, v_j with $\Sigma_{i,j}\,\langle u_i, v_j\rangle^2 \geq NM/r$.

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Lemma (Forster). Can choose u_i, v_j with $\Sigma_{i,j} \langle u_i, v_j \rangle^2 \geq NM/r$.

Idea. For any $B \in \text{GL}(r)$, can transform

$$u_i \to \frac{1}{\| Bu_i \|} Bu_i, \qquad v_j \to \frac{1}{\| (B^{-1})^\mathsf{T} v_j \|} (B^{-1})^\mathsf{T} v_j.$$

Find needed B by compactness.

II. Forster's method

So, $\Sigma_{i,j} \langle u_i, v_j \rangle^2 \geq NM/r.$

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But: $\Sigma_{i,j} \langle u_i, v_j \rangle^2 \leq \| A \|^2 r$

(matrix analysis)

II. Forster's method

Theorem (Forster 2001). *For all* $A \in \{-1,+1\}^{N\times M}$,

$$\text{sign-rank}(A) \geq \frac{\sqrt{NM}}{\|A\|}.$$

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Theorem (Forster, Krause, Lokam, Mubarakzjanov, Schmitt, and Simon 2001). *For all* $A \in \mathbf{R}^{N\times M}$,

$$\text{sign-rank}(A) \geq \frac{\sqrt{NM}}{\|A\|} \cdot \min |A_{ij}|.$$

III. Sign-rank vs. PH

[Babai, Frankl, and Simon, 1986]

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$$\text{UPP} = \left\{ \text{functions } f : \{0,1\}^n \times \{0,1\}^n \to \{-1,+1\} \text{ with } U(f) \le \text{polylog}(n) \right\}$$

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$$= \Big\{ 2^n \times 2^n \text{ sign matrices } F \text{ with sign-rank}(F) \le 2^{\text{polylog}(n)} \Big\}.$$

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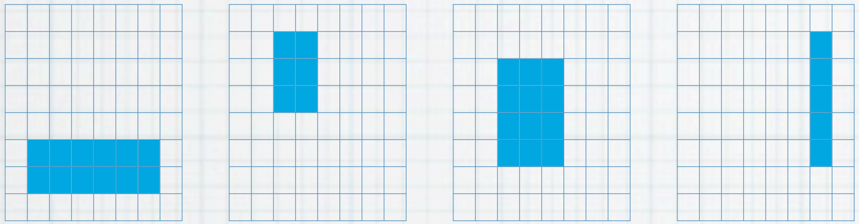

$2^n \times 2^n$

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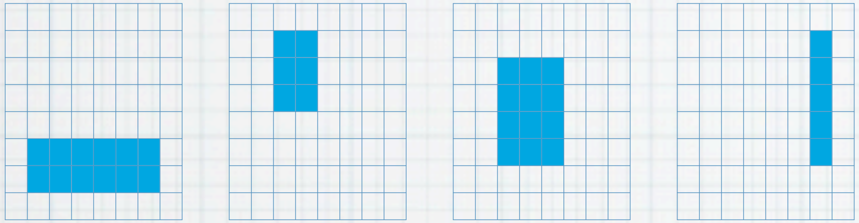
[Babai, Frankl, and Simon, 1986]

Π_0



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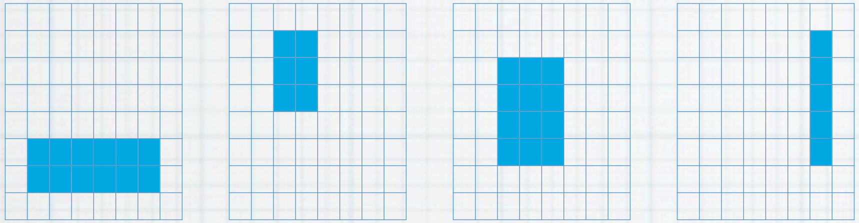

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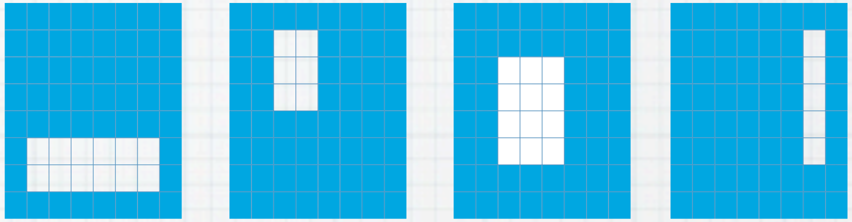

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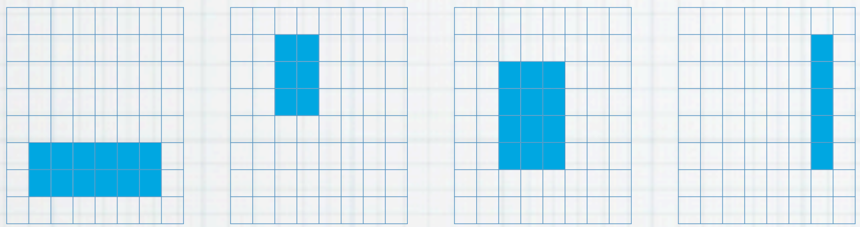

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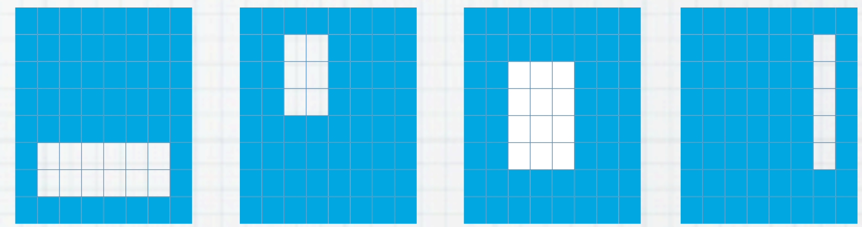

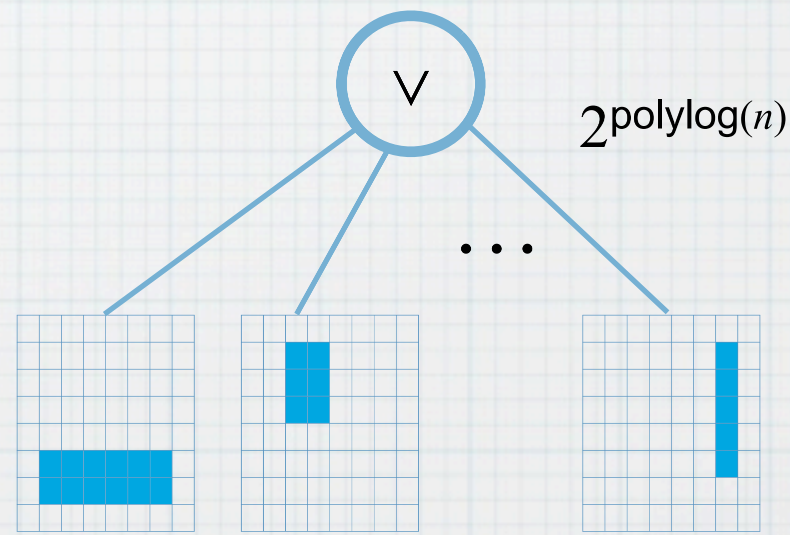

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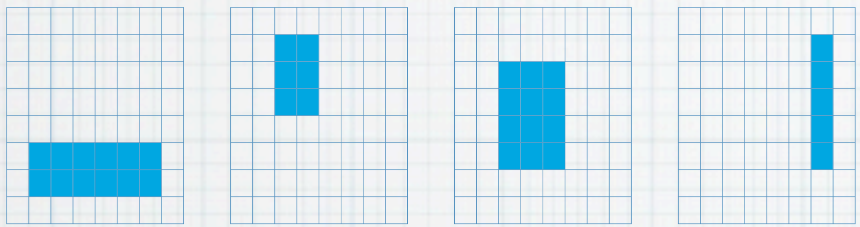

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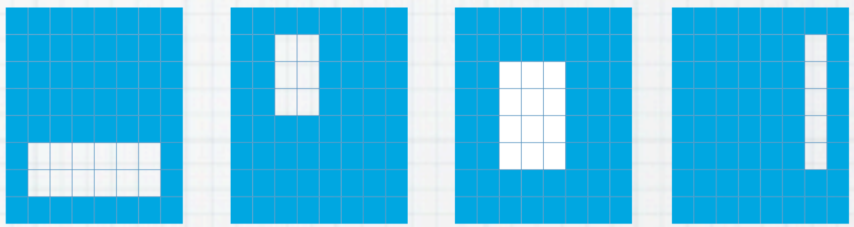
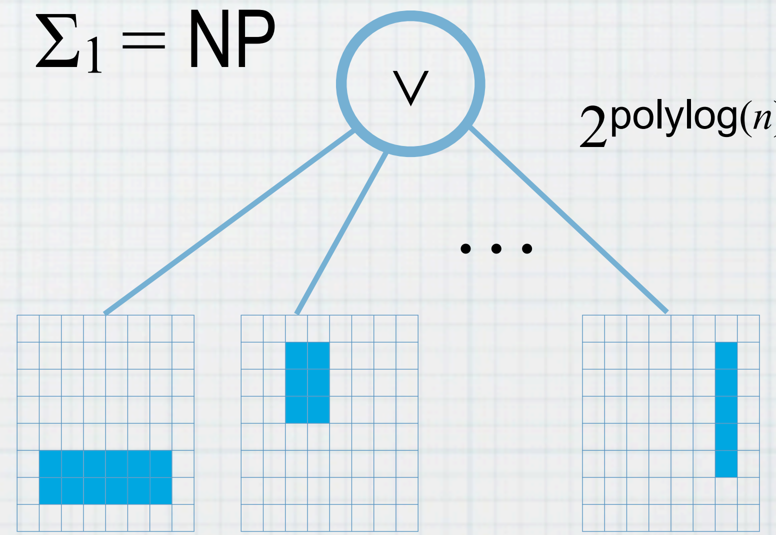


$\Sigma_1 = \mathrm{NP}$



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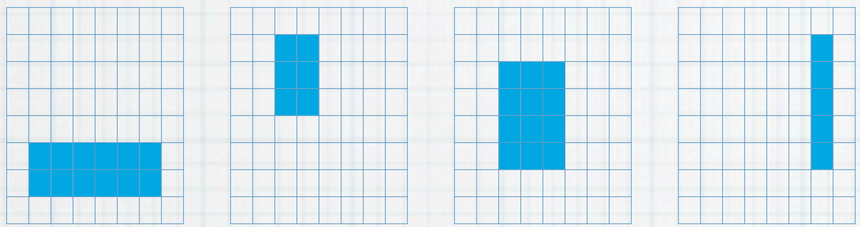

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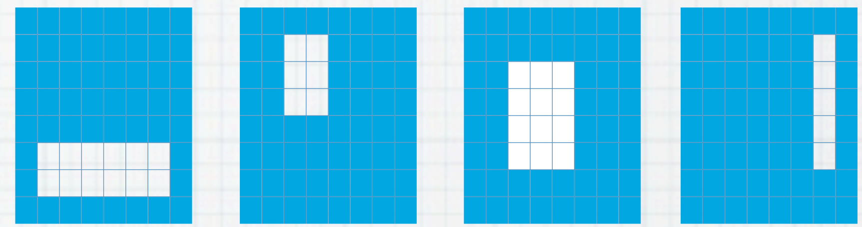
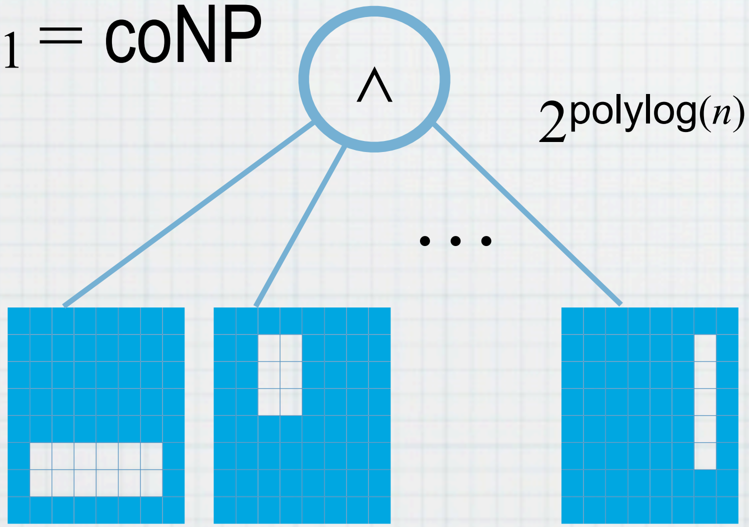


$\Pi_1 =$ coNP



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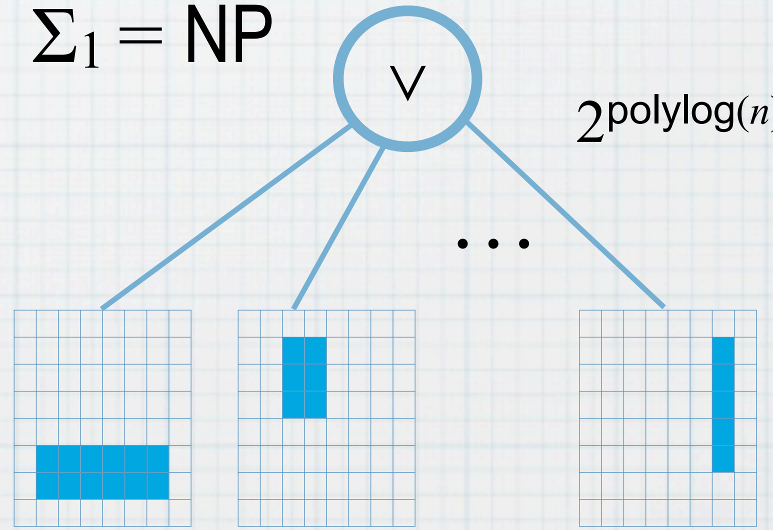

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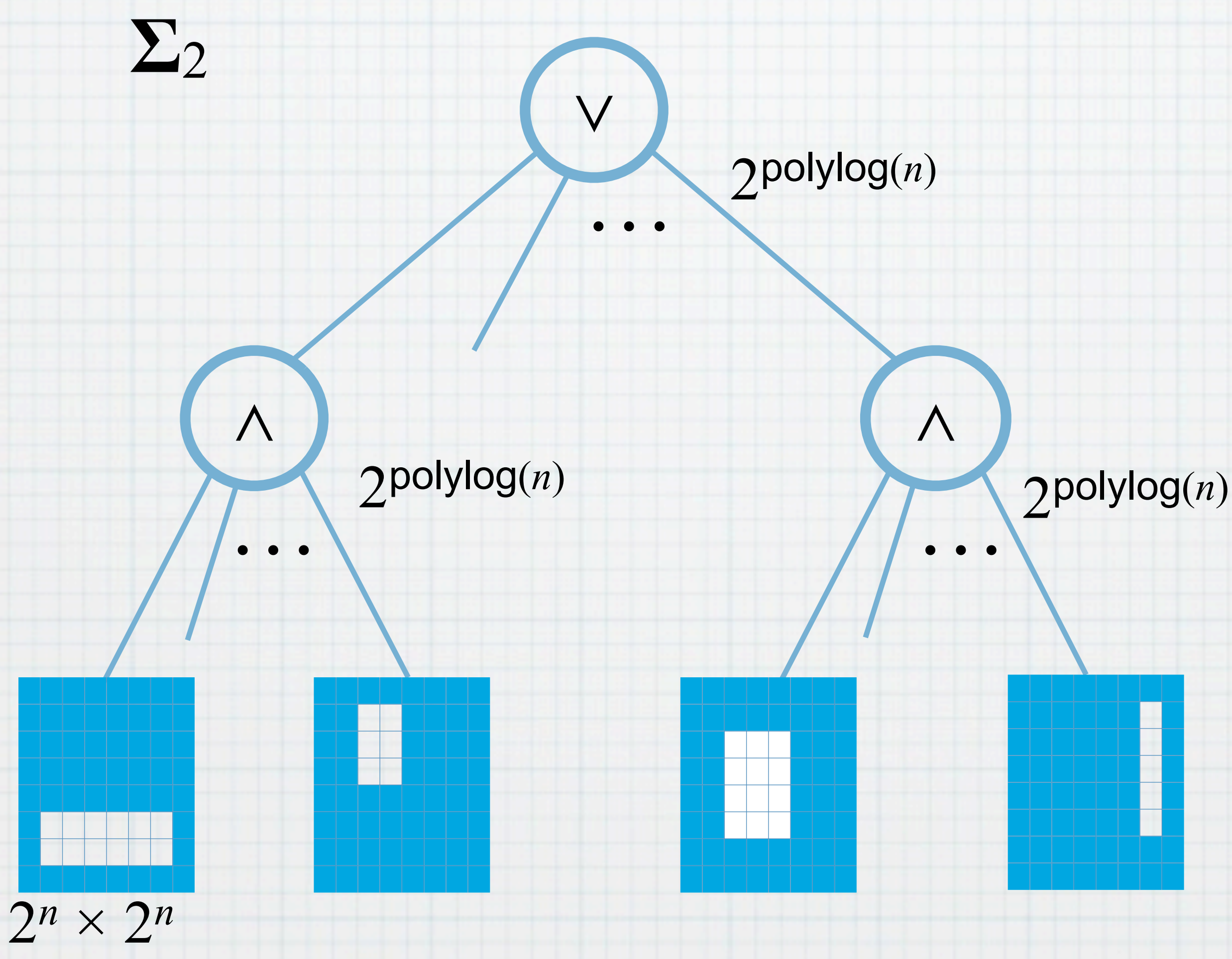

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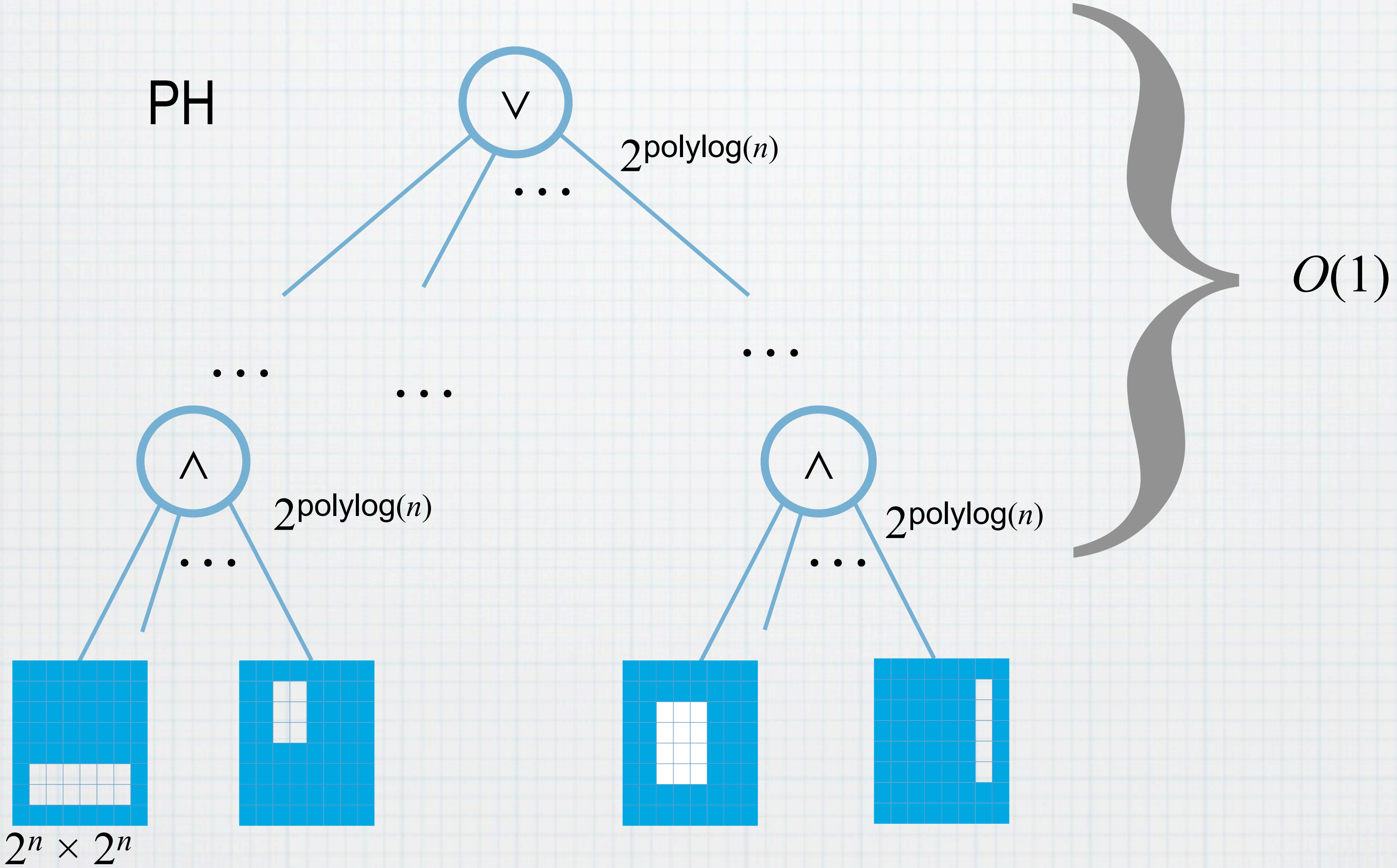

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Theorem (Razborov and S., 2008).
Let

$$f(x,y) = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{m^2} (x_{ij} \wedge y_{ij}).$$

Then the matrix $[f(x,y)]_{x,y}$ has sign rank $2^{\Omega(m)}$.

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Corollary. $\Sigma_2, \Pi_2 \not\subseteq$ UPP.

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analytic property
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f has high
**approximate
degree**

→

**Pattern matrix
method**

[S. 2007, 2008]

→

communication
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$f(x|_S)$ has high
bounded-error
c.c.

III. Sign-rank vs. PH

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analytic property
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find it by solving dual problem

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[Forster 2001]

$[f(x|s)]_{x,y}$

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Open problems

- Lots of results quantum vs. classical in alternate models (one-way, message passing, sampling, partial functions/relations)
  [Raz 1999]
  [Buhrman, Cleve, Watrous, de Wolf, 2001]
  [Bar-Yossef, Jayram, and Kerenidis, 2004]
  [Gavinsky, Kempe, Regev, and de Wolf, 2006]
  [Gavinsky, Kempe, and de Wolf, 2006]
  [Gavinsky, Kempe, Kerenidis, Raz, and de Wolf, 2007]
  [Gavinsky 2008]
  [Gavinsky and Pudlak, 2008]
  [Regev 2010]

  Only a quadratic separation for total functions
  [Razborov 2002]
  [Aaronson and Ambainis, 2005]

Open problems

- Power of entanglement
  [Buhrman and de Wolf, 2001]

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- Power of entanglement
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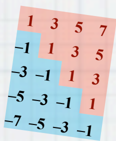
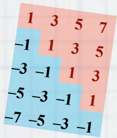
- Alternative to Yao-Kremer-Razborov?

Open problems



If $A_1, A_2, \ldots, A_k \in \{-1, +1\}^{n \times n}$ have low sign-rank, show that $\| A_1 \wedge A_2 \wedge \cdots \wedge A_k \|$ is high.

Open problems



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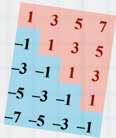

Exhibit $A_n \in \{-1,+1\}^{n \times n}$, $n = 1, 2, 3, \ldots$, with

$$\text{sign-rank}(A_n) = \Omega(n).$$

Open problems

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Best known: $\text{sign-rank}(A_n) \geq n^{1/2}$ [Forster 2001]

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Probabilistic method: sign-rank$(A_n) \geq n - 6.$ [Alon, Frankl, Rödl, 1985]

Thanks!

I. Real algebraic geometry

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Object of study: geometry of $\varphi(\mathbf{R}^n)$

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How many quadrants of $\mathbf{R}^N$ does $\varphi(\mathbf{R}^n)$ intersect?

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Equivalently:

How large is the set

$$\Sigma\varphi = \left\{ \text{sign}\left(\varphi(x)_1, \varphi(x)_2, \ldots, \varphi(x)_N \right) \ : \ x \in \mathbf{R}^n \right\} \ ?$$

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$|\,\Sigma\varphi\,| \leq 2^N$

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Studied by J. Milnor (1964), R. Thom (1965).

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Theorem (Alon, Frankl & Rödl, 1985).
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I. Real algebraic geometry

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$\ll 2^{n^2}$ for $r = o(n)$. $\square$