Coloring Simple Hypergraphs

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$$|S| \geq 2n - 1.$$ 

Then some three points in $S$ determine a right angle.
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If true, then sharp by letting
\[
S = \{(x, y) : x = 1 \text{ or } y = 1\} \setminus (1, 1).
\]
Heilbronn Problem/Conjecture (1947)

How large is the smallest triangle among $n$ points in general position in the unit square?
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How large is the smallest triangle among \( n \) points in general position in the unit square?

\( S \) – collection of \( n \) points in general position in the unit square

\[
T(S) = \text{area of smallest triangle}
\]

\[
T(n) = \max_S T(S)
\]
Trivial: \( T(n) < c/n \)

Observation (Erdős): \( T(n) > c/n^2 \)
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Explicit Construction (first lower bound)
\( n = \text{prime}, \quad y = x^2 \mod n \)
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- Every lattice triangle has area at least \( \frac{1}{2} \) (follows from Pick’s Theorem)
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- Shrink by a factor of $n - 1$. Areas shrink by a factor of $(n - 1)^2$. 

Conjecture: $T(n) = \Theta(1/n^2)$
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- \( T(n) > \frac{1}{2(n-1)^2} \)

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Upper Bounds

Roth (1951) \( n \sqrt{\log \log n} \)

Schmidt (1971) \( n \sqrt{\log n} \)

Roth (1972) \( n^{1.117+o(1)} \)

Komlós-Pintz-Szemerédi (1982) \( n^{1.142+o(1)} \)

Lower Bound

Komlós-Pintz-Szemerédi (1982) \( T(n) > c \log n \)
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Ajtai-Komlós-Szemerédi (1981) $(n \log n)^{1/3}$

Ruzsa (1998) $n^{\sqrt{2} - 1 - o(1)}$
A Sidon set is a set whose elements form a system of distinct representatives in the sense that any two distinct elements have different sums. Given a set $S$, denote by $|S \cap [n]|$ the number of elements of $S$ that are also in $[n]$. The Greedy Algorithm shows that there exists an infinite $S$ such that $|S \cap [n]| > cn^{1/3}$ for all $n$. Ajtai-Komlós-Szemerédi (1981) proved that $n \log n^{1/3}$, whereas Ruzsa (1998) improved this to $n^{\sqrt{2}-1-o(1)}$. Erdős conjectured that $n^{1/2-\epsilon}$ is the best possible bound for all $\epsilon > 0$. Conjecture (Erdős)
Fix $k, r \geq 2$. Let $A$ be an $n \times M$ matrix over $\mathbb{Z}_2$ with

- $k$ one's in each column
- every $r$ columns linearly independent over $\mathbb{Z}_2$ (i.e. every set of at most $r$ column vectors does not sum to 0)
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$M := M(n, k, r) =$ maximum number of columns in $A$

In other words, $M$ is the maximum length of a binary linear code with minimum distance at least $r + 1$ and parity check matrix with $n$ rows and every coordinate having at most $k$ check equations.
Lefmann-Pudlak-Savický (1997)

\[ M(n, k, r) > cn^{\frac{kr}{2(r-1)}} \]

Results of Frankl-Füredi on union closed families yield

\[ M(n, k, 4) < cn^{\left\lceil \frac{4k}{3} \right\rceil} \]

so when \( k \equiv 0 \pmod{3} \), \( M(n, k, 4) = \Theta(n^{\frac{2k}{3}}) \)

Kretzberg-Hofmeister-Lefmann (1999)

If \( r \geq 4 \) is even, \( \gcd(r - 1, k) = 1 \), then

\[ M(n, k, r) > cn^{\frac{kr}{2(r-1)}} (\log n)^{\frac{1}{k-1}} \]

Naor-Verstraëte (2009) Improvements for different ranges of \( k, r \);
connections to extremal graph theory
Let $k \geq 2$ be fixed, $n \to \infty$

Fact (Turán’s theorem) Let $H$ be a $k$-uniform hypergraph with average degree $d$. Then

$$\alpha(H) > c_k \frac{n}{d^{1/(k-1)}}.$$
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Proof. Pick vertices randomly; delete a vertex for each edge among picked vertices.
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Sharp. Let $H = K^k_n$, then $d = \binom{n-1}{k-1} = \Theta(n^{k-1})$, $d^{\frac{1}{k-1}} = \Theta(n)$ and

$$\alpha(H) = k - 1 = \Theta(1)$$
What if \( H \) is locally sparse?

2–cycle

\[
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{cycle2}}
\end{array}
\]

3–cycle

\[
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{cycle3}}
\end{array}
\]

4–cycle

\[
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{cycle4}}
\end{array}
\]

girth \( g \) – no cycle of length less than \( g \)

simple or linear – girth 3 or no 2-cycle
Theorem (Komlós-Pintz-Szemerédi $k = 3$, Ajtai-Komlós-Pintz-Spencer-Szemerédi $k \geq 3$ 1982)

Let $k \geq 3$ be fixed. Let $H$ be a $k$-uniform hypergraph with girth at least 5 and (average) maximum degree $\Delta$. Then

$$\alpha(H) > c_k \frac{n}{\Delta^{1/(k-1)}} (\log \Delta)^{1/(k-1)}.$$
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Conjecture (Spencer 1990), Theorem (Duke-Lefmann-Rödl 1995)

Same conclusion holds as long as \( H \) is simple.
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- $T(n) > c \frac{\log n}{n^2}$
- $|S| > c(n \log n)^{1/3}$ (Improved by Ruzsa)
- $M(n, k, r) > cn^{\frac{kr}{2(r-1)}} (\log n)^{\frac{1}{k-1}}$. 

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\[ \Delta = \Delta(G) = \text{max degree of } G \]

Greedy Algorithm: \[ \chi(G) \leq \Delta + 1 \]

Brook’s Theorem: \[ \chi(G) \leq \Delta \text{ unless } G = K_{\Delta+1} \text{ or } G = C_{2r+1} \]
Graph Coloring

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What if \( G \) is triangle-free?
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Borodin-Kostochka: $\chi(G) \leq \frac{2}{3}(\Delta + 2)$
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What about independence number?
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Ajtai-Komlós-Szemerédi, Shearer: \[ \alpha(G) > c \frac{n}{\Delta} \log \Delta \]
Question (Vizing 1968): What is the best possible bound on the chromatic number of a triangle-free graph $G$ in terms of its maximum degree?
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Theorem (Frieze-M) Let $k \geq 3$ be fixed. Then there exists $c = c_k$ such that every $k$-uniform simple $H$ with maximum degree $\Delta$ has

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- The proof is independent of K-P-S and A-K-P-S-S (and D-L-R) so it gives a new proof of those results.
- The result is sharp apart from the constant $c$. 
Semi-Random or “Nibble” Method

A-K-S, K-P-S and A-K-P-S-S (1980-82) were perhaps the first papers using this approach.

Rodl’s proof (1985) of the ErdHos-Hanani conjecture on asymptotically good designs.

Frankl-Rodl (1985) result on hypergraph matchings.

Pippenger-Spencer (1989) result of hypergraph edge-coloring.

Kahn (1990s) proved many results, list coloring using different approach to P-S.


Johansson (1997) additional new ideas for triangle-free graphs.

Vu (2000+) extended Johansson’s ideas to more general situations.

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More Tools

Concentration Inequalities

- Hoeffding/Chernoff
- Talagrand
- Local Lemma
- Kim-Vu polynomial concentration takes care of dependencies

Example of Kim-Vu: Let $G = G(n, p)$, $p = \frac{1}{\sqrt{n}}$. Fix a vertex $x$ in $G$. $T(x)$ is the number of triangles containing $x$. Then $\mu = \mathbb{E}(T(x)) = \binom{n-1}{2}p^3$ but triangles are not independent. Still $\Pr(\left| T(x) - \mu \right| > \delta \mu) < e^{-c \delta \mu}$. 
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The Algorithm \((k = 3)\)

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The Algorithm ($k = 3$)

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- \(H^t = H[U^t]\) – subgraph of \(H\) induced by \(U^t\)
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- \(p^t_u \in [0, 1]^C, u \in U^t\) – vector of probabilities of colors
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- \(p^0_u = (1/q, \ldots, 1/q)\) – initial color vector
Dhruv Mubayi  |  Coloring Simple Hypergraphs
For $u \in U$, $c \in [q]$, tentatively activate $c$ at $u$ with probability
\[ \Theta \cdot p_u(c). \]
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A color is lost at $u$ if either

- there is an edge $uu_1u_2$ such that $c$ is tentatively activated at $u_1$ and $u_2$, or

- $x$ has been colored with $c$ and $c$ has been tentatively activated.
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Parameters $p_u$ are updated in a (complicated) way to maintain certain properties of $H_t = H[U]$. 

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In this case $p_u(c) = 0$ for all further iterations

Assign a permanent color to $u$ if some color $c$ is tentatively activated at $u$ and is not lost
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Parameters $p_u$ are updated in a (complicated) way to maintain certain properties of $H_t = H[U]$
Parameters \((k = 3)\)

During the process, we must choose update values to maintain the values of certain parameters:

- \(\sum_c p_u(c) \sim 1\)
- \(e_{uvw} = \sum_c p_u(c)p_v(c)p_w(c) \ll \frac{\log \Delta}{\Delta}\)
- \(\deg(v) \leq \left(1 - \frac{1}{\log \Delta}\right)^t \Delta \sim e^{-t/\log \Delta}\)
- Also, entropy is controlled; key new idea of Johansson; don’t need martingales, Hoeffding suffices

Continue till \(t = \log \Delta \log \log \Delta\) and then apply Local Lemma.
What next?

Independence number of locally sparse Graphs

Let $G$ contain no $K_4$
What next?

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$$\alpha(G) > c \frac{n}{\Delta} \log \log \Delta$$
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- Shearer (1995)

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Independence number of locally sparse Graphs

Let $G$ contain no $K_4$

  \[ \alpha(G) > c \frac{n}{\Delta} \log \log \Delta \]

- Shearer (1995)
  \[ \alpha(G) > c \frac{n}{\Delta} \frac{\log \Delta}{\log \log \Delta} \]

- Major Open Conjecture (Erdős et. al.)
  \[ \alpha(G) > c \frac{n}{\Delta} \log \Delta \]
Conjecture (Frieze-M)
Let $F$ be a fixed $k$-uniform hypergraph. Then there exists $c = c_F$ such that every $F$-free $k$-uniform hypergraph $H$ with maximum degree $\Delta$ satisfies

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Algorithms??

Convert our proof to a deterministic polynomial time algorithm that yields a coloring with $c(\Delta / \log \Delta)^{1/(k-1)}$ colors

Moser-Tardos results yield a randomized algorithm
Problem (Erdős 1977)

Do $n^2$ points in the plane always contain $2n - 2$ points which do not determine a right angle?
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If true, then sharp (take $[n] \times [n]$ and use Problem at the beginning)
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Lower bounds on the number of points
Another Geometric Application

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- Elekes (2009) $\Omega \left( \frac{n}{\sqrt{\log n}} \right)$
- Gyárfás-M $\Omega(n)$ if Frieze-M Conjecture holds for $k = 3$ and $F = K_9^3$
Thank You