QUANTUM FIELD THEORY
Jan 2007, Jan 2009

Contents

1 Outline

1.1 Outline of course taught in 2007

Part I:

1. Introduction. Why field theory and some intuition
2. Concrete example: Phonons in a crystal and the continuum limit

   →**Students should now have a physical picture of Quantum Field Theory**
3. Relativistic Classical Field Theory, EOM, Noether charge etc
4. Other aspects: Green's fn. two point correlators, relativistic normalization of states etc
5. Causality, commutators
6. Complex scalar field

   → **Students now know all about free relativistic scalar field theory**

   Part II:
7. Mathematical digression: Lorentz Group and its reps.
8. Dirac Equation, gamma matrices, Weyl, Majorana etc.
9. Quantization of Dirac Field
10. C,P,T

    Part III:
11. Maxwell action, Gauge Invariance and Quantization, propagator..

    →**Now students know all the usual free relativistiv QFT's**

    Part IV:
12. Interactions - various kinds
13. Perturbation theory, Interaction picture, Wick's theorem

14. Feynman Diagrams

15. S-Matrix, Cross Section

16. LSZ Reduction Formula, in states, out states etc.

    → **At this point students can *in principle* calculate S-matrices for all the usual QFT's**

    Part V:

17. QED - some processes. tree level.

    Part VI:

18. Functional formalism for free scalars, fermions and gauge theories, gauge fixing

19. Fnl formalism for interacting theories. generating functional and Γ.

20. One loop ϕ^4 theory.

21. Renormalization: bare vs ren flds and parameters.

    Part VII:

22. One loop QED - explicit calculations

23. Ward Identities and BRS symmetry

    → **At this point the students knows how to do loop calculations and renormalize.**

1.2 Course taught in 2009

The emphasis in 2009 was to teach a course that would be useful both for the high energy physicist as well as the low energy physicist. Field theory as used in particle physics is almost identical to what is used in for intance critical phenomena/ statistical physics. The functional integral (in D space +1 time dimensions) after a Euclideanization (Wick rotation, which is done for calculational purposes anyway in particle physics) is the classical statistical mechanics partition function in D+1 space dimensions. The only difference is that in a particle physics courses the ostensible aim of calculating correlation functions is to calculate the S-matrix, whereas in statistical mechanics

a major aim is to calculate critical exponents. Thus the entire package of field theory techniques is common to both disciplines. This however requires that we use the functional formalism. That was what was done in 2009. Two sections, one on S-matrix and one on critical phenomena, explains briefly the applications of the formalism to both disciplines.

Note that Wick contractions and the usual operator method of perturbation theory in the interaction formalism was not done. If the Dirac theory and rep of Lorentz group are done away with, a couple of lectures on this can be included.

1. **Part 1**

   Thus in 2009 the first two (out of three) parts of the course dealt with scalar field theory. The free theory was motivated by the crystal lattice example and the continuum limit taken. Canonical quantization was done. Green's functions defined. Subsequently path integrals were introduced since the students had not done that. The functional formalism for field theories was developed. The connection to the usual Heisenberg and Schroedinger picture was made. The generating functional and the technique of Feynman diagrams for a perturbative evaluation was derived. The actual evaluation of integrals is done later.

2. **Part II**

   A calculation of the scattering cross section is then given as an application to particle physics and as an application to critical pheneomena Landau theory for ferromagnetism with a scalar field (magnetization) as an order parameter is introduced. Critical exponents are defined and the role of "mass" parameter in fld theory as the inverse correlation length in stat mech is explained.

   Loop integrals are evaluated and renormalization is described. The beta function and running coupling constant is explained. Wilson's picture is given and the understanding of bare coupling and renormalized coupling as parameters defined at two widely different scales is explained. This gives an RG explanation for the process of renormalizing infinities by adding counterterms that the particle physicist is familiar with. Wilson's idea of integrating out momentum shells to obtain a theory with a lower cutoff and a modified coupling constant is described. An explicit example of the free scalar field is given in some detail and the

concept of relevant, irrelevant and marginal operators is explained in the context of this free scalar theory. The non trivial fixed point of ϕ^4 theory in less than 4 dimensions is shown. (4-5 lectures were spent of these aspects of RG) Universality in critical phenomena is explained.

3. **Part III**

   Electrodynamics. Gauge invariance and the Fadeev Popov procedure for gauge fixing is described. Photon propagator is derived. Canonical quantization in Coulomb gauge is done. Ward identities are introduced - using functional methods.

   One lecture on rep of Lorentz roup was given. Dirac theory. Green's functions are defined. Quantization with anticommutators is introduced. Functional formalism form with Grassmann variables is defined and the generating functional is set up.

   Tree level processes in QED are calculated. One loop renormalization. beta function. Ward-Takahashi identitites.

2 Introduction

1. Introduction

   - **Why field theory?** ANS: Start with the example of electromagnetic phenomena. Classical electromagnetic phenomena is well described by the introduction of the concept of an electromagnetic field. $\vec{E}(x,y,z,t), \vec{B}(x,y,z,t)$. There the introduction of the field allows you to do away with "action at a distance". i.e. a charge acts on another charge through the medium of an electric field that it creates everywhere. Very useful. Makes it **local**. The existence of electromagnetic waves - prediction of Maxwell- makes the idea of a field more 'real'. It is hard to describe wave phenomena without the idea of a field pervading space.

     Other examples: whenever there is a continuous medium that has dynamical properties eg water, air, it is useful to introduce the idea of a field to describe properties of the material as a function of space and time. eg. pressure $p(x,y,z,t)$, density $\rho(x,y,z,t)$ ..etc.

The idea of a field in electromagnetism is a little different from the idea of a field in fluid mechanics. In the former we believe it to be an exact description. In the latter it is an approximation. But the techniques of classical field theory are applicable to both.

But what about **quantum field theory?**

It is now a fundamental idea and all fundamental entities in particle physics are described as fields. The application of quantum mechanics to particle physics requires quantum field theory.

- **Connection with Statistical Mechanics- If we Wick rotate $t \to -it$ we get a Euclidean theory that can be related to stat mech where $\hbar$ becomes kT.** This remarkable fact allows all the techniques and concepts of QFT to be used in understanding Critical Phenomena in Condensed matter systems. This connection is easiest to see in the "feynman path integral" approach to field theory, (which is definitely the most flexible approach). Thus we can use the computational techniques of Quantum field theory to calculate correlation functions. In practice even in particle physics these calculations are done in Euclidean space (this is just the mathematical trick of analytic continuation). At the end of the day we rotate back to Minkowski space and interpret these in terms of scattering amplitudes. The same calculation can be done in studying critical phenomena where statistical fluctuations replace quantum fluctuations and the field is an order parameter - an effective object rather than something fundamental.
- **Importance of space time continuum:** Not really fundamental. can be thought of as a mathematical idealization. Useful because differential equations are understood better than difference equations. Thus even though we know that water is made up of discrete entitities - water molecules - we introduce the notion of $\rho(x, y, z, t)$ (density) as if it is a continuous function. This is useful because in practice things are reasonably continuous in some approximation.

  We used to believe that space time is really a continuum - but nowadays it is not taken as sacroscanct - esp in theories of quantum gravity.
- **quantum versus classical:** we believe that everything is quantum mecahnical. Therefore all dynamical variables should be

treated as quantum mechanical - including the fundamental fields. These need not be true for effective fields - which may be described by classical mechanics. This means that after the problem has been treated quantum mecahnically the resultant theory is described by some convenient dynamical variables that obey some differential equations. These can be treated as classical. In practice we may write down these final equations directly by observing nature at macroscopic scales. we just assume that there is a fundamental quantum mechanical description from which these can be derived. Thus the elctromagnetic field is quantum mechanical. The density waves in air are classical.

- **particles versus fields:** when a field is quantized we get discrete excitations of the field. These are photons. Thus field becomes an operator that creates photons out of the vacuum. The formalism thus allows particle production and destruction. Whenever these process are important - as in relativistic particle physics - field theory is useful. Ordinary quantum mechanics formalism is clumsy.
- inasmuch as they are extremely useful to describe nature we can say that they are real (in the ontological sense) entities.

2. **Electromagnetic Field:**

- As a motivation for the formalism let us start with the familiar example of em - photons. Consider an em wave moving in the $+z$ direction. $\omega = kc$. Assume the el fld is in the x-direction.

$$E_x(z,t) = ae^{i(kz-\omega t)} + a^* e^{-i(kz-\omega t)}$$

Intensity $\propto |a|^2$.It is an exptl fact that the number of photons $\propto$ intensity.So treat a as a harmonic oscillator (annihilation) operator and then a^*a becomes the number operator on quantization. So interpretaion of $a^\dagger$ is - creation operator - creates a **photon**. In this case this photon has wave number k. Thus if $|0\rangle$ is the ground state of the harmonic oscillator - which means no photons, then $|1\rangle = a^\dagger|0\rangle$ is the state with one photon and $|2\rangle = a^\dagger a^\dagger|0\rangle$ becomes a state with two photons. Thus the formalism allows you to deal with varying number of particles.

- Relativistic quantum field theory is the way to do QM relativistically.Otherwise there are at least suerficially clashes between QM and causality.
- The notion of an em field has implicit in it the notion of a space time continuum. Thus $E(x,t)$ - here x,t are real numbers.This creates problems because it suggests that an em wave can have arbitrarily small wavelength and therefore arbitrarily high wavenumber. The photon then has arbitrarily high momentum and energy. Also one can fit an infinite number of standing waves in a cavity. All these "infinities" cause problems. So we tend to assume in the intermediate stages of the calculation that things are discrete and later take the "continuum limit". This is subtle.
- In order to illustrate the issues involved in the continuum approximation to a discrete system we will start with a discrete system and derive a continuum description. A crystal where the atoms vibrate about their mean position. These vibrational waves are the pressure waves and the corresponding particles obtained on quantizing them are called **phonons**. Just like photons from em waves.

3 Continuum Limit of a Discrete System: Phonons

3.1 A Crystal

We start by studying a model of a crystal. Impose **periodic boundary conditions** for simplicity. $q_1 = q_{N+1}$.[1]

$$L = \frac{1}{2}\sum_{i=1}^{N} \dot{q}_i^2 - \frac{1}{2}\Sigma_{i=1}^{N}\nu^2(q_{i+1} - q_i)^2 \tag{1}$$

ν is a constant that has dimensions of frequency. **Eqn of Motion (EOM)**

$$\frac{\partial L(t)}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L(t)}{\partial \dot{q}_i(t)} = 0 \tag{2}$$

[1]Some redefinitions of variables have been done that are illustrated by the single harmonic oscillator Lagrangian. If we start with $L = \frac{1}{2}m\dot{X}^2 - \frac{1}{2}kX^2$ and define $\sqrt{m}X = q$, we get $\frac{1}{2}(\dot{q}^2 - \frac{k}{m}X^2) = \frac{1}{2}(\dot{q}^2 - \nu^2 q^2)$

$$-\nu^2(q_{i+1} - q_i) + \nu^2(q_i - q_{i-1}) + \ddot{q}_i \tag{3}$$

$$\ddot{q}_i = -\nu^2(2q_i - q_{i+1} - q_{i-1}) \tag{4}$$

Try: $q_m(t) = Q_K(t)e^{imK}$ - Plane wave

$$\ddot{Q}_K(t)e^{imK} = -2\nu^2 Q_K(t)e^{imK}(2sin^2\frac{K}{2}) \tag{5}$$

$$\ddot{Q}_K(t) = -4\nu^2 sin^2\frac{K}{2}Q_K(t) \tag{6}$$

The solution is:

$$Q_K(t) = Q_K(0)e^{-i\omega_K t} \quad ; \tag{7}$$

$$\boxed{\omega_K = \pm 2\nu sin\frac{K}{2}}$$

Our normal mode solutions are $\boxed{q_m(t) = Q_K(0)e^{-i\omega_K t}e^{imK}}$ (8)

Allowed values of K will be fixed by boundary condns (bc).

Let N be total no.Since $q_{N+1} = q_1$, we must have $e^{iNK} = 1$. So

$$K = \pm\frac{2\pi n}{N}$$

. with $n = 0, 1, ..N-1$. Step size is $\frac{2\pi}{N}$. As $N \to \infty$, K becomes continuous. If a is the lattice spacing between atoms and L is the length of the crystal, then $N = \frac{L}{a}$.

$$K = \frac{2\pi na}{L} = ka = \frac{2\pi}{\lambda}a$$

Since ma denotes a position, we can let $ma = x$.

The normal mode solutions look like:

$$q_m(t) = Q_K(0)e^{i(\frac{2\pi n}{L})(ma)}e^{-i\omega_k t} = e^{ikx}e^{-i\omega_k t}$$

We can then write:

$$q(x,t) = Q(k)e^{i(kx-\omega_k t)}$$

These are sound waves in the solid crystal. $\boxed{\omega_k = 2\nu sin\frac{ka}{2}}$

Finally, by linearity the final solution is a superposition:

$$q(x,t) = \sum_k Q_k e^{i(kx-\omega_k t)} + cc \tag{9}$$

Since $q(x,t)$ is real, we must add cc.

3.2 Continuum Approximation

- From a distance we don't see the graininess of a crystalline solid.

  If a is very small , i.e. $a << \lambda$ then $ka << 1$ and $sin\frac{ka}{2} \approx \frac{ka}{2}$ So $\omega_k = \underbrace{\nu a}_{c} k = kc$. Here c is the velocity of the wave.

  $q_m(t) = q(x,t)$ where $x = ma$. But if take $a \to 0$ (spacing between atoms) keeping L fixed, then for a finite distance x, m will have to be infinity. So it is not convenient to use m. So use continuous **label** x. (IMP. In fld theory, x is a label, not a dynamical variable - unlike in the usual single particle QM). So write everything in terms of x.

- $$q_{i+1} - q_i = q(x+a) - q(x) \approx a\frac{\partial q}{\partial x}$$

  Let us adopt the following convention for interpolating between two lattice points: any function $F(x)$ that has the value $F(ma)$ at $x = ma$ and $F((m+1)a)$ at $x = (m+1)a$, keeps the value $F(ma)$ for $ma \leq x < (m+1)a$. (Step approximation). Then

  $$\lim_{a\to 0} \frac{1}{a} \int_x^{x+a} dx F(x) = F(ma)$$

  $$\Rightarrow \sum_{m=1}^{N} F(ma) = \frac{1}{a} \int_0^L dx F(x)$$

  Thus we see that $\int dx = a \sum_i$. Also

  $$\sum_i \frac{\partial}{\partial q_i}(q_j) = \sum_i \delta_{ij} = 1 \tag{10}$$

Letting $ia = x$ and $ja = y$, so that $q_i = q(x)$ and $q_j = q(y)$ and also $\delta(x-y) = \frac{1}{a}\delta_{ij}$

because

$$\int dx\ \delta(x-y) = a\sum_i \frac{1}{a}\delta_{ij} = 1$$

we see that

$$a\sum_i \frac{1}{a}\frac{\partial}{\partial q_i}(q_j) = \int dx\ \frac{\delta}{\delta q(x)}q(y) = \int dx\ \delta(x-y) = 1$$

from which we also get the relation between the functional derivative and ordinary derivative. Now write action in terms of $q(x)$.

$$\frac{1}{2a}[\int_0^L dx\ (\dot{q}(x)^2 - \nu^2 a^2(\frac{\partial q}{\partial x})^2)]$$

$$= \frac{1}{2a}[\int_0^L dx\ (\dot{q}(x)^2 - c^2(\frac{\partial q}{\partial x})^2)]$$

- The last step is to redefine the dynamical variable q to get rid of the negative power of a in front. Let $\phi(x) = \frac{1}{\sqrt{a}}q(x)$. We get

$$= \frac{1}{2}[\int_0^L dx\ (\dot{\phi}(x)^2 - c^2(\frac{\partial \phi}{\partial x})^2)]$$

This last step involving defining a rescaled field variable is called in technical jargon **"wave function renormalization" or "field renormalization"**.

- EOM: $\frac{\delta S}{\delta \phi} = \frac{\partial^2 \phi}{\partial t^2} - c^2\frac{\partial^2 \phi}{\partial x^2} = 0$
- There can be many discrete versions for the same continuum version. Try adding $(2q_i - (q_{i+1} + q_{i-1}))^2$to the crystal action. Does the continuum limit change? This is the issue of "universality" in the theory of critical phenomena. More on this later.

3.3 Quantization

- Step 1: Write as a sum of decoupled HO
- Step 2 "Quantize" each mode
- Step 3: Superpose "quantum" normal modes to get "quantum" field.
- Step 4: take $a \to 0$ and $L \to \infty$ limits.

Preliminary Simple Harmonic Oscillator:

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$$
$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$$

Classical:

$$x(t) = A sin\ \omega t + B cos\ \omega t$$

$x(0) = B\ ; \dot{x}(0) = \omega A.$

So $\boxed{x(t) = \frac{p(0)}{\omega} sin\ \omega t + x(0) cos\ \omega t}$

This can be written as:

$$\frac{1}{\sqrt{2\omega}}\Big\{ \underbrace{\Big[\frac{\sqrt{\omega}x(0) - i\frac{p(0)}{\sqrt{\omega}}}{\sqrt{2}}\Big]}_{a^*} e^{i\omega t} + \underbrace{\Big[\frac{\sqrt{\omega}x(0) + i\frac{p(0)}{\sqrt{\omega}}}{\sqrt{2}}\Big]}_{a} e^{-i\omega t}\Big\}$$

Thus $\boxed{x(t) = \frac{1}{\sqrt{2\omega}}\big[a^*(0)e^{i\omega t} + ae^{-i\omega t}\big] \equiv \frac{1}{\sqrt{2\omega}}[a^*(t) + a(t)]}$

Also $p = \dot{x}$ and $\boxed{H = \omega a^* a}$

Quantum HO: Quantize $[x, p] = i\hbar \Rightarrow [a, a^\dagger] = \hbar$ Rescale $a, a^\dagger \to \sqrt{\hbar}a, \sqrt{\hbar}a^\dagger$ to get $[a, a^\dagger] = 1$.

Also $H = \frac{1}{2}\hbar\omega(aa^\dagger + a^\dagger a) = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega \equiv (N + \frac{1}{2})\hbar\omega$. N is called the number operator.

Digression: Schroedinger Formalism and Heisenberg Formalism

In S.. formalism time dependence is in the wave function. In H.. formalism it is in the operator.

Schroedinger Picture

x, p satisfy the usual comm. relns. there is no reference to time. If we work in the x basis, we have wave functions $\psi(x,t)$ on which p acts as $-i\frac{\partial}{\partial x}$. ψ evolves in time as $|\psi,t\rangle = e^{-iHt}|\psi,0\rangle$.

Heisenberg Picture The time dependence is in the operator.

Time evolution:

$$i\hbar\frac{\partial O}{\partial t} = [O,H]$$

- heisenberg eqns.

Thus $a^\dagger(t) = e^{i\omega t}a(0)$ follows from this. In general $O(t) = e^{iHt}O(0)e^{-iHt}$. Thus the "equal time commutation relations" hold, if they hold at $t = 0$. i.e $[x(t), p(t)] = i\hbar$, $[a(t), a^\dagger(t)] = 1$

Connection between the two:

$$|\psi,t\rangle_S = e^{-iHt}|\psi,0\rangle_S = e^{-iHt}|\psi\rangle_H$$

Thus at $t = 0$ the Heisenberg and Schroedinger states are the same. Also

$${}_S\langle\psi,t|O_S|\psi,t\rangle_S = {}_S\langle\psi,0|e^{iHt}O_Se^{-iHt}|\psi,0\rangle_S = {}_H\langle\psi|e^{iHt}O_H(0)e^{-iHt}|\psi\rangle_H = {}_H\langle\psi|O_H(t)|\psi\rangle_H$$

$O_H(t) = e^{iHt}O_Se^{-iHt}$.

End of digression

Now we can perform the four steps of quantization.

- **Step 1: Normal mode decomposition:**

$$q_m(t) = \sum_{K=0}^{2\pi} Q_K(0)e^{-i\omega_K t}e^{imK} + c.c$$

$K = \frac{2\pi n}{N}.K_{max} = 2\pi$ (when $n = N$).

One finds that (Using $\sum_m e^{im(K+K')} = \delta_{K,-K'}$)

$$\sum_m q_m^2(t) = N \sum_{n=-\frac{1}{2}N}^{n=+\frac{1}{2}N} \big\{ -\omega_K^2[Q_K Q_{-K} + Q_K^* Q_{-K}^*] + \omega_K^2[Q_K Q_K^* + c.c] \big\}$$

The factor N comes from the number of sites.

Similarly one finds that

$$\nu^2 \sum_m (q_{m+1} - q_m)^2 = N\nu^2 \sum_{n=-\frac{1}{2}N}^{n=+\frac{1}{2}N} \big\{ (e^{iK}-1)(e^{-iK}-1)[Q_K Q_{-K} + Q_K^* Q_{-K}^*]$$

$$+ (e^{iK}-1)(e^{-iK}-1)[Q_K Q_K^* + c.c] \big\}$$

Using $(e^{iK}-1)(e^{-iK}-1) = 4sin^2\frac{K}{2}$,

$$\nu^2 \sum_m (q_{m+1} - q_m)^2 = N \sum_{n=-\frac{1}{2}N}^{n=+\frac{1}{2}N} \omega_K^2 \big\{ [Q_K Q_{-K} + Q_K^* Q_{-K}^*] + [Q_K Q_K^* + c.c] \big\}$$

Adding we get:

$$\frac{1}{2} \sum_m q_m(t)^2 + \frac{1}{2}\nu^2 \sum_m (q_{m+1} - q_m)^2 = N \sum_K \omega_K^2 [Q_K Q_K^* + c.c]$$

Finally rescaling : $\boxed{a_K = \sqrt{N 2\omega_K} Q_K}$

we get

$$H = \frac{1}{2} \sum_K \omega_K [a_K a_K^* + c.c.]$$

$$q_M(t) = \frac{1}{\sqrt{N}} \sum_{n=-\frac{1}{2}N}^{n=+\frac{1}{2}N} \frac{1}{\sqrt{2\omega_K}} [a_K(t) e^{imK} + cc] \;\; ; \quad p_m(t) = \dot{q}_m(t) \quad (11)$$

END OF STEP 1.

- **Step 2: Quantize each mode:**

$$a_K(0) \to \sqrt{\hbar} a_K \quad a_K^*(0) \to \sqrt{\hbar} a_K^\dagger$$

. $\boxed{[a_K, a_K^\dagger] = \delta_{K,K'} \quad ; \quad H = \hbar \sum_K \omega_K (a_K a_K^\dagger + a_K^\dagger a_K)}$

(**IMP: Zero point energy** $= \infty$ **(as** $N \to \infty$**)!!!**)

END of STEP 2

- **Step 3: Superpose:**

Calculate $[q_m(0), p_{m'}(0)]$ using the com relns of $a, a^\dagger$ and eqn (11) and also using $\sum_K e^{i(m-m')K} = N\delta_{m,m'}$ we get $\boxed{[q_m, p_{m'}] = i\delta_{m,m'}\hbar}$

and other commutators $[q, q]; [p, p] = 0$.

Using time evolution one gets $\boxed{[q_m(t), p_{m'}(t)] = i\delta_{m,m'}\hbar}$

Equal Time Commutator (ETC) of QFT.

END of STEP 3.

Summary of what has been done so far: We start off with q_m which is the displacement of an individual atom of the crystal. There are N of these. We could just demand that q_m, p_m have the usual comm reln. But instead we preferred to first separate into normal modes. The **amplitude of a normal mode** (travelling wave with a definite wave number) is our **new** position coordinate. There are N such modes and thus N of these coordinates. (So either way we have N dynamical variables and their conjugate momenta.) It has the advantage that it has the dynamics of a harmonic oscillator, which we can easily quantize. We prefer to work with creation and annihilation operators for this harmonic oscillator rather than its position and momentum. These are

$a_K, a_K^\dagger$. This is then quantized. When we work our way back to q_m, p_m we find that they also obey the standard comm relns.

Interpretation of q_m is easy. Interpretation of $a_K^\dagger$ - creates an excitation corresponding to a wave with definite wave number. Collective mode. The wave is the sound wave and these discrete excitations ae **phonons**. This is the usual particle - wave duality of QM. Analog of this for EM is a photon.

All that is left is to take the **continuum limit,** $a \to 0$

Exercise:1. Understand why $\omega_K \to 0$ as $K \to 0$ 2. What would happen if we add a term $\frac{1}{2}\mu \sum_m q_m^2$ to H of the crystal? What would this represent physically?

- **Step 4: Continuum Limit:**

  Let $a \to 0$ and $L \to \infty$.

  We defined $\phi(x,t) = \frac{1}{\sqrt{a}} q(x,t) = \frac{1}{\sqrt{a}} q_m(t)$ with $x = ma$. Define $\Pi(x,t) = \frac{1}{\sqrt{a}} p(x,t) = \frac{1}{\sqrt{a}} p_m(t) = \frac{1}{\sqrt{a}} \dot{q}_m(t) = \dot{\phi}(x,t) = \frac{\delta L}{\delta \dot{\phi}(x,t)}$

  Then

$$[\phi(x,t), \Pi(x',t)] = \frac{1}{a}[q_m(t), p_{m'}(t)] = \frac{1}{a}\hbar\delta_{m,m'}$$

  What is $\lim_{a\to 0} \frac{1}{a}\hbar\delta_{m,m'}$?

$$\begin{aligned} \lim_{a\to 0} \frac{1}{a}\hbar\delta_{m,m'} &= 0, \quad m \neq m' \\ &= \infty, \quad m = m' \\ &= \hbar\delta(x - x') \end{aligned}$$

  Check: Using $\int dx\ \delta(x - x') = a \sum_m \frac{\delta_{m,m'}}{a} = 1$ We see thus that

$$\boxed{[\phi(x,t), \Pi(x',t)] = \hbar\delta(x - x')}$$

  Now go to momentum space:

$$\phi(x,t) = \frac{1}{\sqrt{a}} q_m(t) = \frac{1}{\sqrt{Na}} \sum_K \frac{1}{\sqrt{2\omega_K}} [a_K e^{imK} + a_K^\dagger e^{-imK}]$$

Use $\frac{K=2\pi n}{N}$, $k = \frac{K}{a} = \frac{2\pi n}{L}$ and also $\Delta k = \frac{2\pi}{L} \Rightarrow \sum_K = \sum_k = L \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{dk}{2\pi}$

$$\phi(x,t) = \frac{1}{\sqrt{L}} L \frac{1}{\sqrt{2\omega_K}} \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{dk}{2\pi} [a_K e^{imK} + a_K^\dagger e^{-imK}]$$

Finally let $\boxed{a_K\sqrt{L} = a(k) \quad , a_K^\dagger \sqrt{L} = a^\dagger(k)}$

We get $[a(k), a^\dagger(k')] = L\delta_{K,K'}$. $\lim_{L\to\infty} L\delta_{K,K'} = c\delta(k-k')$. For some c. $\int dk\ c\delta(k-k') = \frac{2\pi}{L}\sum_k L\delta_{k,k'} = 2\pi$ So $\Rightarrow c = 2\pi$. Thus

we get $\boxed{[a(k), a^\dagger(k')] = 2\pi\delta(k-k')}$

$$\phi(x,t) = \frac{1}{\sqrt{2\omega_k}} \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{dk}{2\pi}\ (a(k,t)e^{ikx} + a^\dagger(k,t)e^{-ikx})$$

$$= \frac{1}{\sqrt{2\omega_k}} \int_{\frac{-\pi}{a}}^{\frac{\pi}{a}} \frac{dk}{2\pi}\ (a(k)e^{-i\omega t+ikx} + a^\dagger(k)e^{i\omega t-ikx}) \tag{12}$$

End of Step 4

3.4 Interpretation

- We have a quantum theory of a crystal. We also have the continuuum limit of this. This gives us a "Quantum Field Theory".
- The physical interpretation for the various objects was also given. $\phi(x,t)$ is a field whose value gives the displacement of the atom. $a(k)^\dagger$ is the creation operator for the harmonic oscillator labelled by the wave number k. These are normal modes of the original system. Because it is quantized we have discrete amounts of excitations in each mode. We call these particles.

- Thus let $|0\rangle_p$ be the ground state of the harmonic oscillator asssociated with wave number p. Then $a^\dagger(p)|0\rangle_p \equiv |1\rangle_p$ is a state with one "particle" with wave number p (or momentum $\hbar p$). $a^\dagger(p)a^\dagger(p)|0\rangle_p$ is a state with two particles. The HO is in it's second excited state.

- The state $|1\rangle_p$ is what in ordinary QM we call $|p\rangle$ - a momentum eigenstate for the given particle. It's wave function is proportional to $e^{ikx-i\omega_k t}$. What about $|2\rangle_p$? It describes two particles. In ordinary QM we would introduce two coordinates and two momenta and label the wave functions as $\psi(x_1, x_2)$ or the kets as $|p_1, p_2\rangle$ or $|x_1, x_2\rangle$ etc. In this notation the state $|2\rangle_p$ is $|p, p\rangle$. Note that both momenta are the same.

- If we want particles with different momenta, we have to consider two different normal modes and thus two different HO's. Thus we consider the state $|0, 0\rangle_{p_1,p_2} \equiv |0\rangle_{p_1} \otimes |0\rangle_{p_2}$ in the direct product Hilbert Space - which is the ground state of the combined system. Then $a^\dagger(p_1)a^\dagger(p_2)|0, 0\rangle_{p_1,p_2} \equiv a^\dagger(p_1)|0\rangle_{p_1} \otimes a^\dagger(p_2)|0\rangle_{p_2} = |1, 1\rangle_{p_1,p_2}$

- Note an important difference with ordinary QM. The state $a^\dagger(p)a^\dagger(p) = \sqrt{2}|2\rangle_p$ is a two particle state. In field theory it is in the Hilbert space of one HO. Whereas $|1, 1\rangle_{p_1,p_2}$ (also a two particle state) is in a direct product Hilbert space of two HO's. In ordinary QM **both** these states belong to the direct product Hilbert space (of two particles).

- Thus in field theory the state $|n\rangle_p$ is in the Hilbert space of one HO labelled by the wave number p, whereas in ordinary QM it would be in the direct product Hilbert space of n particles: $\underbrace{|p\rangle \otimes |p\rangle \otimes |p\rangle \otimes ...|p\rangle}_{n\ times}$.

  Direct product space enters in field theory when we consider different wavenumbers, because there is one HO for each wave number. Thus if all n particles have different momenta, then we would be in the direct product space: $|1\rangle_{p_1} \otimes |1\rangle_{p_2} \otimes .....|1\rangle_{p_n}$ would be the state $|p_1, p_2, ..., p_n\rangle$ in ordinary QM.

- Consider the discrete space where there are N possible wavenumbers and hence N oscillators. The ground state of this system is

$$|\mathbf{0}\rangle \equiv |0\rangle_{\frac{-N}{2}} \otimes |0\rangle_{\frac{-N-1}{2}} \otimes .....|0\rangle_{\frac{N-1}{2}} \otimes |0\rangle_{\frac{N}{2}}$$

This state is annihilated by all the operators $a_K, \forall K$ i.e. $a_K|\mathbf{0}\rangle = 0$.

Then

$$a_K^\dagger|\mathbf{0}\rangle = |0\rangle_{\frac{-N}{2}} \otimes |0\rangle_{\frac{-N-1}{2}} \otimes ...|1\rangle_K \otimes ...|0\rangle_{\frac{N-1}{2}} \otimes |0\rangle_{\frac{N}{2}}$$

we can use the label $|K\rangle$ for this state. It describes a phonon with wave number K.

All the excitations of the system (with arbitrarily large number of particles) belong to the direct product Hilbert space of these N HO's. This is called the **Fock Space**.

- Now we can go back and see what $\phi(x,t)$ is precisely. It satisfies (using the mode expansion for ϕ and the properties of a's:

$$\langle\mathbf{0}|\phi(x,t)|\mathbf{p}\rangle = \langle\mathbf{0}|\phi(x,t)a^\dagger(p)|\mathbf{0}\rangle = \frac{1}{\sqrt{2\omega_p}}e^{-i\omega t+ikx}$$

Thus we can say that $\phi(x,t)$ is an operator that annihilates a particle at time t and position x. Thus $\phi^\dagger(x,t)$ creates a particle at x,t. $\phi(x,t)^\dagger|\mathbf{0}\rangle$ is a state with a particle at x,t. It undergoes the usual time evolution given by the Schroedinger equation. In the Heisenberg formalism however the time dependence is in the operator. Thus the Schroedinger state of a particle at x is denoted by $|\mathbf{x},\mathbf{t}\rangle$. In the Heisenberg picture we just take the state at time $t = 0$ and stay with that.

Note that if ϕ is a real field then $\phi = \phi^\dagger$ so ϕ can both create and annihilate particles at a point.

- In the example of the crystal the field had a geometric interpretaion as a displacement of an atom. And the waves were thus sound waves and the particles are phonons. In particle physics for every particle that is observed in nature, we introduce a field. It usually does not have such a geometric interpretation. The only thing we care about is that the field operators create and destroy a particle.

4 Relativistic Classical Field Theory

4.1 Action and EOM

- **Action**: $S = \int dt\ L = \int d^4x\ \mathcal{L}$ (L is Lagrangian, $\mathcal{L}$ is Lagrangian

density)

- $$\delta S = \int d^4x \ \{\frac{\partial \mathcal{L}}{\partial \phi(x,t)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\} = 0$$

$$= \int d^4x \ \{\frac{\partial \mathcal{L}}{\partial \phi(x,t)}\delta\phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + \underbrace{\partial_\mu[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi]}_{Surface\ term}\}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi(x,t)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

if $\delta\phi|_B = 0$.

These are the Lagrangian EOM. In "scalar field theory" $L = -\frac{1}{2}\partial_\mu\phi\partial_\mu\phi$ then EOM is $\partial_\mu\partial_\mu\phi = 0$ or $\partial_t^2\phi - c^2\,\partial_i^2\phi = 0$. Relativistic wave equation.

- Hamiltonian:

$$H = \sum_m p_m\dot{q}_m - L(q_m, \dot{q}_m) = \sum_m \frac{1}{2}p_m^2(t) + \frac{1}{2}\nu^2(q_{m+1}(t) - q_m(t))^2$$

- Continuum limit:

$$H = [\frac{1}{2}\int \frac{dx}{a}[p(x,t)^2 + \underbrace{\nu^2 a^2}_{c^2}(\frac{\partial q(x,t)}{\partial x})^2]$$

Using $p = \sqrt{a}\Pi,\ q = \sqrt{a}\phi$ we get

$$= \frac{1}{2}\int dx[\Pi^2(x,t) + c^2(\frac{\partial \phi(x,t)}{\partial x})^2$$

In 3-space dim $H = \int d^3x\ [\Pi^2(x,t) + |\nabla\phi(x,t)|^2]$.

$$H = \int d^3x\ [\Pi(x,t)\dot{\phi}(x,t) - \mathcal{L}(\phi, \partial_\mu\phi)]$$

where $\Pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,t)}$

- Heisenberg formalism operators are time dependent: $O(t) = e^{iHt}O(0)e^{-iHt}$. Thus formally the Hamiltonian is too but $H(t) = H$ in the time independent case.

- If $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2$ then H will have an additional term : $+\frac{1}{2}m^2\phi^2$. m can be shown to correspond to the mass of the particle. Study dispersion relation: find that $E^2 = \vec{p}.\vec{p} + m^2c^4$.

- Pts to emphasize: 1. Functional derivatives versus ordinary derivatives

$$\sum_m \frac{\partial q_m}{\partial q_{m'}} = \sum_m \delta_{mm'} = 1 \tag{13}$$

$$\int dx\ \frac{1}{a}\frac{\partial q_m}{\partial q_n} = \underbrace{\int dx\ \frac{1}{a}\frac{\partial q(x)}{\partial q(x')} = \int dx\ \frac{\delta q(x)}{\delta q(x')}}_{change\ of\ notation} = \int dx\ \delta(x-x') = 1 \tag{14}$$

Thus

$$\frac{\delta}{\delta q(x')}\int dx\ q(x) = 1 \tag{15}$$

This can be obviously generalized:

$$\frac{\delta}{\delta q(x')}\int dx\ L(q(x)) = \frac{\partial L(x')}{\partial q(x')} \tag{16}$$

2. H in Schr. form has no t. 3. Surface terms. 4. metric convention. 5. $\hbar = c = 1$ convention.

4.2 Noether's Theorem and Symmetries

Sometimes the action is invariant under global symmetries. In that case the equation of motion is left unchanged. Sometimes the Lagrangian is invariant, sometimes the Lagrangian density itself is invariant.

- Infinitesimal symmetry transf: $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$. e.g.:

  1. $\phi \to \phi + w$. So $\delta\phi = w$.
  2. If ϕ is complex, $\phi \to e^{i\omega}\phi$ $\phi^* \to e^{-i\omega}\phi^*$. $\delta\phi = i\omega\phi$
  3. $x^\mu \to x^\mu - a^\mu = x'^\mu$ and $\phi(x) \to \phi(x+a) = \phi + a^\mu\partial_\mu\phi$. Then $\delta_a\phi = a^\mu\partial_\mu\phi$. Note: We take ϕ to be a **scalar** so that $\phi(x) = \phi'(x') = \phi'(x-a)$ or $\phi(x+a) = \phi'(x)$. In this case the Lagrangian density is not invariant $\mathcal{L} \to \mathcal{L} + a^\mu\partial_\mu\mathcal{L}$ but the action is.

- Noether's theorem: for each such symmetry there is a conserved current and a conserved charge. To find the current a simple method is to consider a **local** variation :$\delta_v\phi$ where $v(x)$ is some parameter. Thus

$$\delta_v\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta_v\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta_v\phi)$$

Note that $\delta_v\partial_\mu\phi \equiv \partial_\mu\delta_v\phi$. Integrating by parts

$$= \underbrace{[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}]}_{=\ 0\ by\ EOM}\delta_v\phi + \partial_\mu[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_v\phi]$$

$$= \ \partial_\mu[vj^\mu] = \ v\partial_\mu j^\mu + \partial_\mu v j^\mu$$

The action is invariant for constant v, then the total change in the Lagrangian density must be of the form $\partial_\mu[vN^\mu]$. Thus we find for constant v

$$v\partial_\mu j^\mu = v\partial_\mu N^\mu$$

So

$$\partial_\mu J^\mu = \ 0]$$

where $J^\mu = j^\mu - N^\mu$.

This also defines j^μ as the coefficient of $\partial_\mu v$ when we evaluate $\partial_\mu[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_v\phi]$. Note that the conservation of J^μ is only after using EOM. Also note that if the Lagrangian density $\mathcal{L}$ is invariant then N^μ is zero and in that case j^μ is the conserved Noether current. If we define $Q = \int d^3x j^0$ then $\frac{dQ}{dt} = \ 0$ is easy to see. This is the conserved Noether charge. In the examples above:

1. $j_\mu = \partial_\mu\phi$
2. $j_\mu = i[\phi\partial_\mu\phi^\dagger - \phi^\dagger\partial_\mu\phi]$
   N is zero for the usual scalar Lagrangians.
3. If scale invariant then $\delta\mathcal{L} = \partial_\mu(\epsilon x^\mu\mathcal{L}) = \partial_\mu(a^\mu\mathcal{L})$ with $a^\mu = \epsilon x^\mu$.
   Dilatation Current in more detail: We are considering: $x'^\mu = \frac{1}{\lambda}x^\mu$ We let $\lambda = 1+\epsilon$ with $\epsilon << 1$. Thus $\delta_D x^\mu = -\epsilon x^\mu$.
   (Note: When $\lambda > 1$, this means the new coordinates are smaller. This means the unit of length is bigger. System has shrunk. Momenta are stretched.)

Let $\delta_D\phi(0) = d\phi(0)$ - the field has a scaling dimension d. (If $d > 0$ it is like momentum.) Define $\mathcal{R}X = \frac{1}{\lambda}X$ the finite abstract transformation. $\phi'(X') = \phi'(\mathcal{R}X) = R\phi(X) = \lambda^d\phi(x)$. Thus ϕ is not a scalar under this transf.

$$\phi'(x) = R\phi(\mathcal{R}^{-1}x)$$
$$\phi'(x) - \phi(x) = \delta_D\phi = \lambda^d\phi(\lambda x) - \phi(x)$$

$$\delta_D\phi(x) = \epsilon(d + x.\frac{\partial}{\partial x})\phi(x) \tag{17}$$

Now consider a Lagrangian density $\mathcal{L}$. If $\int d^4x\ \mathcal{L}$ is to be invariant:

4. In this case $\delta\mathcal{L} = a^\mu\partial_\mu\mathcal{L} = a^\nu\partial_\mu[g^\mu_\nu\mathcal{L}]$. Thus The curren t $N^\mu_\nu \equiv g^\mu_\nu\mathcal{L}$ corresponds to the transformation a^ν. $j^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi$ and thus
$$J^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - g^\mu_\nu\mathcal{L}$$

Note that the role of the index ν is to label the transformation (a^ν) and μ is the usual vector index associated with the current. For internal symmetries ν will be replaced by some internal index. The current defined above is in fact the **energy momentum tensor** T^μ_ν.

Note that the charge associated with a^0 (time translation) is $\int d^3x\ T^0_0 = \int d^3x\ [\Pi\partial_t\phi - \mathcal{L}] = \int d^3x\ \mathcal{H}$ - Hamiltonian - as one would expect.

4.3 Green's Functions

- $$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi$$

We have added a source - external force for a harmonic oscillator.

EOM
$$-\partial_\mu\partial^\mu\phi + m^2\phi = J$$

Green's fn $G(x, x')$

$$[-\partial_\mu\partial^\mu + m^2]G(x, x') = \delta^4(x - x')$$

- Soln: Then
$$\phi(x) = \phi_0(x) + \int d^4x' \; G(x - x')J(x')$$
where $\phi_0(x)$ is a soln of the hom eqn and satisfies the bc and then G goes to zero at the boundary. Eg. If bc is the value of $\phi(x, 0)$ then set $\phi_0(x, 0) = \phi(x, 0)$. Then $G(x, x')$ should vanish at $t = 0$.

- Momentum space:
$$G(x, x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} G(p)$$
$$\delta^4(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}$$
$$\Rightarrow [p_\mu p^\mu + m^2]G(p) = [-p_0^2 + p_i^2 + m^2]G(p) = 1$$
$$G(p) = \frac{1}{p^2 + m^2}$$
So
$$G(x - x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2 + m^2} = -\int_c \frac{dp_0}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip_0(x-x')^0 + ip_i(x-x')^i}}{(p_0 - E_p)(p_0 + E_p)}$$
Here $E_p = \sqrt{(p^i)^2 + m^2}$.
The contour c needs to be specified and will decide bc obeyed by G.

1. C_1 Contour goes beneath both poles. If $x^0 > x'^0$ then convergence requires closing below so no poles are included. So $G(x - x') = 0$ if $x^0 > x'^0$. Propagates backwards in time. Hence $G_{advanced}$.
$$G_{adv}(x-x') = i\theta(x'^0 - x^0) \int \frac{d^3p}{(2\pi)^3} \frac{(e^{iE_p(x-x')^0} - e^{-iE_p(x-x')^0}) e^{ip^i(x-x')^i}}{2E_p}$$

2. C_2 Contour goes over both poles. Thus G is non zero only when $x^0 > x'^0$. Thus we get $G_{retarded}$.
$$G_{ret}(x-x') = i\theta(x^0 - x'^0) \int \frac{d^3p}{(2\pi)^3} \frac{(e^{-iE_p(x-x')^0} - e^{iE_p(x-x')^0}) e^{ip^i(x-x')^i}}{2E_p}$$

Note that G_{ret} and G_{adv} are both real. G_r is what is used in class mech. It is causal. G_a answers the question of what should the fields be now in order to reach such and such state in the future. This is an unusual question in CM. We usually ask what is the state in the future given it is such and such now.

3. C_3 Contour goes below $-E_p$ and above E_p. This gives: $x^0 > x'^0$

$$i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-iE_p \underbrace{(x-x')^0}_{+ve} + ip^i(x-x')^i}$$

and when $x'^0 > x^0$ we get

$$i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{iE_p \underbrace{(x-x')^0}_{-ve} + ip^i(x-x')^i}$$

Thus positive energy modes go forward in time and negative energy modes go backward in time. It can be written as:

$$G_{Feynman} = \; i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-iE_p|(x-x')^0| + ip^i(x-x')^i}$$

This contour prescription is equivalent to $m^2 \to m^2 - i\epsilon$.
This is the Green's fn that one uses in QM.

4.4 Two point functions

- These Green functions of the classical theory can be related to "two point correlators" of QFT. (Expand on the idea of correlation functions - also Stat Mech connection.)

  Thus consider

$$\langle 0|\phi(x,t)\phi(x',t')|0\rangle$$

.

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_p e^{-iE_p t + ip.x} + a_p^\dagger e^{+iE_p t - ip.x}]$$

$$\phi(x',t') = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} [a_q e^{-iE_q t' + iq.x'} + a_q^\dagger e^{+iE_q t' - iq.x'}]$$

We get

$$\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\langle 0|\frac{1}{\sqrt{2E_p}}a_p e^{-iE_pt+ip.x}\frac{1}{\sqrt{2E_q}}a_q^\dagger e^{+iE_qt'-iq.x'}|0\rangle$$

$$\langle 0|\phi(x,t)\phi(x',t')|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-iE_p(t-t')+ip.(x-x')} \equiv D(x-x')$$

Note that in this notation

$$G_{ret}(x-x') = i\theta(t-t')[D(x-x')-D(x'-x)] = i\theta(t-t')\langle 0|[\phi(x,t),\phi(x',t')]|0\rangle$$

- Similarly

$$G_{Feynman}(x-x') = i[\theta(t-t')D(x-x')+\theta(t'-t)D(x'-x)]$$

$$= i\langle 0|T(\phi(x,t)\phi(x',t'))|0\rangle$$

where "T" stands for time ordering, i.e. the operator at the earlier time is to the right.

- Finally consider $\langle 0|[\phi(x,t),\phi(x',t')]|0\rangle$ when x, x' are spacelaike separated. In that case we can choose a frame where $t = t'$ - D is Lorentz invariant. Therefore $D(x-x') = D(x'-x)$ because one can perform a rotation, and D is only a fn of the distance. Therefore the commutatot vanishes for spacelike separation. (If not spacelike one cannot relate $x-x'$ and $x'-x$ by a rotation/boost). This means $\phi(x)$ and $\phi(x')$ as operators can be simultaneously diagonalized. Measuring one doesn't affect the other. This is required by **causality**. One can think of the influence going from x to x' - by particle propagation and anti particle (=particle going from x' to x), which cancel each other. Fields at spacelike separation are **uncorrelated**.

- $D(r)$ for spacelike r is $\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip.(x-x')}$

  This can be evaluated:

$$\int \frac{p^2dpd(cos\ \theta)}{(2\pi)^3}\frac{1}{2E_p}e^{iprcos\ \theta} = \int \frac{p^2dp}{(2\pi)^2}\frac{1}{2E_p}\frac{e^{ipr}-e^{-ipr}}{ipr}$$

$$= -\frac{i}{4\pi^2}\int_{-\infty}^{\infty} dp\ \frac{p}{r\sqrt{p^2+m^2}}e^{ipr} \approx \frac{1}{2\pi^2 r}\int_m^{\infty} dp \frac{p}{\sqrt{p^2-m^2}}e^{-pr}$$

Using

$$\int_u^{\infty} \frac{xe^{-\mu x}}{\sqrt{x^2-u^2}}dx \ = uK_1(u\mu) \approx u\frac{1}{\sqrt{u\mu}}e^{-u\mu} \quad u \to \infty$$

we get

$$\approx \frac{1}{r}\sqrt{\frac{m}{r}}e^{-mr}$$

for large r.

4.5 Interpretation of Green's Functions - Harmonic Oscillator

Consider the following equations for a HO

$$\frac{d^2y}{dt^2} + \omega^2 y = j(t) \quad y(0) = y(T) = 0 \tag{18}$$

This is the kind of bc that path integral requires. Consider different kinds off Green's fns:

- $G_{Ret}(t,t') : G_{Ret}(t,t') = 0 \quad \forall t < t'$
- $G_{Adv}(t,t') : G_{Adv}(t,t') = 0 \quad \forall t > t'$
- $G_{Fey}(t,t') : G_{Fey}(0,t') = G(T,t') = 0 \quad \forall t,t'$

Let us construct these:

$$G_{ret}(t,t') = Asin\ \omega t + Bcos\ \omega t \quad t > t'$$
$$G_{Ret}(t,t') = 0 \quad t < t' \tag{19}$$

Continuity requires that $G(t',t') = 0$. Thus $B = 0$. A is fixed by $\dot{G}_{Ret}|_{t'-\epsilon}^{t'+\epsilon} = 1$. So $A\omega = 1$. $A = \frac{1}{\omega}$. Thus

$$G_{Ret} = \frac{1}{\omega} sin\omega(t-t')\theta(t-t') \tag{20}$$

.

G_{Adv} can be obtained similarly. The continuity condn is $G_{Adv}(t',t') = 0$ and $\dot{G}_{Adv}(t,t')|_{t'-\epsilon}^{t'+\epsilon} = 1$. This gives

$$G_{Adv} = -\frac{1}{\omega} sin\ \omega(t-t')\theta(t'-t)$$

Let us work out G_{Fey}:

$$\begin{aligned} G_{Fey} &= A_1 sin\ \omega t + B_1 cos\ \omega t \qquad t > t' \\ &= A_2 sin\ \omega t + B_2 cos\ \omega t \quad t < t' \end{aligned} \tag{21}$$

$A_1 sin\ \omega T + B_1 cos\ \omega T = 0$ and also $B_2 = 0$ are the bc's. Continuity at $t = t'$ gives:

$$A_1 sin\ \omega t' + B_1 cos\ \omega t' = A_2 sin\ \omega t'$$

and the remaining condition on derivatives is

$$\omega A_1 cos\ \omega t' - \omega B_1 sin\ \omega t' - \omega A_2 cos\ \omega t' = 1$$

We have three equations for three unknowns. Solve. Get

$$\begin{aligned} G_F &= \frac{1}{sin\ \omega T} sin\ \omega(t-T) sin\ \omega t' \quad t > t' \\ &= \frac{1}{sin\ \omega T} sin\ \omega(t'-T) sin\ \omega t \quad t < t' \end{aligned} \tag{22}$$

This result will be useful later when we do path integrals.

These Green's function can be seen to be 0+1 dimensional versions of the field theory Green's functions constructed earlier.

4.6 Relativistic Normalization

- $\delta^4(p-q)$ is Lorentz scalar. $\delta^3(p-q)$ is not but $2E_p\delta^3(p-q)$ is where $E_p = +\sqrt{\vec{p}.\vec{p} + m^2}$.

  $p'_z = \gamma(p_z + \beta E)$, $E' = \gamma(E + \beta p_z)$.

$$dp_z \delta(p_z - q_z) = dp'_z \delta(p'_z - q'_z)$$

$$\delta(p_z - q_z) = \frac{dp'_z}{dp_z} \delta(p'_z - q'_z)$$

  Use $\frac{dp'_z}{dp_z} = \gamma(1 + \beta\frac{dE}{dp_z}) = \gamma(1 + \beta\frac{p_z}{E}) = \frac{E'}{E}$

- $\langle q|p\rangle = (2\pi)^3\delta^3(p-q)$ in N.Rel QM. Define ${}_{Rel}\langle q|p\rangle_{Rel} = (2\pi)^3 2E_p\delta^3(p-q)$

  Thus $|p\rangle_{Rel} = \sqrt{2E_p}|p\rangle_{NonRel} = \sqrt{2E_p}a_p^\dagger|0\rangle$

- $\int d^4p$ is rel inv but not $\int d^3p$. $\int \frac{d^3p}{2E_p}$ is.

  Thus
  $$1 = \int \frac{d^3p}{(2\pi)^3}|p\rangle\langle p| = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|p\rangle_{rel}\ \ {}_{rel}\langle p|$$

  Finally

- $\int \frac{d^3p}{(2\pi)^3 2E_p} = \int \frac{d^4p}{(2\pi)^4}2\pi\delta(p^2-m^2)|_{p^0>0}$ (Using $2x\delta(x^2-y^2) = \delta(x-y)$)

4.7 Charged Scalar Field

- Here $\phi = \phi_1 + i\phi_2$ is complex, so one can just treat it as two real fields and be done. But typically the Lagrangian is invariant under $\phi \to e^{i\Lambda}\phi$ and there is a conserved charge and current. Thus
  $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial_\mu\phi - \frac{1}{2}m^2\phi^*\phi$$
  .

- The Noether current: $J_\mu = i[\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi]$. Noether charge
  $$Q = \int d^3x\ i[\phi\partial_t\phi^* - \phi^*\partial_t\phi]$$
  $$= \int d^3x\ [\Pi\phi - \Pi^*\phi^*]$$

- Mode expand:
  $$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a(p)e^{-i\omega_p t + ip.x} + b^\dagger(p)e^{i\omega_p t - ip.x}]$$

  Note that coeff of +ve freq part is the annihilation operator. Since field is complex there is no reason for $b = a$. Thus ϕ destroys a type particles and creates b type particles.

- One can check that

$$Q = \int \frac{d^3p}{(2\pi)^3}[a^\dagger(p)a(p) - b^\dagger(p)b(p)] = N_a - N_b$$

  Thus the two particles are of opposite charge, and same mass. Since they are part of the same field we can call them particles and antiparticles.

- $$[\phi(x), \phi^\dagger(y)] = D(x-y) - D(y-x) = 0$$

  for spacelike separataion. One of the D's is from a and the other is from b. One can think of it as a cancellation between particles going from y to x and anti particles from x to y. The mass thus has to be the same. In local cft particle is always accompanied by same mass antiparticle. If the fld is real, then particle = antiparticle.

5 Functional Formalism

This is a generalization of Feynman's **Path Integral** formulation of quantum mechanics. Much more intuitive and conceptually simpler. Also more flexible. We will use the functional formalism for field theories. Begin with some mathematical preliminaries and then path integral formulation of QM.

5.1 Mathematical Digression

Gaussian Integrals

1.
$$\int_{-\infty}^{\infty} dx\ e^{-\frac{(x-x_0)^2}{2\sigma^2}} = \sqrt{2\pi\sigma^2}$$
$$\int_{-\infty}^{\infty} dx\ (x-x_0)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}} = \sqrt{2\pi\sigma^2}\sigma^2$$

   σ^2 is the standard deviation and x_0 is the mean.

   From now on range of integration is assumed to $(-\infty, \infty)$ unless otherwise indicated.

2.

$$\int dx e^{-\frac{1}{2}ax^2+jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}j\frac{1}{a}j}$$

Trick: extremize the exponent wrt x: Get $-ax + j = 0$. $x = \frac{j}{a}$. Plug this back into the exponent and get the answer. "Semiclassical" approximation is the same as this.

3.

$$\int dx_1 \int dx_2 \; e^{-\frac{1}{2}(a_1x_1^2+a_2x_2^2)+j_1x_1+j_2x_2} = \sqrt{\frac{2\pi}{a_1}}\sqrt{\frac{2\pi}{a_2}} e^{\frac{1}{2}(j_1\frac{1}{a_1}j_1+j_2\frac{1}{a_2}j_2)}$$

4. If A is a diagonal $N \times N$ matrix:

$$A = \begin{pmatrix} a_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & a_N \end{pmatrix} \tag{23}$$

$$\int dx_1 \; .... \int dx_N e^{-\frac{1}{2}x^TAx+j^Tx} = \frac{(\sqrt{2\pi})^N}{\sqrt{a_1a_2...a_N}} e^{\frac{1}{2}j^T(A^{-1})j}$$

5. If A is a general real symmetric $N \times N$ matrix. Then let $A = O^TA_DO$ be a diagonalization by an orthogonal matrix. Then $x^TAx = y^TD_Dy$ where $y = Ox$. The Jacobian of the tranformation of the integration measure $[dy_i] = ||\frac{\partial y_i}{\partial x_j}|||[dx_j]$ is just $Det\ O = 1$. Thus we can use the previous formula and write:

$$\int dx_1 \; .... \int dx_N e^{-\frac{1}{2}x^TAx+j^Tx} = \frac{(\sqrt{2\pi})^N}{Det^{\frac{1}{2}}A} e^{\frac{1}{2}j^T(A^{-1})j}$$

6. **Continuum Integrals**

$$I = \int \mathcal{D}x(t) e^{-\frac{1}{2}\int dt \;\; \int dt' \;\; x(t)A(t,t')x(t') + \int dt \;\; j(t)x(t)}$$

To make sense of this - discretize: $t = j\epsilon$ and $t' = k\epsilon$. $\int dt \;\; \rightarrow \epsilon\Sigma_j$ $x(t) = x(j) = x_j$ and $A(t,t') = A(j,k) = A_{j,k}$. $x(t)$ becomes a column

vector and $A(t,t')$ becomes a matrix - very large $N \times N$ matrix. Take $N \to \infty$ in the end as $\epsilon \to 0$.

$$I = \int \Pi_j dx_j \; e^{-\frac{1}{2}\epsilon^2 \Sigma_{j,k} x_j A_{j,k} x_k + \epsilon \Sigma_k J_k x_k}$$

$$= e^{\frac{1}{2}\Sigma_{j,k} J_k (A^{-1})_{j,k} J_k} \sqrt{Det(\frac{2\pi}{A})}$$

where $\Sigma_k A^{-1}_{j,k} A_{k,l} = \delta_{j,l}$.

$$= e^{\frac{1}{2}\frac{1}{\epsilon^2} \int dt \; \int dt' \; J(t)(A^{-1})_{j=\frac{t}{\epsilon}, k=\frac{t'}{\epsilon}} J(t')}$$

Define a $A^{-1}(t,t') \equiv \frac{1}{\epsilon^2} A^{-1}_{j,k}$ where $t = j\epsilon$ and $t' = k\epsilon$.

$$I = e^{\frac{1}{2}\int dt \; \int dt' \; J(t) A^{-1}(t,t') J(t')} \sqrt{Det(\frac{2\pi}{A})}$$

Note also that

$$\Sigma_k A^{-1}_{j,k} A_{k,l} = \delta_{j,l}$$

$$\Rightarrow \epsilon^2 \int \frac{dt'}{\epsilon} A^{-1}(t,t') A(t',t") = \delta_{j,l}$$

where: $j\epsilon = t, k\epsilon = t', l\epsilon = t"$. Thus

$$\int dt' A^{-1}(t,t') A(t',t") = \frac{\delta_{j,l}}{\epsilon} \equiv \delta(t - t")$$

7. Example: $A(t,t') = A(t)\delta(t-t')$

   So $A_{j,k} = \frac{A_j \delta_{j,k}}{\epsilon}$. $A^{-1}_{j,k} = \frac{\epsilon}{A_j}\delta_{j,k}$. $A^{-1}(t,t') = \frac{1}{A_j}\delta_{j,k}\epsilon = \frac{\delta(t-t')}{A(t)}$. Verify that $\int A(t,t')A^{-1}(t',t")dt \; = \delta(t-t")$ Thus

$$\int \mathcal{D} e^{-\frac{1}{2}\int dt \; x^2(t) A(t) + \int dt \; J(t) x(t)} = e^{\frac{1}{2}\int dt \; \frac{J(t)^2}{A(t)}} \sqrt{Det(\frac{2\pi}{A})}$$

5.2 Path Integrals

1. Instead of starting with a wave function one defines directly a probability amplitude for *a particle to go from a point $X_i$at time t_i to a point X_f at time t_f*. Call it $K(X_f, t_f; X_i, t_i)$. Feynman defined the following formula for it: Motivation: double slit experiment.

$$K(X_f, t_f; X_i, t_i) = \int_{x(t_i)=X_i}^{x(t_f)=X_f} \underbrace{\mathcal{D}x(t)}_{sum\ over\ paths} exp(+\frac{i}{\hbar}\int_{t_i}^{t_f} dtL(x(t), \dot{x(t)}) \tag{24}$$

   Note that this is not the probability amplitude of a measurement, it is the probability amplitude of an *event*.

2. Draw pictures and show classical limit. Principle of stationary phase. Derive Lagrange's eqn.

3. How do you actually calculate: What does $\mathcal{D}x(t)$ mean? Divide $t_f - t_i$ into N intervals $\epsilon = t_{j+1} - t_j$ with $t_0 = t_i$and $t_f = t_N$. Let $x_j = x(t_j)$. Then $\mathcal{D}x(t) \approx dx_1 dx_2 ....dx_j ...dx_{N-1}$ There will in general a constant of proportionality (possibly infinite). Thus

$$K(f,i) = K(X_f, t_f; X_i, t_i) = \mathcal{N}\int_{x_0=X_i}^{x_N=X_f} [dx_1 dx_2 ...dx_{N-1}] e^{\frac{i}{\hbar}S(f,i)}$$

   Where S is the action and $\mathcal{N}$ is a normalization constant.

4. The composition law $K(a,b) = \int dx_c K(b,c)K(c,a)$: Draw figure. K is called Kernel. This can be iterated.

5. Get

$$K(X_f, t_f; X_i, t_i) = \int dx_1 \int dx_2 ... \int dx_{N-1} K(f, N-1)K(N-1, N-2)...K(j+1, j)...K(1, i) \tag{25}$$

6. Do the integral $\int dX_i K(j+1, j)K(j, j-1)$

$$e^{\frac{i}{\hbar}(\frac{m\epsilon}{2}[\frac{x_{j+1}-x_j}{\epsilon}]^2 + \frac{m\epsilon}{2}[\frac{x_j - x_{j-1}}{\epsilon}]^2)}$$

=

$$e^{\frac{im}{\hbar\epsilon}[(x_j-\frac{x_{j+1}+x_{j-1}}{2})^2+(\frac{x_{j+1}-x_{j-1}}{2})^2]}$$

=

$$\sqrt{\frac{i\hbar\epsilon 2\pi}{2m}}e^{\frac{i2\epsilon m}{\hbar 2}(\frac{x_{j+1}-x_{j-1}}{2\epsilon})^2}$$

This is clearly proportional to $K(j+1,j-1)$. The factor in square root is the normalization factor.

Suppose the initial wave function corresponds to a particle with zero momentum. So $\psi(X_i,t_i)=\frac{1}{\sqrt{V}}$ where V is the volume of space. After evolution the wave function is $\psi(X_f,t_f)=\int dX_i\ K(X_f,t_f;X_i,t_i)\frac{1}{\sqrt{V}}$ (see point 5 below). We know on physical grounds that $\psi(X_f,t_f)=\frac{1}{\sqrt{V}}$. Thus $\int dX_i\ K(X_f,t_f;X_i,t_i)=1$. Thus if we use the Gaussian normalization factor for each of the unit K's , i.e. $\sqrt{\frac{m}{2\pi\epsilon\hbar i}}$, we get the final result

$$\sqrt{\frac{m}{2\pi 2\epsilon\hbar i}}e^{\frac{i2\epsilon m}{\hbar 2}(\frac{x_{j+1}-x_{j-1}}{2\epsilon})^2}$$

which has the correct normalization.

Clearly this process can be iterated to replace 2ϵ by $N\epsilon=t_f-t_i$. Thus

$$K(X_f,t_f;X_i,t_i)=\sqrt{\frac{m}{2\pi(t_f-t_i)\hbar i}}e^{\frac{i(t_f-t_i)m}{\hbar 2}(\frac{(x_f-x_i)}{(t_f-t_i)})^2} \tag{26}$$

7. Relation to wave functions - evolution operator.

$$\psi(X_f,t_f)=e^{-i\int_{t_i}^{t_f}Hdt}\psi(X_i,t_i)=\int K(X_f,t_f;X_i,t_i)\psi(X_i,t_i)dX_i \tag{27}$$

8. Expansion of $K(X_f,t_f;X_i,t_i)$ in terms of wave functions

$$K(X_f,t_f;X_i,t_i)=\sum_n \psi_n(X_f)\psi_n^*(X_i)e^{-i\frac{E_n(t_f-t_i)}{\hbar}}$$

9. Derivation of Schroedinger's eqn.

Consider infinitesimal evolution from t to $t+\epsilon$. The evolution operator is

$$K(X_f,t_f;X_i,t_i)=\int_{x(t_i)=X_i}^{x(t_f)=X_f}\underbrace{\mathcal{D}x(t)}_{sum\ over\ paths}\ exp(+\tfrac{i}{\hbar}\int_{t_i}^{t_f}dtL(x(t),\dot{x(t)})$$

We set $t_f=t_i+\epsilon$ to get

$$\psi(X_f,t_i+\epsilon)=\int_{x(t_i)=X_i}^{x(t_i+\epsilon)=X_f}\underbrace{\mathcal{D}x(t)}_{sum\ over\ paths}\ exp(\frac{i}{\hbar}\int_{t_i}^{t_i+\epsilon}dtL(x(t),\dot{x(t)})\psi(X_i,t_i)dX_i$$

For infinitesimal evolution

$$\psi(X_f,t_i+\epsilon)=\mathcal{N}\int e^{\frac{i}{\hbar}\frac{m}{2}\epsilon[\frac{X_f-X_i}{\epsilon}]^2}\psi(X_i,t_i)dX_i$$

$\mathcal{N}$ is chosen so that the gaussian integral gives 1. bLHS is $\psi(X_f,t_i)+\epsilon\frac{\partial\psi}{\partial t_i}$. Letting $X_f-X_i=y$ and $\psi(X_f,t_i)=\psi(X_i,t_i)+y\frac{\partial\psi}{\partial y}+\frac{y^2}{2}\frac{\partial^2\psi}{\partial y^2}$ (we get (linear term vanishes by symmetry)

$$i\hbar\frac{\partial\psi}{\partial t_i}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2}$$

(After multiplying by $\hbar$ on both sides.) This is SE. QED.

Note that $K(X_f,t_f;X_i,t_i)$ satisfies SE. Also the bc $\lim_{t_f\to t_i}K(X_f,t_f;X_i,t_i)=\delta(X_f-X_i)$.

10. getting semi classical energy, momentum. Using $\sqrt{\frac{m}{2\pi(t_f-t_i)\hbar i}}e^{\frac{i(t_f-t_i)m}{\hbar 2}(\frac{(X_f-X_i)}{(t_f-t_i)})^2}$ we can understand semi classical limit : Change in phase wrt change in X_f gives momentum and change wrt t_f gives energy. Use $K(X_f,t_f;X_i,t_i)$ and study variation wrt X_f. Prove that $\frac{\partial}{\partial x}S_{cl}=p$.

a)

$$S+\delta S=\int_{t_a}^{t_b}L(x+\delta x,\dot{x}+\delta\dot{x})dt$$

$$\delta S=\int_{t_a}^{t_b}\frac{d}{dt}[\delta x\frac{\partial L}{\partial\dot{x}}]dt+\int dt[eqn\ of\ motion]$$

$$\delta S=\delta x\frac{\partial L}{\partial\dot{x}}\ |_{t_b}^{t_b}$$

$$\frac{\partial S}{\partial x_b} = \frac{\partial L}{\partial \dot{x}} \mid_{t_b}^{t_b} = P_b$$

b)Same thing for energy:

$$S + \delta S = \int_{t_a}^{t_b+\delta t_b} dt L(t, x'_{cl}, \dot{x}'_{cl})$$

x' is the modified classical solution. $x'_{cl}(t_b + \delta t_b) = x_{cl}(t_b) = x_b$.

$$S + \delta S = \int_{t_a}^{t_b+\delta t_b} L(x'_{cl}, \dot{x'}_{cl}) dt$$

$$= \int_{t_a}^{t_b} L(x'_{cl}, \dot{x'}_{cl}) + \delta t_b L(x'_{cl}, \dot{x'}_{cl})$$

$$\delta S = \delta t_b L(x'_{cl}, \dot{x'}_{cl}) + \int_{t_a}^{t_b} [L(x'_{cl}, \dot{x'}_{cl}) - L(x_{cl}, \dot{x}_{cl})] dt$$

The term in square brackets is after integrating by parts and using equations of motion $\delta x_{cl} \frac{\delta L}{\delta \dot{x}}$.

Using bc we get $x'_{cl}(t_b) + \dot{x'}_{cl} \delta t_b = x_{cl}$. So $x'_{cl} - x_{cl} = -\dot{x'}_{cl} \delta t_b$. All this gives:

$$\delta S = L \delta t_b + \int^{t_b} dt [L(t, x'_{cl}, \dot{x}'_{cl}) - L(t, x_{cl}, \dot{x}_{cl})$$

$$= L \delta t_b + \frac{\partial L}{\partial \dot{x}} (x'_{cl} - x_{cl}) = L \delta t_b - p \dot{x}_{cl} \delta t_b = -E \delta t_b.$$

c) Understand normalization: $\frac{m}{2\pi\hbar T} dx = P(b) dx$.

$$\frac{mb}{T} < p < \frac{m(b + dx)}{T}$$

Range of momentum $dp = \frac{m dx}{T}$. Thus the probability is of the form $P(p)dp = const\ dp$ where const is $\frac{1}{2\pi\hbar}$.

11. Do the Gaussian slit - Feynman - and repeat results of wave packet spreading etc. - Perhaps as HW.

12. Include potential term $V(x)$. **Harmonic oscillator** approx. Add $-V(x(t))$ to L. Then calculate PI all over again. Stationary phase gives the usual classical equations of motion. In general cannot

be done exactly. Expand $V(x)$ in power series near minimum. Quadratic term gives harmonic oscillator. Can be done exactly.

The kernel for the harmonic oscillator can be found exactly:

$$\int_{X(0)=X_i}^{X(T)=X_f} \mathcal{D}X(t) e^{\frac{im}{2\hbar}\int_0^T (\dot{x}^2-\omega^2 x^2)dt} \tag{28}$$

Expand $X(t) = X_{classical}(t) + y(t)$, where $x_{cl}(t)$ is the classical solution that satisfies the boundary conditions. Expand. Purely classical piece give the classical action. This is

$$exp\{\frac{im\omega}{2\hbar sin\omega T}[(X_f^2 + X_i^2)cos\omega T - 2X_f X_i]\}$$

What remains is a Gaussian integral over $y(t)$

$$\int_{Y(0)=0}^{Y(T)=0} \mathcal{D}Y(t) e^{\frac{im}{2\hbar}\int_0^T (\dot{Y}^2-\omega^2 Y^2)dt}$$

Expand $y(t) = \sum_n a_n sin(\frac{n\pi t}{T})$

$$KE = T\sum_n a_n^2 \frac{1}{2}(\frac{n\pi}{T})^2$$

$$PE = T\sum_n \frac{1}{2} a_n^2 \omega^2$$

Do integral over a_n (Jacobian is a constant) : the integral is of the form $e^{const a_n^2((\frac{n\pi}{T})^2-\omega^2)}$. The constant is independent of ω and has the same value when $\omega = 0$. This integral is $const' \times (1 - \frac{\omega^2 T^2}{n^2\pi^2})^{-\frac{1}{2}}$.Product over all n gives $(\frac{sin\omega T}{\omega T})^{-1/2}$. Comparing with free particle gives $const' = (\frac{m}{2\pi i\hbar T})^{1/2}$.

The final result:

$$(\frac{m\omega}{2\pi i\hbar sin\omega T})^{1/2} exp\{\frac{im\omega}{2\hbar sin\omega T}[(X_f^2 + X_i^2)cos\omega T - 2X_f X_i]\}$$

13. Do with forcing function. Only classical action will be different. The equation of motion is

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = j(t) \tag{29}$$

Here $j(t)$ is the time dependent force. Thus we need to solve for the Green's function satisfying

$$\frac{d^2G(t,t')}{dt^2} + \omega_0^2 G(t,t') = \delta(t-t') \tag{30}$$

For QM we need the Green's function that satisfies $G(T,t') = G(0,t') =$ 0,i.e. it vanishes at some initial and some final time. This was derived earlier.

$$\begin{aligned} G_F &= \frac{1}{sin\ \omega T} sin\ \omega(t-T) sin\ \omega t' \quad t > t' \\ &= \frac{1}{sin\ \omega T} sin\ \omega(t'-T) sin\ \omega t \quad t < t' \end{aligned} \tag{31}$$

The HO with forcing function has as kernel :

$$\int_{X(0)=X_i}^{X(T)=X_f} \mathcal{D}X(t) e^{\frac{im}{2\hbar}\int_0^T (\dot{x}^2 - \omega^2 x^2)} e^{+\frac{i}{\hbar}\int_0^T dt\ j(t)x} \tag{32}$$

This is the same as eqn (28) except for the addition of the source, and can be done with the same techniques. Write $x(t) = X_{cl}(t) + y(t)$ where X_{cl} now satisfies the classical equation with source, and satisfies the required boundary conditions on x. Thus

$$\frac{d^2X_{cl}}{dt^2} + \omega^2 X_{cl} + \frac{d^2y}{dt^2} + \omega^2 y = \frac{j(t)}{m}$$

Solution without source:

$$X_{cl} = \frac{X_f sin\omega t - X_i sin\omega(t-T)}{sin\omega T} \tag{33}$$

Solution for y with source (and satisfying $y(0) = y(T) = 0$):

$$y(t) = \frac{1}{m}\int_0^T G_F(t,t') j(t') \tag{34}$$

We will set $m = 1$ from now on for convenience. Thus we get

$$K(X_f, t_f; X_i, t_i) = \int_{y(0)=0}^{y(T)=0} \mathcal{D}y \; e^{\frac{i}{\hbar} \; [\frac{\omega}{2sin\omega T}[(X_f^2+X_i^2)cos\omega T - 2X_iX_f]}$$

$$e^{\frac{i}{\hbar}[\frac{X_f}{sin\omega T}\int_0^T dt \; j(t)sin\omega t - \frac{X_i}{sin\omega T}\int_0^T dt \; j(t)sin\omega(t-T)]}$$

$$e^{\frac{i}{\hbar}\int_0^T dt \; \frac{1}{2}[\dot{y}^2 - \omega^2 y^2] + j(t)y} \tag{35}$$

The integral over y remains to be done. This is the same integral that was done earlier except for the term linear in y. This gives the same prefactor as before (as in any Gaussian integral):

$$(\frac{m\omega}{2\pi i\hbar sin\omega T})^{1/2}$$

The exponent is modified due to j. This contribution is obtained by solving for $y(t)$ using the Green function and plugging the solution into the action. The solution is

$$y(t) = \int_0^T dt \; G_F(t, t')j(t') \tag{36}$$

The action can be written after an integration by parts (and using the bc) as

$$\int_0^T dt \; \; -\frac{1}{2}y[\frac{d^2y}{dt^2} - \omega^2 y] + jy$$

Using the EOM and (36) we get

$$\frac{1}{2}\int_0^T dt \; \int_0^T dt' \; j(t)G_F(t, t')j(t') \tag{37}$$

Thus putting all these ingredients together we get

$$K(X_f, t_f; X_i, t_i) = (\frac{m\omega}{2\pi i\hbar sin\omega T})^{1/2} e^{\frac{i}{\hbar} \; [\frac{\omega}{2sin\omega T}[(X_f^2+X_i^2)cos\omega T - 2X_iX_f]}$$

$$e^{\frac{i}{\hbar}[\frac{X_f}{sin\omega T}\int_0^T dt \; j(t)sin\omega t - \frac{X_i}{sin\omega T}\int_0^T dt \; j(t)sin\omega(t-T)]}$$

$$e^{\frac{i}{2\hbar}\int_0^T dt \; \int_0^T dt' \; j(t)G_F(t,t')j(t')} \tag{38}$$

Ground state-Ground state amplitude: We can use this result to calculate the amplitude for the Forced HO to start at $t = 0$ in the ground state and end in the ground state at $t = T$ ($\hbar$ has been set to 1 in some places):

$$\int dX_i \int dX_f e^{-\frac{1}{2}\omega X_i^2} e^{-\frac{1}{2}\omega X_f^2} K(X_f, t_f; X_i, t_i)$$

This is a Gaussian integral of the form:

$$e^{-\frac{1}{2}\underbrace{(X_f X_i)}_{X}\Big[A\Big]\begin{pmatrix} X_f \\ X_i \end{pmatrix} + (X_f X_i)\frac{1}{sin\ \omega T}\underbrace{\begin{pmatrix} i\int_0^T\ j(t) sin\ \omega t \\ -i\int_0^T\ j(t) sin\ \omega(t-T)) \end{pmatrix}}_{J(t)}}$$

with

$$A = \begin{pmatrix} -\frac{\omega}{2} + \frac{i\omega cos\ \omega T}{2sin\ \omega T} & -\frac{\omega}{2sin\omega T} \\ -\frac{\omega}{2sin\ \omega T} & -\frac{\omega}{2} + \frac{i\omega cos\ \omega T}{2sin\ \omega T} \end{pmatrix} \tag{39}$$

This is a Gaussian integral of the form

$$\int dX e^{-\frac{1}{2}X^T A X + X^T J} = e^{\frac{1}{2}J^T A^{-1} J} Det^{-\frac{1}{2}}\ A$$

$DetA = \frac{\omega^2}{2} - i\frac{\omega^2 cos\ \omega T}{2sin\ \omega T} = \frac{\omega^2}{2}[\frac{e^{i\omega T}}{sin\ \omega T}]$. We are primarily interested in the j-dependence so the prefactors do not matter. (They can be obtained by some normalization requirements. But the phase factor $e^{-\frac{i\omega T}{2}}$ is noteworthy as it gives the ground state energy).The answer is:

$$\begin{aligned} & e^{-i\frac{\omega T}{\hbar}} e^{-\frac{1}{2\omega\hbar}\int_0^T dt\ \int_0^t ds\ j(t) e^{-i\omega(t-s)} j(s)} \\ & = e^{-i\frac{\omega T}{\hbar}} e^{-\frac{1}{4\omega\hbar}\int_0^T dt\ \int_0^T ds\ j(t) e^{-i\omega|t-s|} j(s)} \end{aligned} \tag{40}$$

Note that the exponentials in the jj term are of the form $e^{-i|\omega t|}$. This is what was encountered in the field theory Feynman Green function. It was interpreted as propagating +ve energy forward in time or negative energy backward in time. The HO is a "zero (space) dimensional" field theory.

14. Repeat the above in fourier space to mimic the field theory calculation:

We evaluated:

$$\int dX_f \int dX_i \ \psi_0(X_f)\psi_0(X_i)K(X_f,t_f;X_i,t_i)e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \ j(t)x(t)} \qquad (41)$$

Let us take $t_i \to -\infty$ and $t_f \to +\infty$. Thus we are calculating

$$\langle\psi_0|U(+\infty,-\infty)|\psi_0\rangle_j$$

$$\equiv \langle\psi_0,\infty|\psi_0,-\infty\rangle_j \equiv Z[j] \qquad (42)$$

This is the "gnd state to gnd state" amplitude in the presence of a source. In a fld theory this would be called the "vac to vac amplitude" in the presence of a source. The physically interesting thing in fld theory is actually $\frac{Z[j]}{Z[0]}$ - the normalized quantity.

Explain why this is physically interesting/useful

The following can be used to recast $Z[j]$:

$$\psi_0(X_i)\psi_0(X_f) = e^{-\frac{1}{2}(\omega_0 X_i^2+\omega_0 X_f^2)} = lim_{\epsilon\to 0}e^{-\frac{1}{2}\epsilon\int_{-\infty}^{\infty} dt \ \omega_0 x^2(t)e^{-\epsilon|t|}}$$

(Assume $x(-T) = X_i$and $x(+T) = X_f$ for T large and that $\dot{x} = 0$ for $|t| > T$. Then split the integral from $0, T$ and T, ∞ after integrating by parts. Choose $\epsilon T << 1$. Eventually let $T \to \infty$ and $\epsilon \to 0$. In detail:

$$\epsilon\int_{-\infty}^{+\infty} dt \ x^2e^{-\epsilon|t|} = \int_{-\infty}^{+\infty} dt \ x^2\frac{d}{dt}[-sgn(t)e^{-\epsilon|t|}]$$

$$= \int_{-\infty}^{+\infty} dt \ \frac{d}{dt}[x^2(-sgn(t))e^{-\epsilon|t|}] + \int_{-\infty}^{+\infty} dt \ \frac{dx^2}{dt}[sgn(t)e^{-\epsilon|t|}]$$

Now break up the regions into $(-\infty,-T),(-T,0),(0,T),(T,+\infty)$, where $\epsilon T << 1$. But T is large enough that
$\frac{dx^2}{dt} = 0, \ \ t > T, \ t < -T.$

$$\int_0^T dt \ \frac{d}{dt}[x^2(-sgn(t))e^{-\epsilon|t|}] + \int_T^{\infty} dt \ \frac{d}{dt}[x^2(-sgn(t))e^{-\epsilon|t|}] +$$

$$\int_0^{+T} dt \ \frac{dx^2}{dt}[sgn(t)e^{-\epsilon|t|}] + \int_T^{+\infty} dt \ \frac{dx^2}{dt}[sgn(t)e^{-\epsilon|t|}]$$

$$+\int_{-\infty}^{-T} dt\ \frac{d}{dt}[x^2(-sgn(t))e^{-\epsilon|t|}] + \int_{-T}^{0} dt\ \frac{d}{dt}[x^2(-sgn(t))e^{-\epsilon|t|}] +$$
$$\int_{-\infty}^{-T} dt\ \frac{dx^2}{dt}[sgn(t)e^{-\epsilon|t|}] + \int_{-T}^{0} dt\ \frac{dx^2}{dt}[sgn(t)e^{-\epsilon|t|}]$$

Between $0, T$ we can set $e^{-\epsilon|t|} = 1$. So the first and third terms cancel. The fourth term is zero because $\frac{dx^2}{dt} = 0$. The second term is a boundary term. At $t = \infty$, $e^{-\epsilon|t|} = 0$ so we are left with

$$x^2(T) \tag{43}$$

The same arguments give for the sum of the last four terms

$$x^2(-T) \tag{44}$$

Thus, putting back the factor $-\frac{1}{2}\omega_0$ we get

$$-\frac{1}{2}\omega_0 x^2(T) - \frac{1}{2}\omega_0 x^2(-T) = -\frac{1}{2}\omega_0 X_f^2 - \frac{1}{2}\omega_0 X_i^2$$

)

Then

$$Z[j] = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\ [\frac{1}{2}(\dot{x}^2 - \omega_0 x^2 + i\underbrace{\epsilon'\omega_0}_{=\epsilon} x^2) + j(t)x(t)]} \tag{45}$$

Thus the effect of the ground state wave fn is to replace $\omega_0^2 \to \omega_0^2 - i\epsilon$. Introduce $x(t) = \int \frac{d\omega}{2\pi}\ \tilde{X}(\omega)e^{-i\omega t}$.

$$Z[j] = \int \mathcal{D}x(\omega) e^{\frac{i}{\hbar}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\{\frac{1}{2}[\omega^2 - \omega_0^2 + i\epsilon]|\tilde{X}(\omega)|^2 + j(\omega)X(-\omega)\}}$$

Using our formula for Gaussian integrals ($A(\omega) = \frac{i}{4\pi\hbar}[-\omega^2 + \omega_0^2 - i\epsilon]$ and $J(\omega) = \frac{i\ j(\omega)}{2\pi\hbar}$. So

$$Z[j] = e^{-\frac{i}{2\hbar}\int \frac{d\omega}{2\pi}\ \frac{j(\omega)j(-\omega)}{\omega^2 - \omega_0^2 + i\epsilon}} \underbrace{\sqrt{Det\frac{2\pi}{A}}}_{Z[0]} \tag{46}$$

Reintroducing $j(t)$ we get

$$Z[j] = Z[0]e^{-\frac{1}{2\hbar}\int dt\ j(t)\int dt'\ j(t')\underbrace{\frac{[\theta(t-t')e^{-i\omega_0(t-t')}+\theta(t'-t)e^{-i\omega_0(t'-t)}]}{2\omega}}_{\langle 0|T[x(t)x(t')]|0\rangle}} \tag{47}$$

We recognize in the exponent the Feynman two point function introduce in our fld theory discussion. It is also the same expression we found when we discussed the HO with forcing function above - working entirely in the time domain and with the interval being $0, T$ rather than $-\infty, +\infty$.

15. What is the point of calculation Z[j]? It describes the amplitude to go from a specific state to a specific state (gnd state) but *in the presence of an arbitrary disturbance.* That means we can start at $t = -\infty$ in the ground state and excite the oscillator to any particular state by means of special forcing function and then let it evolve and later at large time we can dexcite it by means of some other specific forcing function to the ground state. In this way we can recover the amplitude to go from any state at early times to any other state at late times. Mathematically if we differentiate wrt j we bring down powers of x. x contains creation and annihilation operators that can act on the gnd state and create an excited state at some early time. Similarly one can bring down powers of x at late times to introduce the appropriate number of annihilation operators.

    While this may seem contrived in QM this is precisely what is done in QFT. Note:

$$\begin{aligned}\frac{\delta^2 Z[j]}{\delta j(t_1)\delta(t_2)}|_{j=0} &= \int \mathcal{D}x(t)\ x(t_1)x(t_2)\ e^{\frac{i}{\hbar}\int dt\ L(x,\dot{x})} \\ &=_H \langle 0,+\infty|T[x_H(t_1)x_H(t_2)]|0,-\infty\rangle_H \end{aligned} \tag{48}$$

    The subscript H indicates Heisenberg picture. (More on this below.) Thus $|0,-\infty\rangle_H$ corresponds to the history that has the property that at $t = -\infty$ it is the grnd state. The time ordered product emerges naturally because in the path integral we do the integrals in the correct time order so whichever x occurs earlier is first integrated over.These

time ordered correlation functions are the basic entities that one needs in a QFT because all physical quantities can be evaluated using these. This is also true in the statistical mechanics applications described in the next section.

16. **Schroedinger/Heisenberg formalisms:** The PI evaluates the following:

$$K(X_f, t_f; X_i, t_i) = \langle X_f | e^{-iH(t_f - t_i)} | X_i \rangle$$

We need to distinguish between various kinds of objects and explain notation:

(a) $|X_1\rangle$ is the eigenstate of the usual X operator of QM with ev X_1. (X is also the Scroedinger operator X_S.) i.e. $X|X_1\rangle = X_1|X_1\rangle$.

(b) The Heisenberg operator $X_H(t) = e^{iHt} X_H(0) e^{-iHt}$ where $X_H(0) = X_S$.

(c) The Heisenberg state denoted by $|X_1, t\rangle_H$ is defined to be the eigenstate of $X_H(t)$ i,e, $X_H(t)|X_1, t\rangle_H = X_1|X_1, t\rangle$. Note that by H- state we mean a history. This history is labelled by describing it at some time (usually zero)- here - at time t this state is the eigenstate of $X_H(t)$ with ev X_1 which means at time t it is the ket $|X_1\rangle$ in the usual sense.

Thus we can say that $|X\rangle_H = |X, 0\rangle_H = |X\rangle_S = |X\rangle$ at $t = 0$, whereas $|X, t\rangle_H = |X\rangle_S = |X\rangle$ at $t = t$.

(d) The dependence on t of $|X_1, t\rangle = e^{iHt}|X_1, 0\rangle$ (**not** $e^{-iHt}|X_1, 0\rangle_H$. It is not the time evolved state of $|X_1, 0\rangle$ Check:

$$X_H(t)|X_1, t\rangle_H = e^{iHt} X_H(0) e^{-iHt} e^{iHt} |X_1, 0\rangle_H =$$

$$e^{iHt} X_H(0)|X_1, 0\rangle_H = e^{iHt} X_1 |X_1, 0\rangle_H = X_1 |X_1, t\rangle_H$$

Thus it is not a real time evolution.(In any case it is a Heisenberg state.) It is just a statement of how different states labelled by the time parameters t, t' are related.

In pictures: look at the history of some state A, which can be called $|A, 0\rangle$, because at $t = 0$ it is the state A: Time progresses towards the left: $t = -2T$ at the far right and $t = +2T$ at the far left:

$$e^{-iH2T}A \qquad e^{-iHT}A \qquad A \qquad e^{iHT}A \qquad e^{iH2T}A$$

Compare with the history $|A,T\rangle$.

$$e^{-iHT}A \qquad A \qquad e^{iHT}A \qquad e^{i2HT}A \qquad e^{i3HT}A$$

which is in state A at time T. Clearly $|A,T\rangle = e^{iHT}|A,0\rangle$.

(e) Thus one needs to distinguish between two statements: One is that $e^{iHt_i}|X\rangle_H = |X,t_i\rangle$ is the history that looks like $|X\rangle$ at time $t = t_i$ and the other is that $e^{-iHt_i}|X\rangle_H$ is what the history $|X\rangle_H$ **looks like** to an observer at our time $t = t_i$ (and his clock reads zero).

Thus for instance $e^{-iHt_i}|X,t_i\rangle_H$ is what it (i.e. history $|X,t_i\rangle_H$) looks like to an observer at $t = t_i$. Check: $e^{-iHt_i}|X,t_i\rangle_H = e^{-iHt_i}e^{iHt_i}|X\rangle_H = |X\rangle_H$. In other words the observer whose clock reads zero at our time $t = t_i$, will see this history as $|X\rangle_H$. Equivalently the Schroedinger state at time $t = t_i$ is $|X\rangle$. This is indeed what it should be.

(f) Thus we have the following different notations all standing for the same object:

$$K(X_f,t_f;X_i,t_i) = \langle X_f|e^{-iH(t_f-t_i)}|X_i\rangle =_S \langle X_f|e^{-iH(t_f-t_i)}|X_i\rangle_S$$
$$=_H \langle X_f,0|e^{-iHt_f}e^{iHt_i}|X_i,0\rangle_H$$
$$=_H \langle X_f,t_f|X_i,t_i\rangle_H$$

Now let us understand different notations for the following

$$\int_{x(t_i)=X_i}^{x(t_f)=X_f} \mathcal{D}x(t)\ x(t_1)x(t_2)e^{\frac{i}{\hbar}S}$$
$$\langle X_f|e^{-iH(t_f-t_1)}\ x\ e^{-iH(t_1-t_2)}\ x\ e^{iH(t_2-t_i)}|X_i\rangle \quad t_1 > t_2$$
$$=_S \langle X_f|e^{-iH(t_f-t_1)}\ x_S\ e^{-iH(t_1-t_2)}\ x_S\ e^{-iH(t_2-t_i)}|X_i\rangle_S$$
$$=_H \langle X_f,0|e^{-iH(t_f-t_1)}\ x_H(0)\ e^{-iH(t_1-t_2)}\ x_H(0)\ e^{-iH(t_2-t_i)}|X_i,0\rangle_H$$
$$=_H \langle X_f,t_f|\ x_H(t_1)\ \ x_H(t_2)\ |X_i,t_i\rangle_H$$

In general as explained earlier we get the time ordered product

$$=_H \langle X_i,t_f|\ T[x_H(t_1)x_H(t_2)]\ |X_f,t_i\rangle_H$$

17. Consider

$$ {}_H\langle 0,+\infty|T[x_H(t_1)x_H(t_2)]|0,-\infty\rangle_H \tag{49}$$

Since $|0,t\rangle$ is the vacuum state of the full theory, it evolves by a phase factor. Thus $|0,-\infty\rangle$ and $|0,0\rangle$ differ by a phase $|0,-\infty\rangle = e^{iE_0\infty}|0,0\rangle$. Thus if we calculate

$$ \frac{{}_H\langle 0,+\infty|T[x_H(t_1)x_H(t_2)]|0,-\infty\rangle_H}{{}_H\langle 0,+\infty|0,-\infty\rangle_H} = \frac{{}_H\langle 0,0|T[x_H(t_1)x_H(t_2)]|0,0\rangle_H}{{}_H\langle 0,0|0,0\rangle_H} \tag{50}$$

Assuming the states are correctly normalized:

$$ =_H \langle 0,0|T[x_H(t_1)x_H(t_2)]|0,0\rangle_H $$

which is what we normally write as:

$$ =_H \langle 0|T[x_H(t_1)x_H(t_2)]|0\rangle_H \tag{51}$$

Thus dividing by the gnd state to gnd state amplitude is a good thing to do - it provides the correct normalization.

18. **Perturbation Theory**

$$ K(X_f,t_f;X_i,t_i) = \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ [\frac{m}{2}\dot{x}^2 - V(x(t),t)]} $$

$$ = \int_{x(t_i)=X_i}^{x(t_f)=X_f} \underbrace{\mathcal{D}x(t)}_{sum\ over\ paths} exp(+\tfrac{i}{\hbar}\int_{t_i}^{t_f} dt L(x(t),\dot{x(t)})[1 - \frac{i}{\hbar}\int_{t_a}^{t_b} ds\ [V(x(s),s)] + $$

$$ \frac{1}{2}(\frac{-i}{\hbar})\int_{t_a}^{t_b} ds\ [V(x(s),s)](\frac{-i}{\hbar})\int_{t_a}^{t_b} ds'\ [V(x(s'),s')] \tag{52}$$

Consider the second term. Rewrite as:

$$ -\frac{i}{\hbar}\int_{t_a}^{t_b} ds\ \int \mathcal{D}x(t) e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\ [\frac{m}{2}\dot{x}^2]}[V(x(s),s)] \tag{53}$$

The integrand of $\int ds$ is an ordinary free path integral, except at the time $t = s$ when $V(x(s),s)$ multiplies the whole thing. Let us call this time $t = s = t_c$ and $x(s) = x_c$. So we have free propagation from

t_a to t_c. Then multiply by $V(x_c, t_c)$ then propgagate to t_b. x_c has to be integrated. Finally the whole thing is a fn of t_c and this has to be integrated. Thus we get

$$-\frac{i}{\hbar}\int dt_c \int dx_c\ K_0(b,c)V(x_c,t_c)K_0(c,a) \tag{54}$$

Give diagram and illustrate higher order terms and Feynman rules.

19. **Several degrees of freedom**. $K(x_f, X_f, t_f; x_i, X_i, t_i)$. The convenience of the formalism. Separable systems. $S(x, X) = S_1(x) + S_2(X)$. The concept of "integrating out" degrees of freedom. When would you want to do that: unobservables : eg ren group - effective actions, thermodynamic heat bath or the rest of the universe,

5.3 Statistical Mechanics and the Path Integral

The path integral makes particularly clear two connections with Statistical Mechanics.

I. Classical Statistical Mechanics

Take the Feynman Path Integral for a free particle and substitute $t = -i\tau$. We get

$$\int \mathcal{D}x(\tau) e^{-\frac{1}{\hbar}\int_0^{iT} d\tau \frac{1}{2}(\frac{dx}{d\tau})^2}$$

We analyticllay continue t to pure imaginary so that τ is real and let $T = -iL$. So we get

$$\int \mathcal{D}x(\tau) e^{-\frac{1}{\hbar}\int_0^{L} d\tau \frac{1}{2}(\frac{dx}{d\tau})^2} \tag{55}$$

This is a partition function of a string or a polymer of length L. The expenent is to be identified with $-\frac{H}{kT} = -\beta H$ (where here T is the temperature). Thus the Hamiltonian is $\int_0^L d\tau \frac{1}{2}(\frac{dx}{d\tau})^2$ and $kT = \hbar$. Thus quantum fluctuations that are important when $\hbar$ is large compared to the typical action is replaced by thermal fluctuatuations which are important when kT is larger than typical energies in the system.

Note that the time direction has become a space direction. In general **if we start with a $D+1$ dimensional quantum theory (1 time D space) and rotate $t = -i\tau$ we get the (equilibrium) stat mech of theory in**

D+1 space dimenions. Note that we are dealing with a system in thermal equilibrium so there is no need for a time variable.

II. Quantum Statistical Mechanics

1.Elementary Quantum Stat Mech: Expectation value of an operator in equilibrium so that states are weighted with Boltzmann factor $< A >= \sum_i p_i A_i$ where $p_i = \frac{1}{Z} e^{-\beta E_i}$

Z The partition fn. Free energy.$F(T,V,N)$or $E(S,V,N)$.

2.Other infmn $P(x)$? Need the unintegrated form of the partition fn i.e. density matrix.

3. $P(x) = \frac{1}{Z} \sum_i \phi_i^*(x)\phi_i(x)e^{-\beta E_i}$

Similarly

$$< A >= \frac{1}{Z} \sum_i A_i e^{-\beta E_i}$$

$$= \frac{1}{Z} \sum_i \phi_i^*(x) A \phi_i(x) e^{-\beta E_i} = \frac{1}{Z} \sum_i < \phi_i \mid A \mid \phi_i > e^{-\beta E_i}$$

Define

$$\rho(x', x) = \sum_i \phi_i(x')\phi_i^*(x)e^{-\beta E_i}$$

$$\rho = \sum_i \mid \phi_i >< \phi_i \mid e^{-\beta E_i}$$

$$= \underbrace{\sum_i \mid \phi_i >< \phi_i \mid}_{1} e^{-\beta H}$$

$$= 1.e^{-\beta H}$$

"Density Matrix".

$$< A >= \frac{1}{Z} Tr[A\rho] = \int dx A \rho(x', x)\delta(x - x')$$

where $Z = Tr[\rho] = \int dx \rho(x, x)$

4. Consider

$$K(X_f, t_f; X_i, t_i) = \sum_n \psi_n(X_f)\psi_n^*(X_i)e^{-\frac{i}{\hbar}E_n(t_f - t_i)}$$

If we let $i(t_f - t_i) = \beta\hbar$ we have the density matrix!

Thus can use path integral with $it\hbar$ replaced by u to evaluate ρ:

$$\rho(x',x) = K(x',\beta\hbar;x,0) = \int_{x(0)=x,x(\beta\hbar)=x'} (exp\{-\frac{1}{\hbar}\int_0^\beta \hbar[\frac{m}{2}\dot{x}^2(u)+V(x)]du\})\mathcal{D}x(u)$$

To calculate $Z = Tr[\rho]$ set $x' = x$ and integrate over x, i.e. sum over all *periodic* paths.

Thus **with the substitution $i(t_f - t_i) = \beta\hbar$ and $t = -i\hbar u$ the quantum mechanics of a D+1 (space-tme) dimensional system becomes the equilibrium quantum stat mech of a D-(space) dimensional system.** If we keep periodic paths (effectively the Euclidean time is a circle) then we get the partition function.

Free particle

$$K(X_f,T;X_i,0) = e^{\frac{i}{2\hbar}\frac{m(X_f-X_i)^2}{T}}\sqrt{\frac{m}{2\pi T\hbar i}} \tag{56}$$

Replace $T \to -i\beta\hbar$ to get

$$\rho(x',x) = e^{-\frac{m(X'-X)^2}{2\beta\hbar^2}}\sqrt{\frac{m}{2\pi\beta\hbar^2}} \tag{57}$$

$$Z = \int dx\ \rho(x,x) = \int dx\ \sqrt{\frac{m}{2\pi\beta\hbar^2}} = L\sqrt{\frac{mkT}{2\pi\hbar^2}} \tag{58}$$

L being the size of the box and T here is the temperature.

5. Density operator in general:

a) Pure case : $\rho_k = \mid \psi_k >< \psi_k \mid$. Assume normalized. $\rho^2 = \rho$. $Tr\rho = 1$.

In terms of some energy eigenstates (say):$\mid \psi_k >= \sum_n c_n \mid \phi_n >$ with $\sum_n c_n^* c_n = 1$. So

$$\rho_k = \sum_{n,m} c_n^* c_m \mid \phi_m >< \phi_n \mid$$

$Tr\rho = 1$ clearly. Off diagonal elements are "coherences".

Time evolution: $\rho_k(t) = \mid \psi_k(t) >< \psi_k(t) \mid$ So $\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$.

Note that only coherences have non zero time dependence.

b) Mixed case

$$\rho = \sum_k p_k \rho_k$$

p_k is a probability :$\sum_k p_k = 1$. Motivation for this can be from thermo or from integrating out.

$Tr\rho = 1$ obviously. But $\rho^2 \leq \rho$. Equals sign only in pure case.

Time evolution: same as pure case. In the case of $e^{-\beta H}$ obviously time dependence is not there.

6. Several variables and partial traces - that discussion can be carried over to density matrices. Tensor product. In general the density matrix is not a direct product of two density matrices. If the systems are physically independent it will be a direct product.

i)Direct product:$\rho = \rho_\phi \otimes \rho_\xi$

$\rho_\phi = p_1 \mid \phi_1 >< \phi_1 \mid + p_2 \mid \phi_2 >< \phi_2 \mid$

$\rho_\xi = q_1 \mid \xi_1 >< \xi_1 \mid + q_2 \mid \xi_2 >< \xi_2 \mid$

Partial trace over ϕ gives ρ_ξ and vice versa.

ii) Consider

$\rho = p_1 \mid \phi_1 >\mid \xi_1 >< \phi_1 \mid< \xi_1 \mid + p_2 \mid \phi_2 >\mid \xi_2 >< \phi_2 \mid< \xi_2 \mid$.

$Tr_\phi \rho = \rho_\xi$ defined above, and vice versa but this ρ is not a direct product.

iii) Start with pure state dm: $\frac{1}{\sqrt{2}}(\mid \phi_1 >\mid \xi_1 > + \mid \phi_2 >\mid \xi_2 >)(< \phi_1 \mid< \xi_1 \mid + < \phi_2 \mid< \xi_2 \mid)\frac{1}{\sqrt{2}}$

This is a pure state. But $Tr_\phi \rho = \frac{1}{2}(\mid \xi_1 >< \xi_1 \mid + \mid \xi_2 >< \xi_2 \mid)$

which is not a pure state.

6 Mathematical Digression: Lorentz Group, Poincare Group

Note: A good concise mathematical description of the Lorentz Group and its representations is given in Ramond's book - which is followed here. Also Weinberg's book has a very logically complete description of the Little Group and the implementation of symmetries in the context of QFT

The Lorentz group along with translations is the Poincare Group. These are symmetries of (flat) space-time. Hence it is useful to consider fields that have well defined transformation properties under these symmetries. Generalization of the rotation group (or Galilean group) in non rel qm. In general we take fields to transform as follows: Let $x' = \mathcal{R}x$ be the action of a group element R on x.

$$\Phi'(x') = \tilde{R}\Phi(x)$$

Here $\mathcal{R}$ represents the way coordinates transform. Thus for the Lorentz group they are 4-vectors and $\mathcal{R}$ represents the usual four by four matrices. $\tilde{R}$ on the other hand are matrices that correspond to any of the infinite number of representations of the Lorentz group. Thus

$$\Phi'(x) = \tilde{R}\Phi(\mathcal{R}^{-1}x)$$

If we want a scalar field ϕ we let

$$\phi'(x) = \phi(\mathcal{R}^{-1}x)$$

. For a vector field A we would have

$$A'(x) = \mathcal{R}A(\mathcal{R}^{-1}x)$$

etc.

6.1 Poincare Group

- Generators: (Hermitian)

  $$P_\mu = -i\frac{\partial}{\partial x^\mu} \quad M_{\mu\nu} = \underbrace{-i(x_\mu\partial_\nu - x_\nu\partial_\mu)}_{Orbital} + \underbrace{S_{\mu\nu}}_{Spin} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}$$

  Commutation:

  $$[M_{\mu\nu}, P_\rho] = -ig_{\mu\rho}P_\nu + ig_{\nu\rho}P_\mu, \quad [M_{\mu\nu}, M_{\rho\sigma}] = ig_{\mu\rho}M_{\nu\sigma} - ig_{\mu\sigma}M_{\nu\rho} + ...$$

  We will use $M_{\mu\nu} = iJ_{\mu\nu}$ where J is anti Hermitian.

- Casimirs: i) P^2

  ii) Another manifest 4-vector is $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$. This is the Pauli Lubanski vector. For a massive particle in the rest frame $P^0 = m, \quad P^i = 0$ and it reduces to $\frac{1}{2}m\epsilon^{ijk}M_{jk} = mS_i$ which is just proportional the ordinary spin angular momentum. One can check that $[W_\mu, P_\alpha] = 0$. This uses the antisymmetry of ϵ symbol. Since W is manifestly a four vector, $W.W$ commutes with M. Thus $W.W$ commutes with the Poincare Group generators and is a Casimir.

- If $P^2 = m^2$, then $W^2 = m^2 S.S = m^2 s(s+1)$. So it is the usual spin as mentioned before. Thus one can go to rest frame and this configuration is left invariant by S_i and therefore the states are in reps of S. This is called the "little group" and is $SO(3)$ in this case.

- If $P^2 = m^2 = 0$, then taking the $m \to 0$ limit of the previous case we see that $W^2 = 0$. Also $W.P = 0$ (using antisymmetry of ϵ). Thus W and P are proportional. eg let $P^0 = P^3 = p$. Then $pW^0 = pW^3$. Thus $W^0 = W^3$. Since $W^2 = 0$ the other componenets of W are zero. Also $W^3 = -M_{12}P_0 = +M_{12}P^0 = S_3 P^0$. Thus W is proportional to the spin in the direction of motion (z-axis in this example) and the proportionality is the energy. This config is left invariant only by S_z and states are simply reps of S_z which are one dimensional. Spin in the direction of motion is called helicity. The little group is $SO(2)$.

- The physical degrees of freedom of the particle are classified by the little group. The other generators merely boost the particle etc. Thus the Lor Gr is non unitary and its unitary reps are infinite dimensional. They require all the x-dependence to form unitary a rep - a wave fn of a particle. However the little group is compact and has finite dimensional unitary reps - they classify the nature of the wave fn for physical states at a point. The fields don't have to be unitary reps of the Lorentz group. Since we need to construct Actions from the fields we need to study the reps of the Lor Group.

  So we begin by classifying all the representations $\tilde{R}$ for the Lorentz gr

6.2 Lorentz Group

- The Lorentz group is the group of transformations that leave the proper distance invariant:

$$s^2 = x^i x^i - t^2 = x'^i x'^i - t'^2$$

$$x^\mu x^\nu \eta_{\mu\nu} = s^2$$

  η is the flat space metric tensor.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{59}$$

$$x' = \Lambda\ x$$

- viewed as a matrix equation

$(x' = \gamma(x - \beta t), \quad t' = \gamma(t - \beta x))$ in units where $c = 1$.

Thus

$$\Lambda_\mu{}^\rho \eta_{\rho\sigma} \Lambda_\nu{}^\sigma = \eta_{\mu\nu}$$

viewed as a tensor equation.

$$\Lambda \eta \Lambda^T = \eta$$

viewed as a matrix equation.

This also implies $det\ \Lambda = \pm 1$.

- Near the identity : $\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \epsilon_\mu{}^\nu$.

  Substitute to get

$$(\delta_\mu{}^\rho + \epsilon_\mu{}^\rho)\eta_{\rho\sigma}(\delta_\nu{}^\sigma + \epsilon_\nu{}^\sigma) = \eta_{\mu\nu}$$

  This gives

$$\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0.$$

  Thus we get (i, j stand for space indices):

  $\epsilon_0{}^i = \epsilon_i{}^0$ and $\epsilon_i{}^j = -\epsilon_j{}^i$

  these six matrices parametrixe the Lie Algebra of SO(3,1).

- The three boost generators have the form

$$J_0{}^1 = J_1{}^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

$$J_0{}^2 = J_2{}^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{61}$$

$$J_0{}^3 = J_3{}^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{62}$$

With this convention note the symmetry properties: $-J_{i0} = J_i{}^0 = J_0{}^i = J_{0i}$

- And the rotation generators have the form

$$J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{63}$$

$$J_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{64}$$

$$J_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{65}$$

- If we adopt the convention that $J_{\mu\nu} = -J_{\nu\mu}$ they obey the commutation relations:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}$$

- Note that the rotation group generators (compact) are anti Hermitian whereas boost generators (non compact) are Hermitian. Thus $e^{J_{ij}}$ is unitary whereas $e^{J_{0i}}$ is not. This is because the non compact generators cannot be represented by finite dimensional unitary matrices. SO(3,1) is a non compact form of SO(4) - rotations in 4 Euclidean dimensions.

- Change notation: $J_{0i} = iK_i$ (K_i is now anti Hermitian) s and $\frac{1}{2}\epsilon_{ijk}J_{jk} = -iJ_i$. J_i is Hermitian and obeys the same comm relns as $\frac{1}{2}\sigma_i$ (Pauli matrices):

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

Further

$$[J_i, K_j] = i\epsilon_{ijk}K_k$$

which shows that K_i are a vector under rotations. Finally,

$$[K_i, K_j] = -i\epsilon_{ijk}J_j$$

This shows that K_i are elements of $\frac{SO(3,1)}{SO(3)}$. The negative sign in the structure constant is achieved by multiplying by i- which makes K antiHermitian. Thus e^{iK} is not unitary and this is because K are non compact generators of the Lorentz group - the boosts.

- One can make two mutually commuting sets of generators: $N_i = \frac{1}{2}(J_i + K_i)$ and $N_i^\dagger = \frac{1}{2}(J_i - K_i)$

$$[N_i, N_j] = i\epsilon_{ijk}N_k$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0$$

N defines $SU(2)_L$ and $N^\dagger$ defines $SU(2)_R$. Thus $SO(3,1) = SU(2)_L \otimes SU(2)_R$.

Complex conjugation interchanges left and right. (**Note:** If the group had been $SO(4)$ the two SU(2)'s would have been independent.)

6.3 Representations of the Lorentz Group

- Since we know the representations of SU(2) can be labelled by the "spin" i.e. $0, 1/2, 1, 3/2...$ we can use these same labels for $SU(2)_L \otimes SU(2)_R$ i.e. they are labelled by two numbers (n, m) where $2n, 2m \in \mathbf{Z}$. Thus $(0, 0)$ is the scalar - one dimensional trivial representation. $(0, 1/2)$ and $(1/2, 0)$ are two (complex) dimensional representations.

They are two dimensional spin half representations. They are left or right handed spinors. $(0,1)+(1,0)$ are six real (or three complex +c.c) dimensional representation. The two index antisymmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$ is an example of this. $(1/2,1/2)$ has four real components and corresponds to the four vector. The SU(2) properties can be made manifest by combining the components of a four vector into a 2 X 2 matrix: $A^\mu(\sigma_\mu)_{\alpha\dot{\beta}}$. Here $\sigma^\mu = (1,\sigma_x,\sigma_y,\sigma_z)$. An $SU(2)_L$ group element acts by multiplying on the left by a 2X2 matrix. It acts on the undotted index. Similarly $SU(2)_R$ acts by right multiplication and acts on the dotted index.

- **The spinor representation**. The group elements can be represented by Pauli matrices. Thus $J_i = \frac{1}{2}\sigma_i$ can act as the rotation group. K_i can be represented by $\frac{i}{2}\sigma_i$. Thus $\Lambda_L = e^{\frac{i}{2}\vec{\sigma}.(\vec{\omega}+i\vec{\nu})}$ and $\Lambda_R = e^{\frac{i}{2}\vec{\sigma}.(\vec{\omega}-i\vec{\nu})}$. Note that Λ_L and Λ_R act on different spaces. Thus they should be thought of as

$$\begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$$

Also note that $\Lambda_L^{-1} = \Lambda_R^\dagger$.

In Euclidean space $SO(4)$ both J, K would be compact and represented by $\frac{1}{2}\sigma_i$. In that case $\Lambda_L = e^{\frac{i}{2}\vec{\sigma}.(\vec{\omega}+\vec{\nu})}$ and $\Lambda_R = e^{\frac{i}{2}\vec{\sigma}.(\vec{\omega}-\vec{\nu})}$ would be completely independent.

Some useful properties:

1. $$(\sigma^2)^\dagger = \sigma^2$$

. Thus σ^2 is unitary.

$$\sigma^2\sigma^i\sigma^2 = -(\sigma^i)^*$$

Therefore: $\boxed{\sigma^2\Lambda_L\sigma^2 = \Lambda_R^*}$

So L and R spinors can be related by complex conjugation:

2. $$\sigma^2\psi_L^* \to \sigma^2\Lambda_L^*\psi_L^* = \sigma^2\Lambda_L\sigma^2\sigma^2\psi_L^* = \Lambda_R\sigma^2\psi_L^*$$

Thus $\chi_R = \sigma^2\psi_L^*$ transforms like a right handed spinor $(0,\frac{1}{2})$ whereas ψ_L is $(\frac{1}{2},0)$.

3. σ^2 is an invariant tensor:

$$\sigma^2 \Lambda_L^T \sigma^2 = (\Lambda_R^*)^T = \Lambda_R^\dagger = \Lambda_L^{-1}$$

So $\boxed{\Lambda_L^T \sigma^2 \Lambda_L = \sigma^2}$

4. Thus if χ_L and ψ_L are both $(\frac{1}{2}, 0)$ then,

$$\chi_L^T \sigma^2 \psi_L \to \chi_L^T \Lambda_L^T \sigma^2 \Lambda_L \psi_L = \chi_L^T \sigma^2 \psi_L$$

Thus $\chi_L^T \sigma^2 \psi_L$ is invariant. We can also set $\chi = \psi$ and make the invariant $\psi_L^T \sigma^2 \psi_L$.

5. $\sigma^2 = -i\epsilon^{\alpha\beta}$ with $\epsilon^{12} = 1 = -\epsilon^{21}$. Thus $-i\chi_\alpha(\epsilon^{\alpha\beta}\psi_\beta$ is manifestly invariant. The SU(2) group matrices are $(\Lambda_L)_\alpha{}^\beta$. Complex conjugation changes: $(\psi_\alpha)^* = (\psi^*)^{\dot\alpha}$. Then $(\psi^*)^{\dot\beta} i\epsilon_{\dot\alpha\dot\beta} = (\chi)_{\dot\alpha}$ converts it into a $(0, \frac{1}{2})$. This is equivalent to the statement $\chi_R = \sigma^2 \psi_L^*$ made earlier.

On the group matrices one finds similarly:

$$(\Lambda_{L\alpha}{}^\beta)^* = \Lambda^{\dot\alpha}{}_{\dot\beta}$$

And now we need to raise and lower indices which can be done using $i\epsilon$ tensor:

$$i\epsilon^{\dot\alpha\dot\gamma} \Lambda^{\dot\beta}{}_{\dot\gamma} i\epsilon_{\dot\delta\dot\beta} = (\Lambda_R)_{\dot\delta}{}^{\dot\alpha}$$

(We can adopt the convention that the right index is used to raise and lower: $i\epsilon^{\alpha\beta} A_\beta = A^\alpha$,$i\epsilon_{\beta\alpha} A^\alpha = A_\beta$. This is consistent with $\epsilon\epsilon = -\mathrm{I}$.)

This is equivalent to the earlier relation: (Use $\sigma^2 = -i\epsilon = -\sigma_2^T$) $\sigma^2 \Lambda_L \sigma^2 = \Lambda_R^*$.

6. Note that $\epsilon_{ab}\epsilon^{bc} = -\delta_a^c$. Thus $\epsilon_{12} = +1 = -\epsilon_{21}$. This is also consistent with $\epsilon_{ab}\epsilon^{bc}\epsilon_{cd} = -\epsilon_{ad}$. Similar relns hold for the dotted indices.

7. $\psi_{L\alpha}\epsilon^{\alpha\beta}\psi_{L\beta}$ is invariant and non zero because ψ is Grassmann.

8. Another invariant: Let $\chi_L = \sigma^2 \psi_R^*$. The invariant $\chi_L^T \sigma^2 \psi_L$ becomes

$$(\sigma^2 \psi_R^*)^T \sigma^2 \psi_L = \psi_R^\dagger \psi_L$$

In terms of indices : $(\psi_{R\dot\alpha})^* \psi_{L\alpha} = (\psi_R^*)^\alpha \psi_{L\alpha}$.

9. Note that ψ_L and ψ_R are two independent fields. Thus one can form invariants with just one spinor or with two spinors.

10. ψ_L and ψ_R are called Weyl Spinors. They have two components. One can make a four component Dirac spinor:

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

This transforms as $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$.

11. 4-vectors out of spinors:
$\psi_L^\dagger \bar{\sigma}^\mu \psi_L = (\psi_L^\dagger, -\psi_L^\dagger \sigma^i \psi_L)$ is a 4-vector.
Under boost:

$$\psi_L^\dagger \bar{\sigma}^\mu \psi_L \to \psi_L^\dagger e^{-\frac{1}{2}\vec{\sigma}.\vec{\nu}} \bar{\sigma}^\mu e^{-\frac{1}{2}\vec{\sigma}.\vec{\nu}} \psi_L$$

We can specialize to a boost in the y-direction and consider the $0, 2$ components. We use $e^{-\frac{1}{2}\sigma^2\nu^2} = cosh\ \frac{\nu}{2} - sinh\ \frac{\nu}{2}\ \sigma^2$. So

$$\psi_L^\dagger \psi_L \to \psi_L^\dagger (cosh\ \nu - sinh\ \nu\ \sigma^2)\psi_L$$

$$-\psi_L^\dagger \sigma^2 \psi_L \to -\psi_L^\dagger (cosh\ \nu\ \sigma^2 - sinh\ \nu\ )\psi_L$$

This is the usual Lorentz boost

$$t \to \gamma(t + \beta y) \quad ; \quad y \to \gamma(y + \beta t)$$

where $\gamma = cosh\ \nu$ and $\beta = tanh\ \nu$
On the other hand $\Lambda_R = e^{\vec{\sigma}.\vec{\nu}}$ and transforms as

$$\psi_R^\dagger \psi_R \to \psi_R^\dagger (cosh\ \nu + sinh\ \nu\ \sigma^2)\psi_R$$

$$\psi_R^\dagger \sigma^2 \psi_R \to \psi_R^\dagger (cosh\ \nu\ \sigma^2 + sinh\ \nu\ )\psi_R$$

Therefore, in order to get the same transformation law, we identify $\psi_R^\dagger \sigma^\mu \psi_R = (\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma^i \psi_R)$ as the 4-vector - without the minus sign.

12. Working with Dirac spinors: Since $\psi_R^\dagger \psi_L$ is Lorentz invariant the Hermitian combination $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ is a legitimate scalar. This is equal to $\psi_D^\dagger \gamma^0 \psi_D$ where γ^0 is the matrix

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\psi_D^\dagger \gamma^0 \psi_D$ is commonly written as $\bar{\psi}_D \psi_D$. Similarly if we let

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

the four vector can be written as $\bar{\psi}_D \gamma^\mu \psi_D$.

13. The four component Dirac spinor is built out of ψ_L and an independent ψ_R. If we chose ψ_R to be $\sigma^2 \psi_L^*$ - a legitimate choice - we get a four component object with the same fild content as a two component weyl spinor ψ_L. This is called a Majorana representation. It can be made real by a unitary transformation. i.e 4 real components instead of two complex components as in a Weyl spinor. Thus

$$\psi_M = \begin{pmatrix} \psi_L \\ -\sigma^2 \psi_L^* \end{pmatrix}$$

14. One can define a spinor

$$\psi_D^c = \begin{pmatrix} \sigma^2 \psi_R^* \\ -\sigma^2 \psi_L^* \end{pmatrix}$$

which is called a charge conjugate of ψ_D. It has the opposite electric charge (because of complex conjugation), is otherwise a Dirac spinor, and also obeys $(\psi_D^c)^c = \psi_D$. In the case of the Majorana spinor - it obeys $\psi_M^c = \psi_M$. The Majorana spinor is self conjugate. It must necessarily have no electric charge.

15. The matrix $i\gamma^0\gamma^1\gamma^2\gamma^3$ is called γ^5.

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

It has the property that ψ_L has eigen value -1 and ψ_R has ev +1. It measures "chirality" or handedness of a spinor.

16. The gamma matrices satisfy:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

This is often used as the defn of gamma matrices. They are said to form a "Clifford Algebra".

17. Given that $\bar{\psi}_D\gamma^\mu\psi_D$ is a four vector it is clear that $\bar{\psi}_D\gamma^\mu\partial_\mu\psi_D$ and $\partial_\mu\bar{\psi}_D\gamma^\mu\psi_D$ are scalars. Thus $i\bar{\psi}_D\gamma^\mu\partial_\mu\psi_D$ is Hermitian upto a total derivative. The combination $i\bar{\psi}_D\gamma^\mu\partial_\mu\psi_D - i\partial_\mu\bar{\psi}_D\gamma^\mu\psi_D$ is Hermitian. Note that γ^0 is not Hermitian but γ^i are. (Since $(\gamma^0)^2 = -1$ it must have imaginary ev's and hence cannot be Hermitian.)

- **The four vector** $(\frac{1}{2}, \frac{1}{2})$: A_μ and $\partial_\mu\phi(x)$ where ϕ is a scalar are examples of 4-vectors.

- **Spin 3/2.** This can be obtained by $(1/2, 1/2) \otimes ((0, 1/2) \oplus (1/2, 0))$. We have made it parity invariant by including both (0,1/2) and (1/2,0). $(1/2, 1/2) \otimes (1/2, 0) = (1, 1/2) \oplus (0, 1/2)$. The (1,1/2) is the spin 3/2. It is thus represented by ψ^μ_L which is a spinor with a four vector index, provided we can get rid of the (0,1/2) extra spin 1/2 contamination. This can be done by imposing $\sigma^\mu\psi_{L\mu} = 0$. In Dirac notation this becomes ψ_μ with $\gamma^\mu\psi_\mu = 0$. This is also called the Rarita-Schwinger field. It occurs in supergravity theories.

- **Spin 2.** This can be obtained as $(1/2, 1/2) \otimes (1/2, 1/2) = [(0, 0) \oplus (1, 1)]_S \oplus [(1, 0) \oplus (0, 1)]_A$. The antisymmetric part of two spin 1/2 gives a spin 0. Thus antisymmetrized on both spins gives an overall symmetrized piece - this is the subscript S. Thus $A_{\mu\nu} = A_{(\mu\nu)} + A_{[\mu\nu]}$. The former gives the spin two part. The anti symmetric tensor is a spin one piece. The symmetric part has ten components, which includes a trace, which is actully a scalar. The graviton is an example of spin two. The metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The fluctuation $h_{\mu\nu}$ represents a spin two field and is called a graviton.

- String theories contain higher spin fields. But phenomenologically observed fundamental fields are only spins 1/2,1 and 2. A spin 0 field is required in the standard model but has not yet been observed.

7 Dirac Field

7.1 Classical Dirac Equation and Solutions

- We have already seen that $i\bar{\psi}_D\gamma^\mu\partial_\mu\psi_D$ is Lorentz invariant. So is $m\bar{\psi}_D\psi_D$. Thus we can write an action

$$\int d^4x\ \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

The subscript D is dropped from now on. Weyl spinors will have the subscript L, R. $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Thus γ^0 is Hermitian and the others are anti-Hermitian. This is also clear from the fact that $(\gamma^0)^2 = 1$ and so it's ev's are ± 1 whereas the rest have ev's $\pm i$.

- The Dirac eqn is

$$(i\gamma^0\frac{\partial}{\partial x^0} + i\gamma^i\frac{\partial}{\partial x^i} - m)\psi = 0$$

$i\frac{\partial}{\partial x^0} = -p_0 = +p^0 = E$ and $i\frac{\partial}{\partial x^i} = -p_i = -p^i$. Thus the eqn becomes

$$(\gamma^0 E - \gamma^i p^i - m)\psi = 0$$

- Simple way is to solve in the rest frame and then boost to a moving frame. Remember that $p^\mu = \gamma m(1, \beta^i)$ Also $\gamma = cosh\,\eta \quad , \quad |\beta| = tanh\,\eta$ so $(p^0, p^z) = m(cosh\ \eta, sinh\ \eta)$ for boosts in the z direction. This can be obtained by the matrix $e^{\eta J_{03}} = cosh\ \eta + J_{03} sinh\ \eta$. Thus

$$e^{\eta J_{03}}\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E \\ 0 \\ 0 \\ p^z \end{pmatrix}$$

The analogous boost on ψ_L is performed by the matrix $\Lambda_L = e^{-\frac{1}{2}\sigma^z\eta} = \sqrt{e^{-\sigma^z\eta}}$ and for ψ_R, by $\Lambda_R = e^{\frac{1}{2}\sigma^z\eta} = \sqrt{e^{\sigma^z\eta}}$.

$$\sqrt{e^{-\sigma^z\eta}} = \frac{1}{\sqrt{m}}\sqrt{m(cosh\ \eta - sinh\ \eta\sigma^z)} = \frac{1}{\sqrt{m}}\sqrt{p^0 - p^z\sigma^z}$$

Thus the general boost matrix for ψ_L is $\frac{1}{\sqrt{m}}\sqrt{p^0 - p^i\sigma^i} = \frac{1}{\sqrt{m}}\sqrt{-p_\mu\sigma^\mu}$ Similarly for ψ_R it is $\frac{1}{\sqrt{m}}\sqrt{-p_\mu\bar{\sigma}^\mu}$.

- In the rest frame $E = m,\ p^i = 0$, so the equation becomes

$$(\gamma^0 m - m)\psi = m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\psi = 0$$

Clearly it is satisfied by any spinor of the form

$$u^s = \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$$

Here $s = 1, 2$. We can choose

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus there are two independent positive energy solutions.

If we put the time dependence back in, and choose a convenient normalization factor $\sqrt{m}$, the two solutions are:

$$\psi^s(t) = u^s(m)e^{-imt} = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}e^{-imt}, \qquad s = 1, 2$$

- Interestingly there are also solutions with $E = -m$. The equation becomes

$$(-\gamma^0 m - m)\psi = m\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\psi = 0$$

The solutions are correspondingly

$$\psi^s(t) = v^s(m)e^{+imt} = \sqrt{m}\begin{pmatrix} \\ \eta^s \\ -\eta^s \end{pmatrix} e^{+imt}$$

with $s = 1, 2$. η^s can be chosen the same as ξ^s, but in fact we will make a different choice for later convenience:

$$\eta^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\eta^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

These are the famous negative energy solutions.

- Both solutions can be generalized to the moving frame by multiplying the L component by $\frac{1}{\sqrt{m}}\sqrt{-p_\mu\sigma^\mu}$ and R componenets by $\frac{1}{\sqrt{m}}\sqrt{-p_\mu\bar{\sigma}^\mu}$. Thus the solutions are

$$\psi^s(x,t) = u^s(p)e^{ipx} = \begin{pmatrix} \sqrt{-p_\mu\sigma^\mu}\xi^s \\ \sqrt{-p_\mu\bar{\sigma}^\mu}\xi^s \end{pmatrix} e^{ipx}, \quad p^0 > 0$$

$$\psi^s(x,t) = v^s(p)e^{ipx} = \begin{pmatrix} \sqrt{-p_\mu\sigma^\mu}\eta^s \\ -\sqrt{-p_\mu\bar{\sigma}^\mu}\eta^s \end{pmatrix} e^{ipx}, \quad p^0 < 0$$

7.2 Various Properties of the solutions

- $$\bar{u}(p)^r u(p)^s = u(p)^{r\dagger}\gamma^0 u(p)^s = 2\sqrt{-p_\mu\sigma^\mu}\sqrt{-p_\mu\bar{\sigma}^\mu}\xi^{rT}\xi^s$$

$$= 2\sqrt{(p^0)^2 - (p^i)^2}\delta^{rs} = 2m\delta^{rs}$$

- $$u(p)^{r\dagger}u(p)^s = \sqrt{-p_\mu\sigma^\mu}\sqrt{-p_\mu\sigma^\mu}\xi^{rT}\xi^s + \sqrt{-p_\mu\bar{\sigma}^\mu}\sqrt{-p_\mu\bar{\sigma}^\mu}\xi^{rT}\xi^s = 2p^0\delta^{rs} = 2E_p\delta^{rs}$$

- $$\bar{v}(p)^r v(p)^s = v(p)^{r\dagger}\gamma^0 v(p)^s = -2\sqrt{-p_\mu\sigma^\mu}\sqrt{-p_\mu\bar{\sigma}^\mu}\eta^{rT}\eta^s$$

  $$= -2\sqrt{(p^0)^2 - (p^i)^2}\delta^{rs} = -2m\delta^{rs}$$

- $v(p)^{r\dagger}v(p)^s = \sqrt{-p_\mu\sigma^\mu}\sqrt{-p_\mu\sigma^\mu}\eta^{rT}\eta^s + \sqrt{-p_\mu\bar{\sigma}^\mu}\sqrt{-p_\mu\bar{\sigma}^\mu}\eta^{rT}\eta^s = 2p^0\delta^{rs} = 2E_p\delta^{rs}$

- **Spin sums**

  $$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \begin{pmatrix} \sqrt{-p_\mu\sigma^\mu}\sqrt{-p_\mu\bar{\sigma}^\mu} & \sqrt{-p_\mu\sigma^\mu}\sqrt{-p_\mu\sigma^\mu} \\ \sqrt{-p_\mu\bar{\sigma}^\mu}\sqrt{-p_\mu\bar{\sigma}^\mu} & \sqrt{-p_\mu\bar{\sigma}^\mu}\sqrt{-p_\mu\sigma^\mu} \end{pmatrix}$$

  $$= \begin{pmatrix} m & -p.\sigma \\ -p.\bar{\sigma} & m \end{pmatrix} = -\gamma^\mu p_\mu + m$$

  Similarly

  $$\sum_{s=1,2} v^s(p)\bar{v}^s(p) = -\gamma^\mu p_\mu - m$$

7.3 Gamma Matrix Properties

- The matrices $1, \gamma^\mu, \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu], \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}, \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$ are irreps under the Lorentz group. (i.e. the Lor Gr doesn't mix them up.) The total number of these is 1+4+6+4+1 =16. Thus any 4×4 real matrix Γ can be written in terms of these. Thus $\bar{\psi}\Gamma\psi$ can be written in terms of these and each is a well defined tesnor. Thus $\bar{\psi}\gamma^\mu\psi$ is a 4-vector. $\bar{\psi}\gamma^{\mu\nu}\psi$ is an antisymmetric tensor (6 dim rep of Lor gr.).

  $$\frac{1}{6}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$$

  $$\frac{1}{24}\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$$

  Thus the three index antisymmetric tensor $\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\psi$ can be written in terms of $\bar{\psi}\gamma^\sigma\gamma^5\psi$ and the four index one in terms of $\bar{\psi}\gamma^5\psi$. These are thus equivalent to a 4 -vector and a scalar respectively. However because of the γ^5 in them thay have the opposite parity properties so they are called an axial vector and pseudo scalar respectively.

- There is a "Majorana Representation" for gamma matrices. Since the Majorana spinor is real (in some basis) there must exist a basis where the Lorentz generators are all real. The Lor. generators are $\gamma^{\mu\nu}$ upto normalization. So either all the gamma's are real or all imaginary. In the present representation γ^2 is imaginary and the rest ae real. Let $\gamma\prime^0 = \gamma^0\gamma^2$, $\gamma\prime^1 = \gamma^2\gamma^1$, $\gamma\prime^2 = -\gamma^2$ and $\gamma\prime^3 = \gamma^2\gamma^3$. Then $\gamma\prime^5 = -\gamma^5\gamma^2$. This is a Majorana rep. The gamma matrices are all imaginary and the Lor gr generators are real.

- Lorentz Gr generators in terms of gamma matrices (i.e. spin representation)

$$S_{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

  Incidentally this works in any dimension. These are Hermitian.

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

  (eg. in 3 dimension $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$ is satisfied by $\gamma^i = i\sigma^i$. The Lor Gen are $-\frac{i}{4}[\sigma^i, \sigma^j] = \frac{1}{2}\epsilon^{ijk}\sigma^k$. This is as expected.)

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$[\gamma^\mu, S^{\rho\sigma}] = \overbrace{(iJ^{\rho\sigma})^\mu_\nu}^{Hermitian}\ \gamma^\nu$$

- Terms of the form

$$\bar{\psi}(1+\gamma^5)\gamma^\mu\psi$$

  involve only the right handed part ψ_R and are called chiral currents. $\bar{\psi}\gamma^\mu\psi$ is called a vector current and $\bar{\psi}\gamma^\mu\gamma^5\psi$ is called an axial current.

- Fierz Identities: Using $(\sigma_\mu)_{\alpha\beta}(\sigma^\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} = -(\sigma_\mu)_{\alpha\delta}(\sigma^\mu)_{\gamma\beta}$ to write

$$\overline{\psi}_{1L}\sigma_\mu\psi_{2L}\overline{\chi}_{1L}\sigma^\mu\chi_{2L} = -\overline{\psi}_{1L}\sigma_\mu\chi_{2L}\overline{\chi}_{1L}\sigma^\mu\psi_{2L}$$

7.4 Quantization of the Dirac Field

- $$L = \int d^3x \ \{\psi^\dagger i\partial_t\psi + \psi^\dagger\gamma^i i\partial_i\psi - m\bar{\psi}\psi\}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = i\psi^\dagger$$

$$\{\psi(x), i\psi^\dagger(y)\} = i\delta^3(x-y) \qquad x^0 = y^0$$

Check:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_{s=1,2}\frac{1}{\sqrt{2E_p}}[\underbrace{a^s(p)}_{annihilation}\underbrace{u^s(p)e^{ipx}}_{+ve\ energy} + \underbrace{b^{\dagger s}(p)}_{creation}\underbrace{v^s(p)e^{-ipx}}_{-ve\ energy}]$$

$$\psi^\dagger(y) = \int \frac{d^3q}{(2\pi)^3}\sum_{s=1,2}\frac{1}{\sqrt{2E_q}}[a^{\dagger s}(q)u^{\dagger s}(q)e^{-iqy} + b^s(q)v^{\dagger s}(q)e^{+iqy}]$$

Note that ψ is a 4-component spinor with index a which is being suppressed. So the anticommutation relns that we expect is actually:

$$\{\psi_a(x), i\psi_b^\dagger(y)\} = i\delta^3(x-y)\delta_{ab}$$

So assume $\{a^s(p), a^{\dagger s}(q)\} = (2\pi)^3\delta(p-q)\delta^{rs} = \{b^s(p), b^{\dagger s}(q)\}$ we get

$$\int \frac{d^3p}{(2\pi)^3}\sum_{s=1,2}\frac{1}{2E_p}[u^s(p)u^{\dagger s}e^{ip_i(x-y)^i} + v^s(p)v^{\dagger s}(p)e^{-ip_i(x-y)^i}]$$

Using the spin sums

$$\sum_{s=1,2}u^s(p)u^{\dagger s}(p) = (-\gamma.p + m)\gamma^0 \qquad \sum_{s=1,2}v^s(p)v^{\dagger s}(p) = (-\gamma.p - m)\gamma^0$$

$$\therefore \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}[(-\gamma_0p^0 - \gamma_ip^i + m)e^{ip_i(x-y)^i} + (-\gamma_0p^0 - \gamma_ip^i - m)e^{-ip_i(x-y)^i}$$

($\gamma^i p_i \quad and \quad m$ cancel.)

$$= \int \frac{d^3p}{(2\pi)^3} e^{ip_i(x-y)^i} \mathbf{1} = \delta^3(x-y)\delta_{ab}$$

All other anti comm =0.

Finally:

$$H = \int \frac{d^3p}{(2\pi)^3}[-i\psi^\dagger\gamma^0\vec{\gamma}.\vec{\nabla}\psi + m\bar{\psi}\psi]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_p(a^{\dagger s}(p)a^s(p) + b^{\dagger s}(p)b^s(p))$$

The vacuum is defnd. by

$$a(p)|0\rangle = b(p)|0\rangle = 0$$

and so the energy is positive. $a^\dagger$ creates particles of positive energy and $b^\dagger$ creates anti particles of positive energy.

7.5 Properties of spin $\frac{1}{2}$ states

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}(i\gamma^0\frac{\partial}{\partial t} + i\gamma^i\frac{\partial}{\partial x^i} - m)\psi = \psi^\dagger(i\frac{\partial}{\partial t} + i\gamma^0\gamma^i\frac{\partial}{\partial x^i} - m\gamma^0)\psi$$

$$\partial_\mu[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\omega\psi] = \partial_\mu(\omega_A J^{\mu A})$$

$$\omega_A J^{\mu A} = i\psi^\dagger\delta_\omega\psi$$

$$\psi \to e^{-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi$$

$$\delta_\omega\psi = -\frac{i}{2}\omega_A S^A\psi$$

This is only the spin part. (The full orbital part + spin part - HW)

Let $\omega_{12} = -\omega_{21} = \theta$.

$$\delta_\theta\psi = -i\theta S^{12}\psi$$

$$S^{12} = \frac{1}{2}\epsilon^{123}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \frac{1}{2}\Sigma^3$$

$$\therefore \delta_\theta \psi = -i\frac{\theta}{2}\Sigma^3\psi$$

$$J^0_\theta = \frac{\theta}{2}\psi^\dagger\Sigma^3\psi$$

Charge : $\int d^3x J^0_\theta = \int d^3x \frac{1}{2}\psi^\dagger\Sigma^3\psi = J_3$ - spin-angular momentum.

Mode expansion for ψ:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2}E_p}\sum_{s=1}^{2}[a^s(p)u^s(p)e^{ipx} + b^{\dagger s}(p)v^s(p)e^{-ipx}]$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2}E_p}\sum_{s=1}^{2}[a^{\dagger s}(p)u^{\dagger s}(p)e^{-ipx} + b^s(p)v^{\dagger s}(p)e^{ipx}]$$

Consider the state $a^{\dagger s'}(0)|0\rangle$. We want $J^3 a^{\dagger s'}(0)|0\rangle$. We can use the fact that $J^3|0\rangle = 0|0\rangle$ to convert this to $[J^3, a^{\dagger s}(0)]|0\rangle$. Only terms in J^3 that involve $a^s(0)$ will contribute.

$$\int d^3x \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2}E_p}\frac{1}{\sqrt{2}E_q}[a^{\dagger s}(p)a^r(q), a^{\dagger s'}(0)]u^{\dagger s}(p)\frac{\Sigma^3}{2}u^r(q)e^{-ipx}e^{iqx}|0\rangle$$

x integral gives delta fn in $p-q$ and commutator gives delta fn in p. So we get

$$\frac{1}{2}\delta^{rs'}a^{\dagger s}(0)|0\rangle\frac{1}{2m}u^{\dagger s}(0)\Sigma^3 u^r(0)$$

Doing the spin sums over r:

$$= \frac{1}{2}\sum_s a^{\dagger s}(0)|0\rangle(\sigma^3)^{ss'}$$

This confirms the expected value of $+\frac{1}{2}$ for $s=1$ and $-\frac{1}{2}$ for $s=2$.

Finally if we had started with $b^{\dagger s'}(0)|0\rangle$ the calculation would have been exactly the same except for a minus sign: J^3 has $b^s(0)b^{\dagger r}(0)$. On anticommuting we get $-b^{r\dagger}(0)b^s(0) + \ infinite\ constant$. The infinite constant doesn't contribute to the commutator with $b^{\dagger s'}$. Thus at the

end of the day we get the same answer with an overall minus sign: $s = 1$ has spin $-\frac{1}{2}$ and $s = 2$ has spin $+\frac{1}{2}$.

This can be understood as follows: The state $b^{\dagger 1}|0\rangle$ can be thought of as the result of removing an electron with negative energy $(-m)$ and spin $+\frac{1}{2}$, and charge $-e$ from the Dirac sea. Thus the state effectively has positive energy m and spin $-\frac{1}{2}$ and charge $+e$ (positron with opposite spin and charge).

7.6 Green's Function

- Dirac eqn is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

So the Green's fn satisfies:

$$(i\gamma^\mu \partial_\mu - m)S(x,y) = \mathbb{I}\delta^4(x-y)$$

Define

$$\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} S(p) = S(x-y)$$

$$\Rightarrow (-\gamma^\mu p_\mu - m)S(p) = \mathbb{I}$$

$$\therefore S(p) = -\frac{1}{\gamma^\mu p_\mu + m} = \frac{\gamma^\mu p_\mu - m}{p^2 + m^2}$$

As before one can get a retarded, advanced or Feynman propagator by choosing the contour suitably. The Feynman propagator is obtained also by changing $m^2 \to m^2 - i\epsilon$.

Finally the two point function $\langle \psi(x)\bar{\psi}(y)\rangle$ can be calculated exactly as in the case of the scalar and one finds the same function $D(x-y)$ except for the factor of $\gamma^\mu p_\mu - m$:

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\gamma^\mu p_\mu - m}{2E_p} e^{ip(x-y)}$$

And again as in the scalar case the time ordered two point fn gives the Feynman propagator.

$$\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = iS_F(x-y)$$

with the importanat difference that there is a sign change when we change the order of the field:

$$T(\psi(x)\bar{\psi}(y)) = \theta(x^0 - y^0)\psi(x)\bar{\psi}(y) - \theta(y^0 - x^0)\bar{\psi}(y)\psi(x)$$

8 Discrete Symmetries:C,P,T

- Wigner's Theorem says that if there is a symmetry one can think of it as viewpoints of two different observers, O and O'. O sees states $R_1, R_2, ...R_n$ and O' sees correspondingly $R'_1, R'_2, R'_3, ...R'_n$. Then $P(R_i \to R_j) = P(R'_i \to R'_j)$ i.e. the probabilities of transition must be equal. This means the operator $U : R \to R'$ must be either **1) Unitary and Linear** or **2) Anti-unitary and Anti-linear**. Thus either

  1. $$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle, \quad U(\xi\phi + \eta\psi) = \xi U\phi + \eta U\psi$$

  or

  2. $$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle^*, \quad U(\xi\phi + \eta\psi) = \xi^* U\phi + \eta^* U\psi$$

- If L is linear the adjoint is defined by:

  $$\langle\phi|L^\dagger\psi\rangle \equiv \langle L\phi|\psi\rangle = \langle\psi|L\phi\rangle^*$$

  Note that LHS and RHS are both anti linear in ϕ. If L were anti linear then RHS would be linear in ϕ.

  For anti-linear we must have the definition

  $$\langle\phi|A^\dagger\psi\rangle \equiv \langle A\phi|\psi\rangle^* = \langle\psi|A\phi\rangle$$

- $$\langle\phi|A^\dagger\psi\rangle = \langle\phi|A^\dagger|\psi\rangle = \langle A\phi|AA^\dagger|\psi\rangle^* = \langle A\phi|\psi\rangle^*$$

  This means $AA^\dagger = I$. Thus both for unitary and antiunitary $UU^\dagger = I$.

- **P, T**

  Define

$$\mathcal{P}^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

  This is a discrete element of the Poincare group. so is T below.

$$\mathcal{T}^{\mu}{}_{\nu} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- If this is a symmetry $\exists P$, a unitary operator, that commutes with H. The generators $iJ^{\mu\nu}$ and iP^{μ} are mapped into themselves. Thus

$$PiJ^{\rho\sigma}P^{-1} = \mathcal{P}^{\rho}{}_{\mu}\mathcal{P}^{\sigma}{}_{\nu}iJ^{\mu\nu}$$

  and similary momentum and also for $\mathcal{T}$.

  Special cases: $H = P^0$.

$$PiHP^{-1} = iH, \quad THT^{-1} - iH$$

  If P is unitary then $E \to E$. which is OK. If T is unitary then $E \to -E$ which is not OK. $\therefore$ **T must be antiunitary.** It changes i to $-i$.

- Let $\psi_{k,\sigma}$ be a state with $\vec{P} = 0$ spin σ.

$$P\psi_{k,\sigma} = \eta_{\sigma}\psi_{k,\sigma}$$

  . η is called intrinsic parity. Note J is an axial vector, so σ is not reversed.

  We can now act with a Lorentz boost: $L(\vec{p})$. $PL(\vec{P})P^{-1} = L(\mathcal{P}\vec{p})$. Thus

$$\psi_{p,\sigma} = U(L(\vec{p}))\psi_{k,\sigma}$$

$$P\psi_{p,\sigma} = PUP^{-1}P\psi_{p,\sigma} = U(L(\mathcal{P}\vec{p}))\eta_{\sigma}\psi_{k,\sigma} = \eta_{\sigma}\psi_{\mathcal{P}\vec{p},\sigma}$$

  Thus the intrinsic parity is a property of the particle.

- Similarly $T\psi_{k,\sigma} = \xi_\sigma \psi_{k,-\sigma}$ - spin is reversed.

- $$T(J_1 + iJ_2)T^{-1} = -J_1 + iJ_2$$
  $$T(J_1 - iJ_2)T^{-1} = -J_1 - iJ_2$$
  because of the i.

- $$T(J_1 + iJ_2)\psi_{k,\sigma} = T\sqrt{(j-\sigma)(j+\sigma+1)}\psi_{k,\sigma+1} = T(J_1 + iJ_2)T^{-1}T\psi_{k,\sigma}$$
  $$= (-J_1 + iJ_2)\xi_\sigma \psi_{k,-\sigma} = -\xi_\sigma \sqrt{(j-\sigma)(j+\sigma+1)}\psi_{k,-\sigma-1}$$
  Thus $T\psi_{k,\sigma+1} = \xi_{\sigma+1}\psi_{k,-\sigma-1}$ Thus $\xi_{\sigma+1} = -\xi_\sigma$.
  Now σ ranges from $+j$ to $-j$. Thus if we let $\xi_j = \xi$ then $\xi_{-j} = \xi(-1)^{2j}$ and for general σ, $\xi_\sigma = \xi(-1)^{j-\sigma}$.

- Unlike intrinsic parity ξ is not significant. Thus let $e^{i\theta}\psi_{k,\sigma}$ be the state. Then
  $$Te^{i\theta}\psi_{k,\sigma} = e^{-i\theta}T\psi_{k,\sigma} = e^{-i\theta}\xi\psi_{-k,-\sigma}$$
  Choose θ so that $e^{2i\theta} = \xi$. Then $Te^{i\theta}\psi_{k,\sigma} = e^{i\theta}\psi_{-k,-\sigma}$ Thus we have intrinsic phase is zero and $\xi = 1$.

- Note also that is $2j$ is odd integer then $T^2\psi_{k,\sigma} = (-1)^{2j}\psi_{k,\sigma} = -\psi_{k,\sigma}$. In particular $T\psi \neq \psi$ because if $T\psi = \xi\psi$ then $T^2\psi = \xi^*\xi\psi = \psi$ -contradiction. This is important: Even if background $\vec{E}$ is there so there is no rotational inv, there cannot be a dipole moment prop to spin. Because this would remove the degeneracy between $+\frac{1}{2}$ and $-\frac{1}{2}$. But these states have to be degenerate, if T is a good symmetry. This is **Kramer's degeneracy**.

8.1 P, T, C in QFT:

$$P\psi_{\vec{p},\sigma} = \eta\psi_{-\vec{p},\sigma}$$

- **spin 0**

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}[a(p)e^{ipx} + b^\dagger(p)e^{-ipx}]$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}[a^\dagger(p)e^{-ipx} + b(p)e^{ipx}]$$

$a^\dagger(p)|0\rangle = \psi_{\vec{p},0}$. $Pa^\dagger(p)|0\rangle = \eta\psi_{-\vec{p},0}$

$$\therefore Pa^\dagger(p)P^{-1}P|0\rangle = \eta a^\dagger(-\vec{p})|0\rangle$$

$$\Rightarrow Pa^\dagger(p)P^{-1} = \eta a^\dagger(-\vec{p}), \quad Pa(p)P^{-1} = \eta^* a(-\vec{p})$$

Similarly $Pb^\dagger(\vec{p})P^{-1} = \eta_b b^\dagger(-\vec{p})$. Plugging into mode expansion, (and defining the integration ariable $p' = -p$ where required) if $\eta^* = \eta_b$ the field ϕ has a well defined transformation : $P\phi(\vec{x},t)P^{-1} = \eta^*\phi(-\vec{x},t)$. η is the intrinsic parity of the state.

Note that requiring that the **field** ϕ have a well defined transformation under P relates the phase of the particle and antiparticle. This is one of the consequences of the field theory hypothesis.

- **spin** $\frac{1}{2}$ $P\psi_{\vec{p},\frac{1}{2}} = \eta\psi_{-\vec{p},\frac{1}{2}}$.

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2}E_p}\sum_{s=1}^{2}[a^s(p)u^s(p)e^{ipx} + b^{\dagger s}(p)v^s(p)e^{-ipx}]$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2}E_p}\sum_{s=1}^{2}[a^{\dagger s}(p)u^{\dagger s}(p)e^{-ipx} + b^s(p)v^{\dagger s}(p)e^{ipx}]$$

$a_p^{\dagger s}|0\rangle$ is a fermion state. $Pa_p^{\dagger s}(\vec{p})P^{-1} = \eta a_p^{\dagger s}(-\vec{p})$. We assume that the phase for b is η_b:$Pb_p^{\dagger s}(\vec{p})P^{-1} = \eta_b b_p^{\dagger s}(-\vec{p})$.

$$P\psi(x)P^{-1} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2}E_p}\sum_{s=1}^{2}[\eta^* a^s(-\vec{p})u^s(p)e^{ipx} + \eta_b b^{\dagger s}(-\vec{p})v^s(p)e^{-ipx}]$$

$$u^s(-\vec{p}) = \begin{pmatrix} \sqrt{-p.\bar{\sigma}}\xi^s \\ \sqrt{-p.\sigma}\xi^s \end{pmatrix} = \gamma^0 u^s(\vec{p})$$

Similary $v^s(-\vec{p}) = -\gamma^0 v^s(\vec{p})$. Let $\vec{p}' = -\vec{p}$

$$P\psi(x)P^{-1} = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2}E'_p}\sum_{s=1}^{2}[\eta^* a^s(\vec{p}')\gamma^0 u^s(p')e^{-iEt-ip'x} + \eta_b b^{\dagger s}(\vec{p}')(-\gamma^0)v^s(p')e^{iEt+ip'x}]$$

So if we choose $\eta^* = -\eta_b$ we have the transf rule:

$$P\psi(\vec{x},t)P^{-1} = \gamma^0\psi(-\vec{x},t)$$

The minus sign between η and η_b implies that a fermion and anti fermion have opposite intrinsic parity. Thus a fermion anti fermion bound state will have intrinsic negative parity (if they are in s-wave). The parity due to orbital ang momentum is over and above this.

Note that $\bar{\psi}\gamma^5\psi$ is a pseudo scalar etc...

9 Functional Formalism for Field Theories

For a point particle (one degree of freedom) we wrote down an expression for the amplitude of propagation:

$$K(X_f,t_f;X_i,t_i) = \int_{x(t_i)=X_i}^{x(t_f)=X_f} \underbrace{\mathcal{D}x(t)}_{sum\ over\ paths} \quad exp(+\tfrac{i}{\hbar}\int_{t_i}^{t_f} dtL(x(t),\dot{x(t)}) \tag{66}$$

There is also the more general case where there is an external force:

$$K(X_f,t_f;X_i,t_i)_j = \int_{x(t_i)=X_i}^{x(t_f)=X_f} \underbrace{\mathcal{D}x(t)}_{sum\ over\ paths} \quad exp(+\tfrac{i}{\hbar}\int_{t_i}^{t_f} dt[L(x(t),\dot{x(t)}) + j(t)x(t)] \tag{67}$$

This was explicitly worked out in the case of the HO. As explained there this contains all the information about correlation functions.

We calculate the amplitude that the field at $t = t_i$ in some specific configuration $\phi_i(x,y,z)$ evolves to another configuration $\phi_f(x,y,z)$ at

a later time t_f. Note that the initial and final configuration in the case of a field is a *function of* x, y, z. In the point particle case it was a number (or a set of numbers) giving the coordinates of the particle. Thus the integral is one over all possible paths in configuration space. This is called a functional integral. Thus instead of $\int \mathcal{D}x(t)$ we have $\int \mathcal{D}\phi(x, y, z, t)$.

Thus

$$K(\phi_f(x,y,z), t_f; \phi_i(x,y,z), t_i) = \int_{\phi(x,y,z,t_i)=\phi_i(x,y,z)}^{\phi(x,y,z,t_f)=\phi_f(x,y,z)} \mathcal{D}\phi(x,y,z,t) e^{i\int_{t_i}^{t_f}\int d^3x \mathcal{L}[\phi,\partial_\mu\phi]}$$

As in the point particle case we can construct the vacuum to vacuum amplitude - where the gnd state is the gnd state of the free theory - and get

$$\begin{aligned}
&\langle 0|U(+\infty,-\infty)|0\rangle_J \equiv {}_H\langle 0,+\infty|0,-\infty\rangle_{H\,J} \\
&= \int \mathcal{D}\phi(x,y,z,t) e^{i\int dt \ \int d^3x[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(m^2-i\epsilon)\phi^2 + J\phi]} \\
&= Z[0] e^{\frac{i}{2\hbar}\int \frac{d^4p}{(2\pi)^4} \ \frac{J(p)J(-p)}{p^2+m^2-i\epsilon}} \\
&= Z[0] e^{\frac{i}{2\hbar}\int d^4x_1 \int d^4x_2 \ J(x_1)J(x_2)G_F(x_1,x_2)} \\
&= Z[J]
\end{aligned} \tag{68}$$

Also from (44) $Z(0) = \sqrt{Det \frac{2\pi}{-(p^2+m^2-i\epsilon)}}$. Furthermore:

$$\begin{aligned}
\frac{1}{Z[0]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}|_{J=0} &= iG_F(x_1,x_2) = \frac{\langle 0,+\infty|T[i\phi(x_1)i\phi(x_2)]|0,-\infty\rangle}{\langle 0,+\infty|0,-\infty\rangle} \\
&= -\langle 0|T(\phi(x_1)\phi(x_2))|0\rangle
\end{aligned} \tag{69}$$

Thus we recover the usual results.

9.1 Interactions - Functional Formalism: Z[J]

We have seen how the vacuum to vacuum amplitude in the presence of an external source - $Z[J]$- can be calculated in the functional formalism for the free theory. We now proceed to do this in the interacting theory.

This can be used to evaluate the expectation values of time ordered products of fields, which as we have seen, gives (via the LSZ theorem) the S-matrix. Also for statistical systems it gives the various correlation functions that are of interest.

Thus let us consider the following action:

$$S[\phi] = \int d^4x[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4] \tag{70}$$

We define the **Generating Functional** $Z[J]$:

$$Z[J]_\lambda = \int \mathcal{D}\phi(x,y,z,t)e^{i\int dt \ \int d^3x[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(m^2-i\epsilon)\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi]} \tag{71}$$

$$= \langle 0,\infty|0,-\infty\rangle_J$$

Green's functions can be evaluated by taking functional derivatives just as was done for the free case. Thus $Z_\lambda[J]$ contains all the information about the quantum theory. As in the free particle case the classical configuration is the one that gives the largest contribution to the functional integral and can be obtained by extremizing (minimizing) the action - and this gives the Euler-Lagrange equations of motion. One can also see the role of the equation of motion of the quantum theory by the following procedure: As an identity

$$\int \mathcal{D}\phi(x)\frac{\delta}{\delta\phi(y)}e^{iS} \ = 0$$

This is assuming that the boundaries of field space (infinity) give zero contribution because the action is infinite. Or if the field is compact there is no boundary. This immediately gives:

$$\int \mathcal{D}\phi(x)\frac{i\delta S}{\delta\phi(y)}e^{iS} \ = \langle 0|T[\frac{i\delta S}{\delta\phi(y)}]|0\rangle = 0$$

Thus in the quantum theory the equation of motion holds as an expectation value. We can also consider insertions of ϕ:

$$\int \mathcal{D}\phi(x)\frac{\delta}{\delta\phi(y)}[\phi(z)e^{iS}] \ = 0$$

$$= \int \mathcal{D}\phi(x)[\frac{i\delta S}{\delta\phi(y)}\phi(z) + \delta(z-y)]e^{iS} \;=\; 0$$

$$\Rightarrow \langle 0|T[\frac{i\delta S}{\delta\phi(y)}\phi(z)]|0\rangle + \delta(z-y) = 0$$

For the free theory this would give:

$$i(\partial_\mu\partial^\mu - m^2)\langle 0|T[\phi(y)\phi(z)]|0\rangle + \delta(z-y) = 0$$

9.2 Perturbative Evaluation of $Z_\lambda[J]$

We will use perturbation theory to evaluate $Z[J]$. Thus

$$Z[J]_\lambda = \int \mathcal{D}\phi(x,y,z,t)e^{i\int dt \;\int d^3x[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(m^2-i\epsilon)\phi^2 + J\phi]}[1+$$

$$-\int d^4x\; \frac{i\lambda}{4!}\phi(x)^4 + \int d^4x\;\; \frac{-i\lambda}{4!}\phi^4(x)\int d^4y\; \frac{-i\lambda}{4!}\phi^4(y) + ...] \qquad (72)$$

Now use the fact that $\frac{\delta}{\delta J(x)}$ brings down $i\phi(x)$ from the exponent. Thus the $O(\lambda)$ term can be written as:

$$-\frac{i\lambda}{4!}\int d^4x\; (-i\frac{\delta}{\delta J(x)})^4 Z[J]_0 \qquad (73)$$

Similarly the $O(\lambda^2)$ term can be written as

$$[\frac{-i\lambda}{4!}\int d^4x\; (-i\frac{\delta}{\delta J(x)})^4][\frac{-i\lambda}{4!}\int d^4y\; (-i\frac{\delta}{\delta J(y)})^4]Z[J]_0 \qquad (74)$$

It follows that

$$Z[J]_\lambda = e^{-i\frac{\lambda}{4!}\int d^4x\; (-i\frac{\delta}{\delta J(x)})^4} Z[J]_0 \qquad (75)$$

where $Z[J]_0$ was evaluated earlier

$$Z[J]_0 = Z[0]e^{\frac{i}{2}\int d^4x_1\;\; \int d^4x_2\;\; J(x_1)G_F(x_1-x_2)J(x_2)} \equiv e^{iW_0[J]}$$

Thus we need to evaluate $(-i\frac{\delta}{\delta J(x)})^4 e^{iW_0[J]}$. We need

$$i\frac{\delta W_0}{\delta J(x)} = i\int d^4y\; G_F(x-y)J(y)$$

Figure 1: Feynman Diagrams

and

$$i\frac{\delta^2 W_0}{\delta J(x)^2} = iG(x-x)$$

These can be represented diagrammatically as "Feynman Diagrams". Rules for drawing these diagrams can be made. "Feynman Rules". Combinatoric rules can be made to get the combinatoric factors such as 3 and 6 multiplying these expressions.

We can use these in:

$$\frac{\delta^4 Z[J]}{\delta J(x)^4} = [3(\frac{i\delta^2 W_0}{\delta J^2})^2 + 6i\frac{\delta^2 W_0}{\delta J^2}(i\frac{\delta W_0}{\delta J})^2 + (i\frac{\delta W_0}{\delta J})^4]e^{iW_0[J]}$$

The argument x of J has been suppressed. Thus to $O(\lambda)$, $Z_\lambda[J]$ is given by

$$Z_\lambda[J] = Z_0[J] - i\frac{\lambda}{4!}\int d^4x\ \frac{\delta^4 Z[J]}{\delta J(x)^4} =$$

$$\{1 - i\frac{\lambda}{4!}\int d^4x\ [\underbrace{3(\frac{i\delta^2 W_0}{\delta J^2(x)})^2}_{(1)} + \underbrace{6i\frac{\delta^2 W_0}{\delta J^2(x)}(i\frac{\delta W_0}{\delta J(x)})^2}_{(2)} + \underbrace{(i\frac{\delta W_0}{\delta J(x)})^4}_{(3)}]\}Z_0[J] \tag{76}$$

So to order λ:

$$Z_\lambda[J] = Z_0[J][1 - i\frac{\lambda}{4!}((1)+(2)+(3)]$$

9.3 Green's Functions

Having calculated $Z_\lambda[J]$ we can proceed to evaluate Green's functions.

Example: Calculate

$$\frac{\delta^4 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\ |_{J=0}$$

$$\frac{\delta Z_0}{\delta J(x_1)} = i\frac{\delta W_0}{\delta J(x_1)}e^{iW_0}$$

$$\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)} = (i\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_2)} + i\frac{\delta W_0}{\delta J(x_1)}i\frac{\delta W_0}{\delta J(x_2)})e^{iW_0}$$

$$\frac{\delta^3 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = (i\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_2)}i\frac{\delta W_0}{\delta J(x_3)} + i\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_3)}i\frac{\delta W_0}{\delta J(x_2)} +$$

$$i\frac{\delta^2 W_0[J]}{\delta J(x_3)\delta J(x_2)}i\frac{\delta W_0}{\delta J(x_1)})e^{iW_0}$$

$$\frac{\delta^4 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = i\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_2)}i\frac{\delta^2 W_0[J]}{\delta J(x_3)\delta J(x_4)} +$$

$$i\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_3)}i\frac{\delta^2 W_0[J]}{\delta J(x_2)\delta J(x_4)} + i\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_4)}i\frac{\delta^2 W_0[J]}{\delta J(x_2)\delta J(x_3)} \; + terms \; involving \; \frac{\delta W_0}{\delta J}$$

When we set $J = 0$ only the first three terms survive and what we get is:

$$= iG(x_1 - x_2)G(x_3 - x_4) + iG(x_1 - x_3)G(x_2 - x_4) + iG(x_1 - x_4)G(x_3 - x_2)$$

and what we have evaluated is $\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle$.

Diagrammatically:

$O(\lambda)$ contribution to G:

Now we evaluate the $O(\lambda)$ contribution to the Green's function $\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle$.

Thus we have the sum of three terms. The first one is:

$$\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}[(1)\times Z_0[J]] = (1)\times\frac{\delta^4 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}$$

$$= 3(i)^2(G(x-x))^2 \; \frac{\delta^4 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}|_{J=0}$$

Since there are no J's in (1). This term has already been evaluated above.

Similarly acting on $(2)Z_0[J]$, two of the derivatives act on (2) and the remaining two act on $Z_0[J]$:

$$\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}[(2)\times Z_0[J]] =$$

$$6(i)^3 G(x-x)G(x-x_1)G(x-x_2)i\frac{\delta^2 W_0[J]}{\delta J(x_3)\delta J(x_4)}Z_0[J]|_{J=0}+$$

$$two\ \ other\ \ permutations$$

$$= 6G(x-x)G(x-x_1)G(x-x_2)G(x_3-x_4)Z_0[0]+two\ other\ permutations$$

The third term is (all derivatives act on (3)):

$$\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}[(3)\times Z_0[J]]|_{J=0} = 4!G(x-x_1)G(x-x_2)G(x-x_3)G(x-x_4)Z_0[0]$$

The first two terms are essentially forward scattering terms, i.e there is no interaction between the scattering particles. Only the last term is a genuine scattering and it contributes to the S-matrix. Thus the final result for this terms is

$$-i\lambda\int d^4x\ G(x-x_1)G(x-x_2)G(x-x_3)G(x-x_4)$$

where we have divided by the vacuum to vacuum amplitude.

9.4 Feynman Diagrams and Rules

Above we have explicitly calculated Green functions and represented them diagrammatically. We can directly draw the diagrams if we observe some facts.

1. At the end of the day we are always evaluating $(-i)^n\frac{\delta^n Z_0[J]}{\delta J(1)\delta J(2)...\delta J(n)}|_{J=0}$. This is going to be some products of $\frac{i\delta W_0}{\delta J}$ and $\frac{i\delta^2 W_0}{\delta J(1)\delta J(2)}$. When we set $J=0$, $\frac{\delta W_0}{\delta J}$ becomes zero. So we are left with products of $(-i)^2\frac{i\delta^2 W_0}{\delta J(1)\delta J(2)} = -iG_F(x_1-x_2)$. This called a "contraction of two fields" since it represents a two point function. Thus all we have to do is to make all possible contractions.

2. The $\frac{\delta}{\delta J}$ comes from two sources: i) External fields: Thus if we are evaluating $\langle 0|T[\phi_1\phi_2..\phi_n]|0\rangle$ we will have products of n factors $\frac{\delta}{\delta J_i}$. ii) Vertices: The action contains $S_{int}[\phi] = S_{int}[-i\frac{\delta}{\delta J}]$. Thus we get a factor of $-i\frac{\lambda}{4!}\int d^4x\ (-i)^4\frac{\delta^4}{\delta J(x)^4}$ from each such term. At n'th order we have $\frac{(iS_{int})^n}{n!}$.
3. Thus we can draw a diagram:
   - Draw the external points x_i with one line coming out of each of them.
   - Draw the location of the interaction terms, say $x, y..$. Associated with each there is a factor $-i\frac{\lambda}{4!}\int d^4x$ (*or* $\int d^4y$ ...). There are four lines coming out of each of these.
   - Make contractions between all lines. The contractions can be between any pair of points (a, b) where a, b can be an external point or an interaction vertex. Keep track of the combinatoric factors counting the different possible ways of making these contractions to get the same diagram.
   - For each contraction between a pair of points a, b associate a factor $-iG_F(a-b)$.

   With the above rules one can make diagrams in a simple way.
4. Having drawn a diagram with all the numerical coefficients one can write it in momentum space. The factor $-iG_F(x-y)$ is replaced by $\frac{-i}{p^2+m^2-i\epsilon}$. Momentum conservation is imposed at each vertex. If there are undetermined internal momenta, integrate $\int\frac{d^4p}{2\pi^4}$.
5. **Connected and Disconnected diagrams**

   When we evaluate a green's function (or the generating functionsl) typically there are parts of the diagram that are not connected to each other. So the diagram can be separated into disconnected pieces. Eg Fig 3. If a diagram has no disconnected parts it is called connected.

   The disconnected diagram again may be of two types. If the external lines belong to different external parts, then these two sets of external lines do not see each other. These are typically either forward scattering (where each particle is unaffected) or products of two scattering processes involving smaller number of particles.

These are not interesting (because they have been calculated already in a lower order term).

Sometimes one of the disconnected pieces have no external legs (i.e. factors of J). These are called "vacuum bubbles". An eg of a vacuum bubble is Fig 4. It comes at $O(\lambda)$ and it is equal to $-i3\frac{\lambda}{4!}\int d^4x G_F(x-x)G_F(x-x)$. Since $G_F(x-x) = G(0)$ it doesn't depend on x. So we end up getting $-iVT \times Y$ where Y is some number $O(\lambda)$ and VT is the space time volume.

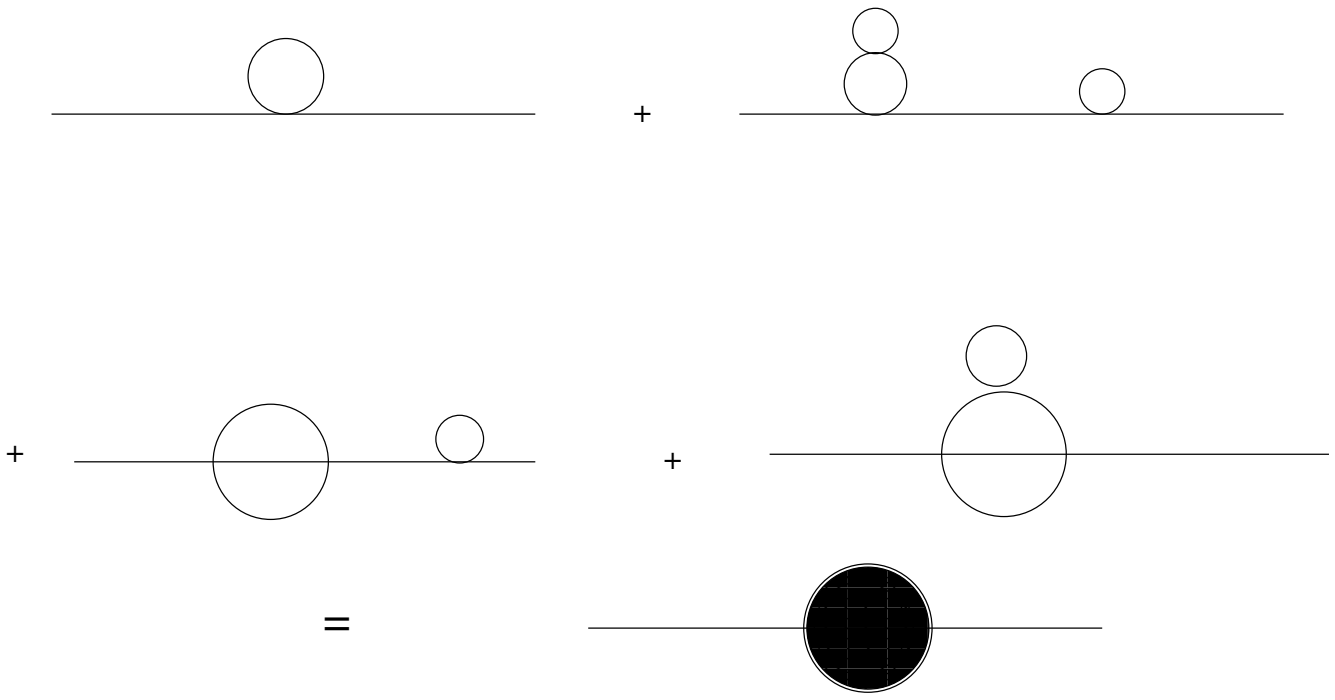
At next order in λ we will have a product of two such terms - so we get $\frac{(-iVTY)^2}{2!}$ and at $n'th$ order we get $\frac{(-iVTY)^n}{n!}$. In fact they exponentiate to give e^{-iVTY}.

The same argument can be made for any vacuum bubble: they all exponentiate. Thus at the end of the day the vacuum bubbles contribute a phase factor $e^{-iVT(Y+Z+...)}$. They multiply *all* Green's functions and is a common factor. In fact they also multiply the identity which means they are just the vacuum to vacuum amplitude that we have talked about. So dividing by the factor $\langle 0, \infty | 0, -\infty \rangle$ gets rid of vacuum bubbles. So *we need not evaluate diagrams with vacuum bubbles.*

6. External Leg Corrections: The diagrams shown in Fig ... are of a special form. They modify the propagator of the particle. On shell they modify the mass of the particle which is defined as the pole of the propagator. So in the S-matrix calculation (fig 3) if one uses the physical mass of the particle then one need not worry about these diagrams. Only the central blob is important. The external leg factors do contribute to wave function renormalization. Thus if one wants the external state to consist of one particle, then one has to multiply by a factor $\sqrt{Z}$ which is less than 1 (formally - in practice these are infinite!). Thus the full propagator has a piece of the form

$$G(p) = \frac{Z}{p^2 + m^2 - i\epsilon} + multi\ particle\ contributions$$

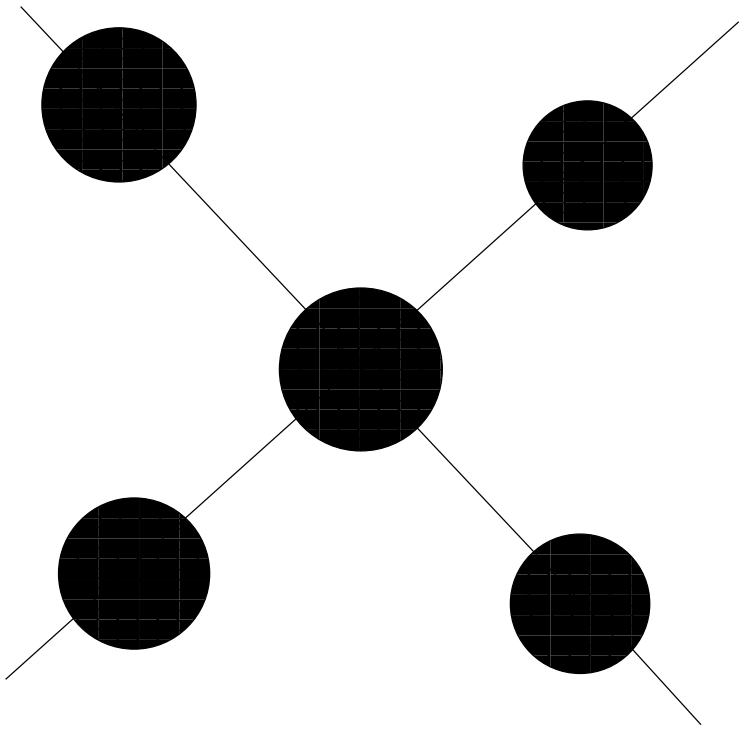
So the requirement that external states be one particle states reduces the final amplitude by a factor $(\sqrt{Z})^n$, when there are n external states.



EXTERNAL LEG CORRECTIONS

Figure 2:

.



External Leg Corrections to Scattering.

Figure 3:

10 Functional Formalism for Fermions

We have seen that for fermions we have anticommutation relations. This means that we cannot represent by ordinary functions in the functional formalism. We need the concept of Grassmann numbers or variables. These are anticommuting versions of ordinary variables.

10.1 Grassmann Calculus

1. **Grassmann Variable**

$$\theta: \quad \{\theta,\theta\} = 0 \Rightarrow \theta^2 = 0$$

(Curly brackets denote anticommutators.)

$$\theta_1,\theta_2 : \{\theta_1,\theta_2\} = 0 \Rightarrow \theta_1\theta_2 = -\theta_2\theta_1 \;\; ; \;\; \theta_1^2 = \theta_2^2 = 0$$

Similarly $\theta_1, \theta_2, \theta_N$ for arbitrary N - maybe infinite.

2. **Differentiation**:

$$\{\frac{d}{d\theta}, \theta\} = 1$$

function $f(\theta) = a + b\theta$ (since higher powers are zero). b must be Grassmann in order for the function to be well defines - commuting - otherwise some part is commuting and some part of it is anti commuting.

$$\frac{df}{d\theta} = \frac{d}{d\theta} b\theta = -b\frac{d}{d\theta}\theta = -b$$

Similarly

$$\{\frac{d}{d\theta_1}, \frac{d}{d\theta_2}\} = \{\frac{d}{d\theta_1}, \frac{d}{d\theta_1}\} = 0$$

Also

$$\frac{d}{d\theta_1}\theta_2 = \frac{d}{d\theta_2}\theta_1 = 0$$

3. **Integration**

$$\int d\theta 1 = 0 \qquad \int d\theta\theta = 1$$

So integration = differentiation.

$$\int d\theta_1 \int d\theta_2 \; \theta_1\theta_2 = -\int d\theta_1 \; \theta_1 \int d\theta_2 \; \theta_2 = -1$$

4. **Integration over functions**

$$f(\theta_1,\theta_2) = a_0 + a_1\theta_1 + a_2\theta_2 + a_{12}\theta_1\theta_2$$

$$\int d\theta_1 \int d\theta_2 \; f = -a_{12}$$

This can be generalized in an obvious way.

Consider

$$e^{-\frac{1}{2}\theta^T M\theta} = e^{-\frac{1}{2}\sum_{i=1}^{N}\theta_i M_{ij}\theta_j}$$

If $N = 2$ this is

$$e^{-\frac{1}{2}\theta_1 M_{12}\theta_2 - \frac{1}{2}\theta_2 M_{21}\theta_1} = e^{-\frac{1}{2}\theta_1(M_{12}-M_{21})\theta_2} = 1 - \frac{1}{2}\theta_1(M_{12} - M_{21})\theta_2$$

$$\int d\theta_1 \int d\theta_2 \; e^{-\frac{1}{2}\theta^T M\theta} = \frac{1}{2}(M_{12} - M_{21}) = M_{12} = \sqrt{Det\ M}$$

Clearly only the antisymmetric part of M contributes. This can easily be generalized to larger values of N, so from now on we take M to be an antisymmetric matrix.

$$\int d\theta_1 \int d\theta_2 \; ... \int d\theta_N \; e^{-\frac{1}{2}\theta^T M\theta} = \sqrt{Det\ M}$$

5. **Correlators**

$$\int d\theta_1 \int d\theta_2 \; ... \int d\theta_N \; \theta_k\theta_l \; e^{-\frac{1}{2}\theta^T M\theta}$$

$$= \int d\theta_1 \int d\theta_2 \; ... \int d\theta_N \; -2\frac{d}{dM_{kl}} e^{-\frac{1}{2}\theta^T M\theta} = -2\frac{d}{dM_{kl}}\sqrt{Det\ M}$$

$$= \sqrt{DetM}(M^{-1})_{kl}$$

6. Add **Grassmann sources**:

$$Z[\chi] = \int d\theta_1 \int d\theta_2 \; ... \int d\theta_N \; e^{-\frac{1}{2}\theta^T M\theta + \chi^T\theta}$$

Shift variables: $\theta' = \theta + M^{-1}\chi$:

$$\theta' T M\theta' = \theta^T M\theta - 2\chi^T\theta - \chi^T M^{-1}\chi$$

So substitute for $\theta^T M\theta$, (dropping the primes on θ)

$$Z[\chi] = \int d\theta_1 \int d\theta_2 \; ... \int d\theta_N e^{-\frac{1}{2}\theta^T M\theta - \frac{1}{2}\chi^T M^{-1}\chi} = \sqrt{Det\ M} e^{-\frac{1}{2}\chi^T M^{-1}\chi}$$

$$\langle\theta_k\theta_l\rangle = \frac{\delta^2}{\delta\chi_k\chi_l} Z[\chi] = \sqrt{DetM}(M^{-1})_{kl}$$

7. Complex notation
   Let $\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}$ and $\theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}$.
   Then as before

$$\int d\theta\ \theta = \int d\theta^*\ \theta^* = 1\ \ ; \qquad \int d\theta^*\ \int d\theta\ (\theta^*\theta) = -1$$

$$\int d\theta_1^*\ \int d\theta_1 \int d\theta_2^*\ \int d\theta_2 e^{-b_1\theta_1^*\theta_1 - b_1\theta_2^*\theta_2} = b_1 b_2$$

   which generalizes to

$$\int d\theta_1^*\ \int d\theta_1 \int d\theta_2^*\ \int d\theta_2 ..... \int d\theta_N^* \int d\theta_N e^{-\theta^{*T} B\theta} = DetB$$

   And adding complex Grassmann sources:

$$\int d\theta_1^*\ \int d\theta_1 \int d\theta_2^*\ \int d\theta_2 ..... \int d\theta_N^* \int d\theta_N e^{-\theta^{*T} B\theta + \eta^{*T}\theta + \theta^{*T}\eta}$$
$$= DetB\ e^{\eta^{*T} B^{-1}\eta}$$

10.2 Dirac Action

Apply this to the Dirac theory we get

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^4x \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta} \tag{77}$$

In the above $B = -i(i\gamma^\mu\partial_\mu - m)$ and $B^{-1} = S_F(x-y)$ and in momentum basis $B^{-1} = \frac{-i}{\gamma^\mu p_\mu + m}$. Also note that our sources are $i\bar{\eta}$ and $i\eta$. So

$$\frac{1}{i}\frac{\delta}{\delta\eta} = -\bar{\psi} \quad , \qquad \frac{1}{i}\frac{\delta}{\delta\bar{\eta}} = \psi$$

and

$$Z[\eta, \bar{\eta}] = DetB\ e^{\int d^4x_1 \int d^4x_2\ \bar{\eta}(x_1) S_F(x_1 - x_2)\eta(x_2)} \tag{78}$$
$$= Det[-i(i\gamma^\mu\partial_\mu - m)]\ e^{i\int \frac{d^4p}{(2\pi)^4}\bar{\eta}(p)\frac{1}{\gamma^\mu p_\mu + m}\eta(-p)}$$

Also

$$\frac{1}{i^2}\frac{\delta^2 Z}{\delta\bar{\eta}(x)\delta\eta(y)} = -\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = -S_F(x-y)$$

$$\Rightarrow \frac{\delta^2 Z}{\delta\bar{\eta}(x)\delta\eta(y)} = \langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = S_F(x-y)$$

10.3 Application: Yukawa Interaction

As an example of the above let us apply this to the Yukawa theory

$$\mathcal{L} = \underbrace{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2\chi^2}_{\mathcal{L}_0} + \underbrace{g\bar{\psi}\psi\phi + h\bar{\chi}\chi\phi}_{\mathcal{L}_{int}}$$

We have the propagators

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = D_F(x-y), \quad \langle 0|T(\chi(x)\chi(y))|0\rangle = \Delta_F(x-y),$$

$$\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle = S_F(x-y)$$

Thus

$$Z[\bar{\eta},\eta,J,j] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi\mathcal{D}\chi e^{i\int d^4x[\mathcal{L}+\bar{\eta}\psi+\bar{\psi}\eta+J\phi+j\chi]}$$

If we define

$$Z_0 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi\mathcal{D}\chi e^{i\int d^4x[\mathcal{L}_0+\bar{\eta}\psi+\bar{\psi}\eta+J\phi+j\chi]}$$

$$= e^{-\int\int(\bar{\eta}S_F(x-y)\eta+\frac{1}{2}JD_FJ+\frac{1}{2}j\Delta_Fj}$$

then

$$Z[\bar{\eta},\eta,J,j] = e^{i\int d^4x\ [g(-\frac{1}{i}\frac{\delta}{\delta\eta_a(x)})(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_a(x)})(\frac{1}{i}\frac{\delta}{\delta J(x)})+h(\frac{1}{i}\frac{\delta}{\delta j(x)})(\frac{1}{i}\frac{\delta}{\delta j(x)})(\frac{1}{i}\frac{\delta}{\delta J(x)})]}Z_0[\bar{\eta},\eta,J,j]$$

The Dirac indices, a, are summed over.

Let us calculate a term in Z that has one J, and to lowest non trivial order in g,h. This comes from:

$$i\int d^4x\,[g(-\frac{1}{i}\frac{\delta}{\delta\eta_a(x)})(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_a(x)})(\frac{1}{i}\frac{\delta}{\delta J(x)})+h(\frac{1}{i}\frac{\delta}{\delta j(x)})(\frac{1}{i}\frac{\delta}{\delta j(x)})(\frac{1}{i}\frac{\delta}{\delta J(x)})]Z_0[\bar{\eta},\eta,J,j]$$

$$= \int d^4x \int d^4y \, [gJ(y)D_F(y-x)Tr(S_F(x-x)) - hJ(y)D_F(y-x)\Delta_F(x-x)]$$

This can be obtained from the Feynman Rules: i) Each propagator gives a factor of S_F, D_F, Δ_F. ii) Each vertex is $ig \times I$ (- in the Dirac index space it is identity) or ih andiii) for each external boson leg a factor i J or i j and iv) a trace and a factor of (-1) for each fermion loop.

11 Effective Action Γ

A very useful object in QFT is the effective action. It is defined as the Legendre transform of $W[J]$. It is the action that is to be used as a tree level (i.e. classical action) to evaluate correlation functions in the full quantum theory. It has all the information of the quantum theory in it.

1. **Definition**

$$Z[J]_\lambda = \int \mathcal{D}\phi \; e^{iS[\phi] + i\int d^4x \;\; J(x)\phi(x)} \tag{79}$$

Let

$$S[\phi] = \int d^4x \{ -\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + m^2\phi^2) - V[\phi] \}$$

$V[\phi] = \frac{\lambda}{4!}\phi^4$ is an example.
Then

$$Z[J]_\lambda \equiv e^{iW[J]} = e^{-iV[\frac{1}{i}\frac{\delta}{\delta J}]} e^{iW_0[J]}$$

and

$$W_0[J] = \frac{1}{2}\int_x \int_y \; J_x G_F(x-y) J_y$$

$$e^{iW[J]} \equiv Z[J] = \int \mathcal{D}\phi \; e^{iS[\phi] + i\int d^4x \;\; J(x)\phi(x)} \equiv e^{i\Gamma[\phi_{cl}] + i\int_x J_x \Phi_{cl}} \tag{80}$$

2. **Legendre Transform**

$$\frac{1}{i}\frac{\delta Z}{\delta J(x)} = \int \mathcal{D}\phi \; \phi(x) e^{iS[\phi] + i\int d^4x \;\; J(x)\phi(x)} = \Phi_{cl} e^{i\Gamma[\phi_{cl}] + i\int_x J_x \Phi_{cl}}$$

Thus

$$\frac{\delta W}{\delta J(x)} = \frac{1}{Z}\frac{1}{i}\frac{\delta Z}{\delta J(x)} = \frac{\langle +\infty, 0|\phi|0, -\infty\rangle_J}{\langle +\infty, 0|0, -\infty\rangle_J} = \Phi_{cl}(x)$$

Thus we get the Legendre transform relation:

$$\boldsymbol{\frac{\delta W[J]}{\delta J(x)} = \Phi_{cl}(x)} \tag{81}$$

So we have to invert this relation, solve for J in terms of Φ_{cl} and substitute in W and in

$$\boldsymbol{W[J] = \Gamma[\Phi_{cl}] + \int dx\ J(x)\Phi_{cl}(x)}$$

to get Γ.

3. **Free Theory**

$$W_0[J] = \frac{1}{2}\int_x\int_y J_x G_F(x-y)J_y; \quad \frac{\delta W_0[J]}{\delta J(x)} = \int_y G_F(x-y)J_y = \Phi_{cl}(x)$$

Acting with the wave operator:

$$(\Box_x - m^2)\frac{\delta W_0[J]}{\delta J(x)} = -J(x) = (\Box_x - m^2)\Phi_{cl}(x)$$

Thus Φ_{cl} obeys the classical EOM.

4. **Inverting the Relation**

$$\Gamma[\Phi_{cl}] = W[J] - \int_x J_x \Phi_{clx}$$

$$\frac{\delta\Gamma}{\delta J(x)} = \int_y \frac{\delta\Gamma}{\delta|Phi_{cl}(y)}\frac{\delta\Phi_{cl}(y)}{\delta J(x)} = \frac{\delta W}{\delta J(x)} - \Phi_{cl}(x) - \int_y J(y)\frac{\delta\Phi_{cl}(y)}{\delta J(x)}$$

Thus using the defn of Φ we get comparing coefficient of $\frac{\delta\Phi}{\delta J(x)}$,

$$\boldsymbol{\frac{\delta\Gamma}{\delta\Phi_{cl}(y)} = -J(y)}$$

5. **Quantum Corrected EOM**

$$Z_\lambda[J] = e^{-iV[\frac{1}{i}\frac{\delta}{\delta J}]} e^{iW_0[J]}$$

$$\frac{1}{i}\frac{\delta Z_\lambda}{\delta J(x)} = e^{-iV[\frac{1}{i}\frac{\delta}{\delta J}]} \int_y G_F(x-y) J(y) Z_0[J]$$

$$(\Box_x - m^2)\frac{1}{i}\frac{\delta Z_\lambda}{\delta J(x)} = e^{-iV[\frac{1}{i}\frac{\delta}{\delta J}]}[-J(x)] e^{iV[\frac{1}{i}\frac{\delta}{\delta J}]} Z_\lambda[J]$$

$$= -J(x) Z_\lambda[J] + [e^{-iV[\frac{1}{i}\frac{\delta}{\delta J}]}, J] e^{iV[\frac{1}{i}\frac{\delta}{\delta J}]} Z_\lambda[J]$$

$$\frac{1}{Z_\lambda[J]}(\Box_x - m^2)\frac{1}{i}\frac{\delta Z_\lambda}{\delta J(x)} = -J(x) + \frac{1}{Z_\lambda[J]} V'[\frac{1}{i}\frac{\delta}{\delta J(x)}] Z_\lambda[J]$$

We have used $[J, \frac{1}{i}\frac{\delta}{\delta J(x)}] = -\frac{1}{i}$. Thus

$$(\Box_x - m^2)\Phi_{cl}(x) = -J(x) + \frac{\lambda}{3!}[\underbrace{(\frac{\delta W}{\delta J(x)})^3}_{\Phi_{cl}(x)^3} \underbrace{-6i\frac{\delta^2 W}{\delta J(x)^2}\frac{\delta W}{\delta J(x)} - \frac{\delta^3 W}{\delta J^3(x)}}_{O(\hbar)}$$

The Φ_{cl}^3 term is part of the interacting classical EOM. The rest of it is non classical. We see that the interacting classical equation is corrected by quantum effects.

12 Application to Particle Physics: S-Matrix

- Having described the properties of free quantum field theories, and listed possible interaction terms, we need to spell out what it is that we seek to calculate. In particle physics the goal is to calculate the S-Matrix. Experiments involve collision of two particles, followed by detection of what comes out of the collision. The probability amplitudes of all the possibilities is the information contained in this matrix. So first we need a precise definition.

- The first ingredient is the definition of the initial state. These are typically states containing widely separated particles. Since the particles are widely separated they do not interact with each other. This is an experimental fact and is also a property of any local QFT. It is called

"cluster decomposition". Thus the state can be thought of as a direct product of single particle states. Single particle states are defined by their quantum numbers: 4-momentum, spin, charge etc. Let E_α be the actual energy, this single particle state. It is an eigenvector of the full H : i.e there is a state Ψ_α that satisfies $H\Psi_\alpha = E_\alpha\Psi_\alpha$. Thus (following Weinberg's notation) let Φ_α denote a single particle eigenstate of the free Hamiltonian with the same energy: $H_0\Phi_\alpha = E_\alpha\Phi_\alpha$. This means that the parameters of H_0 will be chosen so that E_α is the energy of this state. The full Hamiltonian H can be written as $H = H_0 + H_I$, where H_I is the interaction part - it is *defined* by this relation.

For the free theory the multiparticle state is simply a direct product of free single particle state and is also an eigenstate. Thus $E_\alpha = E_1 + E_2 + ...$ where E_i are the various single particle energies. Thus more generally we let Φ_α denote a multi particle eigenstate of H_0. However this multiparticle state is not an eigenstate of the full Hamiltonian.

We are in the Heisenberg formalism where a general state Ψ defines a history and has no time dependence. However at different times, observes will see different things. What they see are related by the evolution operator e^{-iHt}. Thus at time t an observer will see the ket $e^{-iHt}\Psi$ if the observer at time $t = 0$ sees the ket Ψ - and this happens because the operators that the observer uses at time t are related to the ones he uses at $t = 0$ by $O_H(t) = e^{iHt}O(0)e^{-iHt}$. In the Scroedinger formalism one would simply say that the state itself changes.

Note that in the Heisenberg formalism Ψ defines a history - so it is various states (kets) at various times - related by Hamiltonian evolution. So Ψ is a label he uses for the history - the label could be the description of the state at $t = 0$. So we label the history by the Schrodinger state at $t = 0$. An observer who uses a different clock - say his time is zero when mine is t - will see at his time zero in this history a different state - $e^{-iHt}\Psi$. He may label this history like that.

To describe the scattering experiment we have to define a Heisenberg state ("history") that has the property that an observer in the far past would see it as two widely separated free particles. Thus to an observer at $t = -\infty$ it should look like the state Φ_α. Thus we get that our state Ψ_α should satisfy:

$$\lim_{t\to-\infty} e^{-iHt}\Psi_\alpha = \lim_{t\to-\infty} e^{-iH_0t}\Phi_\alpha$$

Such a state is called an **in-state** Ψ^+_α or $|\Psi_\alpha, in\rangle$.

Similarly the outcome of the scattering experiment is a bunch of widely separated particles that one may hope to detect. These are eigenstates of H_0. Thus we let Ψ_β be a state that to an observer at $t = +\infty$ looks like a bunch of widely separated particles - an eigenstate Φ_β of H_0 with some definite energy E_β. Thus

$$\lim_{t\to+\infty} e^{-iHt}\Psi_\beta = \lim_{t\to+\infty} e^{-iH_0t}\Phi_\beta$$

These states are called **out-states** Ψ^-_β or $|\Psi_\beta, out\rangle$.

The matrix elements $\langle out, \Psi_\beta|\Psi_\alpha, in\rangle = S_{\beta\alpha}$ define the **S-Matrix**. One can define the operator $e^{iHt}e^{-iH_0t} = \Omega(t)$ so that $\Psi_\alpha = \lim_{t\to-\infty}\Omega(t)\Phi_\alpha$ Then the S-matrix can also be written as

$$\langle out, \Psi_\beta|\Psi_\alpha, in\rangle = \langle out, \Phi_\beta|\Omega^\dagger(\infty)\Omega(-\infty)|\Phi_\alpha, in\rangle$$

We can also define $\Omega(t, t') = \Omega^\dagger(t)\Omega(t') = e^{iH_0t}e^{-iH(t-t')}e^{-iH_0t'}$, in terms of which the S-Matrix is $S = \Omega(+\infty, -\infty)$. Ω is just the evolution operator with the infinte phases that would be there in free propagation taken out. i.e. if the theory is free, $\Omega = 1$ whereas the usual evolution operator would be $e^{iE(t-t')}$. It thus has only the effects of interaction. Note that for a free theory $S_{\beta\alpha} = \delta(\alpha - \beta)$, by definition.

(Intuitively, In the Schroedinger formalism one would just take the matrix element of the evolution operator e^{-iHt} , for $t \to \infty$ between the multiparticle states $|\alpha\rangle$, $|\beta\rangle$ with the free phase $e^{-iE_\alpha t}$ divided out

$$S_{\beta_\alpha} = \lim_{t\to\infty}\langle\beta|e^{-iHt+iE_\alpha t}|\alpha\rangle$$

This is very rough - it works only if the initial state is an exact energy eigenstate as plane waves. For wave packets it would require using H_0 in place of E_α.)

- Let us illustrate all this with case of a single particle scattering (say off a potential):

  Free case:

  Our initial state is $|k\rangle$ and final state is $|p\rangle$. So the naive FPI would give:

$$K_{fi} = \langle k|e^{-iH_0(t_f-t_i)}|p\rangle$$

$$= e^{-i\frac{p^2}{2m}(t_f - t_i)}\delta_{pk}$$

We take $t_i = -T$ and $t_f = +T$ and let $T \to \infty$. So we get

$$= e^{-i\frac{p^2}{2m}2T}\delta_{pk}$$

Let us write the above in terms of Heisenberg history of states:

The history $|k, t_i\rangle_H$ is (writing the states at $t = -T$ on the extreme right,$t = 0$ in the middle and $t = +T$ on the left:

$$e^{-i\frac{p^2}{2m}2T}|k\rangle \quad e^{-i\frac{p^2}{2m}T}|k\rangle \qquad |k\rangle$$

(Thus $|k, t_i\rangle_H$ is the state that is $|k\rangle$ at $t = t_i$.) Analogously ${}_H\langle p, t_f|$ is

$$\langle p| \qquad \langle p|e^{-i\frac{p^2}{2m}T} \qquad \langle p|e^{-i\frac{p^2}{2m}2T}$$

The innerproduct (can be evaluated at any time) is $e^{-i\frac{p^2}{2m}2T}\delta_{pk}$. And we can write this as

$${}_H\langle p, T|k, -T\rangle_H = e^{-i\frac{p^2}{2m}2T}\delta_{pk}$$

The infinite phase factor is clumsy and we would like to get rid of this. So we need a history such that at $t = 0$ the states are $|p\rangle$ and $|k\rangle$. In other words we need ${}_H\langle p, 0|k, -0\rangle_H = \delta_{pk}$. That corresponds to a different history:

$$e^{-i\frac{p^2}{2m}T}|k\rangle \qquad |k\rangle \qquad e^{i\frac{p^2}{2m}T}|k\rangle \quad = \quad |k, in\rangle \tag{82}$$

$$\langle p, out \mid \; = \qquad \langle p|e^{i\frac{p^2}{2m}T} \qquad \langle p| \qquad \langle p|e^{-i\frac{p^2}{2m}T} \tag{83}$$

Thus our state at $t = -T$ should be $e^{i\frac{p^2}{2m}T}|k\rangle = e^{-iH_0 t_i}|k\rangle$ and at $t = t_f$ should be$\langle p|e^{i\frac{p^2}{2m}T} = \langle p|e^{iH_0 t_f}$. (Or $e^{-iH_0 t_f}|p\rangle$ in terms of kets).

Thus with these initial and final states what we are calculating is

$$S_{pk} = \langle p|e^{iH_0 t_f}\ e^{-iH_0(t_f - t_i)}\ e^{-iH_0 t_i}|k\rangle$$

$$= \langle p|e^{iH_0 t_f}\ U(t_f, t_i)\ e^{-iH_0 t_i}|k\rangle$$

$$= \langle p| \ \Omega(t_f, t_i) \ |k\rangle$$
$$= \delta_{pk}$$

Interacting Case:

The interacting case would have the same equation as above except that $U(t_f, t_i) = e^{-iH(t_f - t_i)}$ where H is the full Hamiltonian rather than the free one.

Thus if we define (with $t_i = -T, \ t_f = T, \ T \to \infty$)

$$e^{iHt_i} e^{-iH_0 t_i} |k\rangle = |k, in\rangle_H$$

This is the history that at $t = t_i$ is the state $e^{-iH_0 t_i}|k\rangle$. and

$$\langle p| e^{iH_0 t_f} \ e^{iHt_f} =_H \langle p, out|$$

then

$$S_{pk} =_H \langle p, out|k, in\rangle_H$$

If we draw the history $|k, in\rangle_H$, we get

$$e^{-H2T} e^{i\frac{p^2}{m}T}|k\rangle \qquad e^{-iHT} e^{i\frac{p^2}{2m}T}|k\rangle \qquad e^{i\frac{p^2}{2m}T}|k\rangle$$

Similarly ${}_H\langle p, out|$

$$\langle p| e^{i\frac{p^2}{2m}T} \qquad \langle p| e^{i\frac{p^2}{2m}T} e^{-iHT} \qquad \langle p| e^{i\frac{p^2}{2m}T} e^{-iH2T}$$

and

$$S_{pk} =_H \langle p, out|k, in\rangle_H = \langle p| e^{i\frac{p^2}{2m}T} e^{-iH2T} e^{i\frac{p^2}{2m}T}|k\rangle$$

Note that when $H = H_0$ we recover the free result. Note also that the state $|k, in\rangle$ corresponds to $|\Psi_\alpha, in\rangle$ and $|p, out\rangle$ corresponds to $|\Psi_\beta, out\rangle$ in the general case described earlier and $|k\rangle$ corresponds to $|\Phi_\alpha\rangle$.

- Differential equation for Ω:

$$\Omega(t, t') = e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}$$

$$\frac{d\Omega}{dt} = e^{iH_0 t} i(H_0 - H) e^{-iH(t-t')} e^{-iH_0 t'}$$

$$\frac{d\Omega}{dt} = \underbrace{e^{iH_0 t} i(H_0 - H) e^{-iH_0 t}}_{-iH_I(t)} \Omega(t,t')$$

$$i\frac{d\Omega}{dt} = H_I(t)\Omega(t,t')$$

Integral equation is $\Omega(t,t') = 1 - i\int_{t'}^{t} dt'' H_I(t'')\Omega(t'',t')$

$$\Omega(t,t_0) = T\{e^{-i\int_{t_0}^{t} dt' \ H_I(t')}\}$$

$H_I(t)$ has a time evolution due to free part only. This is the **interaction picture**. One can thus use the interaction picture field $\phi_I(t)$ that has the property that $\phi(x,0) = \phi_I(x,0)$ and can be expanded in annihilation and creation operators. These are the free field operators. At any given time this can always be done. One can also identify the states of the Heisenberg picture and Scroedinger picture, as well as interaction picture at a given. At a later time, and also at earlier times, ϕ evolves in a complicated way involving H whereas ϕ_I evolves as free field with H_0 and thus always has an expansion in terms of $a, a^\dagger$. (In the Schroedinger picture, ϕ does not evolve at all.) Thus ϕ_I acts on the vacuum of the free theory to create or annihilate a free particle. Thus $\langle 0|\phi_I(x)|p\rangle = e^{ipx}$. Similarly the full field ϕ acts on the vacuum of the full theory to create or annihilate particles: $\langle\omega|\phi(x)|p\rangle = \sqrt{Z}e^{ipx}$. The factor of $\sqrt{Z}$ is because ϕ does more than create single particle states - it also creates multiparticle states in an interacting theory.

- The philosophy in perturbation theory is to work entirely with $|0\rangle$ - the free vacuum and ϕ_I a free field (because it evolves like a free field). The relation between $\phi(x,t)$ and $\phi_I(x,t)$ allows one to write the interactions in terms of ϕ_I. What allows us to write $|\omega\rangle$ in terms of $|0\rangle$ is the following observation: If one starts at $t = -T \approx -\infty$ with $|0\rangle$ and lets it evolve (by H) it will eventually settle down in the true ground state of the theory. Thus

$$e^{-iH(t_0-(-T))}|0\rangle = e^{-iE_0(t_0-(-T))}|\omega\rangle \underbrace{\langle\omega|e^{-iH(T+t_0)}e^{+iE_0(T+t_0)}|0\rangle}_{=\ \langle\omega|0\rangle\ =\ C_0} + \sum_n e^{-iE_n(t_0-(-T))t}|n\rangle C_n$$

In Euclidean time we can easily see that the lowest energy state will dominate the RHS. One can also add a small imaginary part to t to

make the same argument. So after a long time

$$\lim_{T\to\infty}|\omega\rangle = e^{-iH(t_0+T)}e^{iE_0(t+T)}|0\rangle\ (\langle\omega|0\rangle)^{-1}$$

Similarly one can expand $\langle 0|$ in a complete basis and act on it from the right by $e^{-iH(T-t_0)}$. One gets using the Euclideanisation argument:

$$\lim_{T\to\infty}\langle\omega| = \langle 0|e^{-iH(T-t_0)}e^{iE_0(T-t_0)}(\langle 0|\omega\rangle)^{-1}$$

- Thus

$$\langle\omega|T(\phi(x_1)\phi(x_2)...)|\omega\rangle = (\langle 0|\omega\rangle)^{-1}$$
$$\langle 0|e^{iE_0T}e^{-iHT}T(\Omega(t_1)\phi_I(x_1)\Omega^\dagger(t_1)\Omega(t_2)\phi_I(t_2)\Omega^\dagger(t_2)...)e^{-iHT}e^{iE_0T}|0\rangle(\langle\omega|0\rangle)^{-1}$$

Now

$$\langle 0|e^{-i(H-H_0)T}e^{-i(H-H_0)T}|0\rangle =_H \langle 0,T|0,-T\rangle_H$$

(Remember that $e^{-iHT}|0\rangle = e^{-iHT}|0\rangle_H = |0,-T\rangle_H$. Also we assume for simplicity that $H_0|0\rangle = 0$, i.e. the pert vac has zero energy for the free theory.) Now

$$\langle 0|e^{-i(H-H_0)T}e^{-i(H-H_0)T}|0\rangle = \langle 0|e^{-i(H-H_0)T}|\omega\rangle\langle\omega|e^{-i(H-H_0)T}|0\rangle + \textit{higher excited states}$$
$$= \langle 0|e^{-i(H-H_0)T}|\omega\rangle\langle\omega|e^{-i(H-H_0)T}|0\rangle$$
$$= e^{-2iE_0T}\langle 0|\omega\rangle\langle\omega|0\rangle$$

Thus two inverse powers (which are normalization factors)and e^{2iE_0T} can be combined into $\langle 0|\Omega(T,-T)|0\rangle$. The various factors of Ω are easily seen to combine into (remember that factors of e^{iH_0T} can be added at the ends because of our choice of energy definition) the evolution operators $\Omega(t_i,t_j)$ inserted between operators and a factor $\Omega(T,t_1)$ and $\Omega(t_n,-T)$ at the ends. The whole thing is thus

$$\langle 0|T(e^{-i\int_{-T}^{+T}H_I(t)dt}\phi_I(x_1)\phi_I(x_2)....\phi_I(x_n))|0\rangle(\langle 0|\Omega(T,-T)|0\rangle)^{-1}$$

Here $|0\rangle$ is the interaction picture vacuum at $t=0$.

This can also be written as

$$\frac{\langle 0,+T|\phi(x_1)....\phi(x_n)|0,-T\rangle}{\langle 0,+T|0,-T\rangle}$$

which is what is naturally calculated in the FPI formulation.

This achieves the aim of writing everything in terms of interaction picture entities. What remains is to give Wick's theorm for evaluating the time ordered product in perturbation theory. The denominator normalization factor is called the vacuum to vacuum amplitude, or vacuum persistence amplitude.

12.1 LSZ Reduction Formula

- **Scalar Particles**

  This formula gives the S-matrix element in terms of time ordered products: (**Note:** We have used relativistic normalization for the external states in this formula.)

$$\langle p_\beta, out|q_\alpha, in\rangle = Lim_{p_i^2+m^2\to 0,\ \forall i=(1,..,\beta)} \int d^4x_1\, e^{-ip_1x_1} \int d^4x_2\, e^{-ip_2x_2} \ldots \int d^4x_\beta\, e^{-ip_\beta x_\beta}$$
$$Lim_{q_j^2+m^2\to 0,\ \forall j=(1,\ldots,\alpha)} \int d^4y_1\ e^{+iq_1y_1} \int d^4y_2\ e^{+iq_2y_2}\ \ldots \int d^4y_\beta\ e^{+iq_\beta y_\beta}$$
$$\langle 0|T(\phi(x_1)\phi(x_2)\ldots\phi(x_\beta)\phi(y_1)\phi(y_2),,\phi(y_\alpha))|0\rangle \times \prod_i (p_i^2+m^2) \prod_j (q_j^2+m^2) \tag{84}$$

  In this formula $p_i^0 = E_{p_i}$ and $q_j^0 = E_{p_j}$

  The factor $e^{-iE_{q_j}y_j^0}$ picks out the creation operator out of $\phi(y_j)$ and the corresponding factor for x_i^0 picks the annihilaton operator out of $\phi(x_i)$. The integration over t blows up (on shell) but this pole is precisely cancelled by the multiplicative factor of $q_j^2 + m^2$. The proof of this is given in various textboks and we will not worry about it here.

- **Dirac Particle**

  The factors e^{ipx} are the wave functions of the scalar field external states. In the case of the spin 1/2 particle we replace them with corresponding solutions of the Dirac equation.

  The mode expansion of the Dirac field is given below:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \frac{1}{\sqrt{2E_p}} [\underbrace{a^s(p)}_{annihilation\ particle} \underbrace{u^s(p)e^{ipx}}_{+ve\ energy} + \underbrace{b^{\dagger s}(p)}_{creation\ antiparticle} \underbrace{v^s(p)e^{-ipx}}_{-ve\ energy}]$$

$$\psi^\dagger(y) = \int \frac{d^3q}{(2\pi)^3} \sum_{s=1,2} \frac{1}{\sqrt{2E_q}} [a^{\dagger s}(q) u^{\dagger s}(q) e^{-iqy} + b^s(q) v^{\dagger s}(q) e^{+iqy}]$$

Thus if we have an **incoming particle** with spin labelled by s and momentum q we need the operator $a^{\dagger s}(q)e^{-iqx}$ (or $\bar{a}^s e^{-iqx}$). Using $\int d^3x\ \psi^\dagger(x) u^r(q) e^{iqx} = a^{\dagger r}(q)\sqrt{2E_q}$ we see that in the reduction formula we need to multiply the Dirac field by $\int d^4x\ u^s(q)e^{iqx}$ instead of $\int d^4x\ e^{iqx}$ of the scalar field reduction formula. Similarly the factor $q^2 + m^2$ is replaced by $\gamma^\mu q_\mu + m$. Thus for an **incoming particle** the factor is $\int d^4x\ (\gamma^\mu q_\mu + m)u(q)$.

For an **incoming antiparticle** $\int d^4x\ \bar{v}^s(q)e^{iqx}(\gamma^\mu q_\mu + m)$.

For an **outgoing particle** $\int d^4x\ \bar{u}^r(p)e^{-ipx}(\gamma^\mu p_\mu + m)$.

For an **outgoing anti particle** $\int d^4x\ (\gamma^\mu p_\mu + m)v^r(p)e^{-ipx}$.

- **photons**

  This is very similar to the scalar field. For incoming photon the factor is $\int d^4x e^{iqx}\epsilon^r_\mu$. There is a restriction that $k^\mu \epsilon_\mu = 0$.

12.2 S Matrix element for ϕ^4

Let us apply the reduction formula for scalars to the ϕ^4 theory leading order term. We got

$$\langle 0|T[\phi(1)\phi(2)\phi(3)\phi(4)]|0\rangle = -i\lambda \int d^4x\, G_F(x_1-x)G_F(x_2-x)G_F(x_3-x)G_F(x_3-x) \tag{85}$$

We assume that x_1, x_2 are the incoming particles denoted by y_i in the LSZ formula.

$$\int d^4x_1 e^{iq_1x_1} G_F(x_1 - x) = e^{iq_1x} \int d^4x_1 e^{iq_1(x_1-x)} G_F(x_1 - x) = e^{iq_1x} \frac{1}{q_1^2 + m^2}$$

Similarly for the outgoing particles:

$$\int d^4x_3 e^{-ip_3x_3} G_F(x_3-x) = e^{-ip_3x} \int d^4x_3 e^{ip_3(x_3-x)} G_F(x_3-x) = e^{-ip_3x} \frac{1}{p_3^2 + m^2}$$

Thus the LSZ formula gives:

$$S_{q_1,q_2;p_3,p_4} = \int d^4x\ e^{i(q_1+q_2-p_3-p_4)x}(-i\lambda) = -i\lambda(2\pi)^4\delta^4(q_1+q_2-p_3-p_4) \tag{86}$$

12.3 Scattering Cross Section

We will derive expressions for the cross section starting from the S-matrix element, obtained via the LSZ formula. [2]

$$S = \langle \Phi_\beta | \Omega(+\infty, -\infty) | \Phi_\alpha, in \rangle = \langle \Psi_\beta, out | \Psi_\alpha, in \rangle \equiv 1 - iT \tag{87}$$

If we therefore ignore forward scattering it has the form:

$$S_{\beta\alpha} = -(2\pi)^4 i R_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \tag{88}$$

This defines the matrix M. The LSZ reduction formula uses relativistic normalization. If we use ordinary non relativistic normalization we can write

$$S_{\beta\alpha} = -(2\pi)^4 i M_{\beta\alpha} \delta^4(p_\beta - p_\alpha) \tag{89}$$

This defines M.

Since the states are typically plane waves we need to be careful in defining transition probabilities since plane waves are not normalizable in the usual sense. So as an intermediate step we put the whole thing in a box of volume V, and we let the time interval be T. Then we have strictly normalizable quantities. At the end of the calculation one can let both V,T go to infinity.

Box Normalization:

$$\delta^3_V(p - p') = \frac{V}{(2\pi)^3} \delta_{\vec{p},\vec{p}'} \tag{90}$$

$$\delta_T(E_\alpha - E_\beta) = \frac{1}{2\pi} \int_{\frac{-T}{2}}^{\frac{T}{2}} dt \; e^{i(E_\alpha - E_\beta)t} \tag{91}$$

If $\langle x|p\rangle = e^{ipx}$ then $|p\rangle_{box} = \frac{1}{\sqrt{V}}|p\rangle$. $|p\rangle_{box}$ has unit norm. So $\langle \vec{p}|\vec{p}\rangle = V = (2\pi)^3 \delta^3(0)$. If our initial state $\Phi_\alpha\rangle$ has N_α particles then

$$|\Phi_\alpha\rangle_{box} = (\frac{1}{\sqrt{V}})^{N_\alpha} |\Phi_\alpha\rangle$$

$$S_{\beta\alpha} = \langle \Phi_\beta | \Omega | \Phi_\alpha \rangle = V^{\frac{N_\alpha + N_\beta}{2}} \langle \Phi_\beta | \Omega | \Phi_\alpha \rangle_{box}$$
$$= V^{\frac{N_\alpha + N_\beta}{2}} S_{\beta\alpha_{box}}$$

[2] We follow the treatment of Weinberg here. Weinberg defines the S-matrix using non relativistic normalization of states.

Transition Probability

$$P(\alpha \to \beta) = |S_{\beta\alpha_{box}}|^2 = V^{-(N_\alpha + N_\beta)}|S_{\beta\alpha}|^2$$

What remains is the integration over final states. The number of states in the momentum interval d^3p is $V\frac{d^3p}{(2\pi)^3}$. So the total number of states for all N_β particles is $dN_\beta = V^{N_\beta}d\beta = V^{N_\beta}\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}...\frac{d^3p_{N_\beta}}{(2\pi)^3}$

So

$$dP(\alpha \to \beta) = V^{-N_\alpha}|S_{\beta\alpha}|^2 d\beta$$

If we assume the form (88) we get

$$|S_{\beta\alpha}|^2 = [(2\pi)^4\delta_V^3(p_\beta - p_\alpha)\delta_T(E_\beta - E_\alpha)]^2|M_{\beta\alpha}|^2$$

Using (90) we get

$$\delta_V^3(p-p')^2 = \frac{V}{(2\pi)^3}\delta_V^3(p-p')$$

Similarly

$$\delta_T(E-E')^2 = \frac{T}{(2\pi)}\delta_T^3(E-E')$$

Thus

$$|S_{\beta\alpha}|^2 = (2\pi)^4\delta_V^3(p_\beta - p_\alpha)\delta_T(E_\beta - E_\alpha)VT|M_{\beta\alpha}|^2$$

Therefore the final general expression for the transition rate from a state with N_α particles in the initial state to N_β particles in the final state is:

$$d\Gamma = \frac{dP(\alpha \to \beta)}{T} = (2\pi)^4\delta^4(p_\beta - p_\alpha)V^{1-N_\alpha}|M_{\beta\alpha}|^2 d\beta \tag{92}$$

We can now specialize:

$\boldsymbol{N_\alpha = 1}$

This corresponds to a decay:

$$d\Gamma = (2\pi)^4\delta^4(p-p')|M_{\beta\alpha}|^2 d\beta$$

The matrix element M is not rel inv. This is because the states $|p\rangle$ have non rel normalization. $|p\rangle_{rel} = \sqrt{2p_0}|p\rangle$ is rleativistically normalized.

$$R_{\beta\alpha} = \prod_\alpha 2p_0^\alpha \prod_\beta 2p_0^\beta |M_{\beta\alpha}|^2$$

is Lorentz invariant.

Similarly $d\beta$ is not rel inv. $\frac{d\beta}{\prod_\beta 2p_0^\beta}$ is rel invariant. in terms of these the equation is manifestly covariant:

$$d\Gamma = \frac{dP(\alpha \to \beta)}{T} = (2\pi)^4 \delta^4(p_\beta - p_\alpha) \frac{1}{2E_1} \underbrace{R_{\beta\alpha} \frac{d\beta}{\prod_\beta 2p_0^\beta}}_{rel\ invariant}$$

The decay rate transforms like $\frac{1}{E}$ as expected.

$\boldsymbol{N_\alpha = 2}$

$$d\Gamma(2 \to \beta) = (2\pi)^4 \frac{1}{V} |M_{\beta\alpha}|^2 d\beta\ \delta^4(p_\beta - p_\alpha)$$

$\frac{d\Gamma}{flux} = d\sigma$ is the differential **cross section**.

Concept of Cross Section:

Target particles are distributed uniformly over an area (looking from the direction of the incoming beam) A.

We calculate *number of scattering events per incoming particle* and express it as the number of target particles inside an area σ surrounding the incoming particle. This is clearly $\frac{\sigma}{A}$ and defines σ. Total number of target particles is one - so this is a fraction.

Total number of events per unit time then is $\frac{\sigma}{A}$.- assuming there is one target particle and one incoming particle per unit time.

We assume that a total of N incoming particles are scattering.Then rate of scattering is $N\Gamma$ (no of particles scattered per unit time as measured by a detector).

Number of particles per unit time =flux = number density of particles $\times A\times$ velocity of incoming particle-u_α. Note that we are interested in the number of particles that are inside the area A - only these are relevant. Also the particles are plane waves with uniform probability inside the box of $\frac{1}{V}$. So the number density is just $\frac{N_\alpha}{V}$ if we are talking about N particles. So we get flux $=NA\frac{1}{V}u_\alpha$. So total number of scattered particles per unit time per unit flux (unit flux means one incoming particle per unit time per unit area) is $\frac{N\Gamma V}{NAu_\alpha}$. This should equal $\frac{\sigma}{A}$. Thus $\sigma = \frac{\Gamma V}{u_\alpha}$.

Another way of seeing this: Γ gives the rate of scattering for one incoming particle (and one target particle - this is always fixed). The flux associated with this incoming particle is $\frac{1}{V}$ (number density) $\times u_\alpha$ (velocity). So $\sigma = \frac{\Gamma}{flux} = \frac{\Gamma V}{u_\alpha}$.

Note that here u_α is the magnitude of the relative velocity defined in the rest frame of one particle as the velocity of the other particle and vice versa. So our final expression for the differential scattering cross section is:

$$d\sigma = \frac{(2\pi)^4}{u_\alpha}|M_{\beta\alpha}|^2\delta^4(p_\beta - p_\alpha)d\beta$$
$$= \frac{(2\pi)^4}{u_\alpha}\delta^4(p_\beta - p_\alpha)\frac{1}{2E_1 2E_2}\underbrace{R_{\beta\alpha}\frac{d\beta}{\prod_\beta 2p_0^\beta}}_{rel\ \ invariant}$$

$u_\alpha E_1 E_2 = \sqrt{(p_1.p_2)^2 - m_1^2 m_2^2}$ can be verified by going to the rest frame of one particle. Since it is manifestly invariant it can be used in any frame.

$$d\sigma = \frac{(2\pi)^4\delta^4(p_\beta - p_\alpha)}{4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2}}R_{\beta\alpha}\frac{d\beta}{\prod_\beta 2p_0^\beta}$$

In this last form it is manifestly Lorentz invariant.

$\lambda\phi^4$ scalar field theory cross section:

We can apply this to $\lambda\phi^4$ theory for which at tree level

$$S_{\beta\alpha} = -i\lambda(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$$

Thus $R_{\beta\alpha} = \lambda$.

Do the calculation in the C of M frame where $\vec{p}_1 + \vec{p}_2 = 0$. Let $p_1 = (p_1^0, \vec{p}_1)$ and $p_2 = (p_2^0, -\vec{p}_1)$. So $p_1.p_2 = -E^2 - |\vec{p}_1|^2$. So

$$(p_1.p_2)^2 - m_1^2 m_2^2 = (-E^2 - |p_1|^2 + m^2)(-E^2 - |p_1|^2 - m^2) = 4E^2|p_1|^2 = E_{cm}^2|p_1|^2$$

So

$$d\sigma = \lambda^2\int\frac{dp_3^3}{(2\pi)^3 2E_3}\int\frac{dp_4^3}{(2\pi)^3 2E_4}(2\pi)^4\delta(E_{cm} - E_3 - E_4)\delta^3(p_3 + p_4)\frac{1}{4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2}}$$

Energy conservation says $|\vec{p}_1| = |\vec{p}_3| = |\vec{p}_4|$ and $2E_3 = 2E_4 = E_{cm}$. Thus

$$d\sigma = \lambda^2\int\frac{dp_3^3}{(2\pi)^3 2E_3 2E_4}(2\pi)\delta(E_{cm} - E_3 - E_4)\frac{1}{4E_{cm}|p_1|}$$

$$\frac{d\sigma}{d\Omega} = \lambda^2\int\frac{dp_3}{(2\pi)^3}\frac{p_3^2 2\pi}{E_{cm}^2 E_{cm}}\frac{1}{4E_{cm}|p_1|}\delta(2\sqrt{p_3^2 + m^2} - E_{cm})$$

Let $y = 2\sqrt{p_3^2 + m^2}$ and the integral becomes

$$\lambda^2 \int dy \, \frac{y\delta(y - E_{cm})}{(2\pi)^2 16 E_{cm}^3}$$

So

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{\lambda^2}{E_{cm}^2} \tag{93}$$

Finally integrating over Ω gives a factor of 2π (rather than 4π, because the final state has identical particles - we overcount if we integrate over the full 4π) the total cross section:

$$\sigma = \frac{\lambda^2}{32\pi E_{cm}^2}$$

13 Application to Critical Phenomena

Consider a ferromagnet. The spin at a point x is labelled $s(x)$. x is discrete but as discussed earlier if we are only interested in length scales much larger than the lattice spacing then x can be assumed to be continuous. In that case we can treat $s(x)$ as continuum field and write down a continuum Hamiltonian for it. A typical Hamiltonian woule be $H = \int d^3x \, \frac{1}{2}(\nabla s)^2 + Bs^2 + Cs^4 - hs$. Here h is an external magnetic field. This is a gradient expansion. We have seen that typically the nearest neighbour interaction in the dicrete Hamiltonian generates a derivative and next nearest neighbour interaction gives a second derivative term that is multiplied by more powers of a. Thus when a is small we can approximate by keeping just the leading term-which is the sqaure of a single derivative. In fact to the zero'th approximation we can assume s is uniform so that we need not even worry about the gradient term. This means we can effectively work with $M = \int d^3x \; s(x)$ the total magnetization.

Critical phenomena is known to involve long wavelenth fluctuations and to study these we can make this continuum approx. Typically in a ferromagnet there are two phases. Assuming a finite number of egrees of freedom we have : at high temperature, with $h = 0$, $M = 0$ and M is proportional to h in a continuous way. At low temperature, $M \neq 0$ but when $h = 0$ all values of M are equally likely and in equilibrium. If we turn on an infinitesimal h then M aligns with it. So M changes discontinuously when h changes sign. This is a first order phase transition. At a critical temperature T_c $M = 0$

and changing h changes M also in a continuous way. At the critical point there is a second order phase transition. In fact at T_c, M can take any value i.e there is no cost in free energy. That is why there are large fluctuations.

Landau modelled the free energy by (h is the external fld)

$$G(M,H) = A(T) + B(T)M^2 + C(T)M^4 - Mh$$

The minimium is given by

$$\frac{\partial G}{\partial M} = 0 = 2MB(T) + 4C(T)M^3 = h$$

If $h = 0$ either $M = 0$ or $B(T) + 4M^2C(T) = 0$. So $M^2 = -\frac{B(T)}{2C(T)}$.

Clearly if $B(T) > 0$ the only soln is $M = 0$. On the other hand if $B(T) < 0$ we have another soln. So we need $B(T)$ to change sign at $T = T_c$ if we want this to describe a ferromagnet. In particular at $T = T_c$ it must then be zero. So to a first approximation $B(T) = b(T - T_c)$. This immediately gives $M = \sqrt{\frac{b(Tc-T)}{2C(T_c)}}$ for $T \approx T_c$.

This treatment for a ferromagnet can be generalized to many systems. M or $s(x)$ is an order parameter field that characterizes the system. The Landau Ginzburg potential for M that assumes s is uniform is called the Mean Field approximation. The behaviour of the order parameter as a function of temperature is found to be the same for a large class of systems and so we write $M \approx (T_c - T)^\beta$ and β is called a critical exponent. Mean field analysis gives $\boldsymbol{\beta = \frac{1}{2}}$.

In the functional formalism we have approximated the integral that gives the partition fn by a "semi classical" soln that extremizes the classical action. This is mean field theory. We have neglected the thermal fluctuations. Using field theory techniques we can systematically estimate the corrections. Or we can start ab initio and try to solve the full problem using other techniques that do not rely on starting with the Mean field approximation.

The fact that the parameter β is the same in a large class of systems is called universality. Field theory (using Renormalization Group ideas) gives a neat explanation of this phenomenon. It is this that allows us to write down a (continuum) field theory with a *small* number of parameters as a description of the system near the critical point. If there were no universality then we would need a large number of parameters and then the continuum field described by a Hamiltonian itself is not a useful technique.

Continuuing withe same example, when h is non zero, but $T = T_c$ we get

$$4C(T_c)M^3 = h$$

. Thus $M \approx h^{\frac{1}{3}}$. This gives another universal critical exponent $M = h^{\frac{1}{\delta}}$. Thus Mean field theory gives $\boldsymbol{\delta = 3}$.

Similarly $\chi = \frac{\partial M}{\partial h} = \frac{1}{B(T)} = \frac{1}{T - T_c}$ gives the susceptibility - response of M to h. This also diverges at T_c. The critical exponent γ defines this behaviour $\chi \approx (T - T_c)^{-\gamma}$. Mean field theory gives $\boldsymbol{\gamma = 1}$.

We can also ask about the spatial behaviour of $s(x)$. Thus if we turn on a magnetic field $h(x) = h_0\delta^3(x)$ and ask about $s(x)$, mean field theory says we should just solve the classical equation of motion:

$$-(\nabla)^2 s + 2b(T - T_c)s + 4cs^3 = h_0\delta^3(x)$$

The solution to this is given by the Green function $G(x, 0)$ and by now we know that

$$G(x, 0) = \langle s(x)s(0)\rangle$$

The solution is

$$s(x) = \int \frac{d^3k}{(2\pi)^3} \frac{h_0}{|k|^2 + \underbrace{2b(T - T_c)}_{"m^2"}}$$

$$s(x) \approx \frac{h_0}{4\pi} \frac{1}{r} e^{-\frac{r}{\xi}}$$

where $\xi = [2b(T - T_c)]^{-\frac{1}{2}}$ is called the correlation length.

The dependenc of ξ on $T - T_c$ is yet another exponent $\xi = (T - T_c)^{-\nu}$ and MFT gives $\boldsymbol{\nu = \frac{1}{2}}$.

At $T = T_c$, we have a power law fall off $\langle s(x)s(0)\rangle = r^{-(1+\eta)}$ defines the critical exponent η. Mean fld theory gives $\boldsymbol{\eta = 0}$. This is also called the anomalous dimenion of the field. (Note that in d -space dimensions $\langle s(x)s(0)\rangle = r^{-(d-2+\eta)}$ defines η.)

In addition to the statement that these critical exponents are universal, universality makes a stronger statement. It says that

$$\langle s(x)s(0)\rangle = r^{-(d-2+\eta)} f(\frac{r}{\xi})$$

This says for any system (in a universality class) the correlation fucntion has the form given above. The function f (near the critical temperature) depends *only on* $\frac{r}{\xi}$. This follows from the scaling hypothesis that says that all systems depend only on one scale - the correlation length. If there were other scales in the problem, the function would depend on all of them : $f(\frac{r}{\xi}, \lambda_1, \lambda_2...)$ where λ.. are some dimensionless parameters analogous to that in $\lambda\phi^4$ theory, and that depend on the system under consideration. These characterize the microscopic details. The scaling hypothesis says that near a second order phase transition the only quantity we need worry about is the length scale is the scale ξ. This is also experimentally quite successful. Field theory RG arguments and fixed points (thereof) explain why this is so in a fairly natural way.

14 Loops

We now analyse the corrections in detail. Not all diagrams need be calculated. Thus we note the following:

1. We have already seen that vacuum bubbles factorise and equal $\langle 0, \infty | 0, \infty \rangle$ and since this is a normalization factor it cancles out.

2. Disconnected digrams need be evaluated. $Z[J] = e^{W[J]}$. Then

$$Z[J] = 1 + W[J] + \frac{W[J]^2}{2!} + \frac{W[J]^3}{3!} + ...$$

   If we have a connected graph in $W[J]$ then the second term W^2 contains two disconnected copies of this graph and W^3 contains three copies of this graph, and so on. Thus all disconnected graphs can be accounted for in this manner. The conclusion is that we need only calculate connected graphs - which is $W[J]$ and then Z is just the exponential of this.

3. **Vacuum energy**

   $Z_0[0]$ is the vacuum bubble of the free theory. This comes from a determinant.

$$Z_0[0] = Det^{\frac{1}{2}} \frac{2\pi}{A} = e^{-\frac{1}{2} Tr ln \frac{-\partial^2 + m^2}{2\pi}}$$

$$\frac{1}{2}Trln\frac{-\partial^2+m^2}{2\pi} = \frac{1}{2}\int d^4x\langle x|ln\frac{-\partial^2+m^2}{2\pi}|x\rangle$$
$$= \frac{1}{2}VT\int\frac{d^4p}{(2\pi)^4}ln(p^2+m^2)$$

Let

$$I(m^2) = \frac{1}{2}VT\int\frac{d^4p}{(2\pi)^4}ln(p^2+m^2)$$
$$\frac{dI}{dm^2} = \frac{1}{2}VT\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)}$$

Do the contour integral

$$= -\frac{1}{2}VT\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p_0-E_p)(p_0+E_p)}$$
$$= \frac{1}{2}VT\int\frac{d^3p}{(2\pi)^3}\frac{i}{2\sqrt{\vec{p}^2+m^2}}$$
$$I(m^2) = \frac{1}{2}VT\int\frac{d^3p}{(2\pi)^3}i\sqrt{\vec{p}^2+m^2} = \frac{1}{2}VT\int\frac{d^3p}{(2\pi)^3}iE_p$$
$$Z_0[0] \approx e^{-i\frac{1}{2}\underbrace{(V\int\frac{d^3p}{(2\pi)^3}E_p)}_{zero\ point\ energy}T}$$

4. **Propagator corrections**

The two point function has a correction (Do diagrammatically) given by

$$\langle 0|T[\phi(1)\phi(2)]|0\rangle = -\frac{1}{Z_\lambda[0]}\frac{\delta^2 Z[J]}{\delta J(1)\delta J(2)}|_{J=0}$$
$$= -iG_F(x_1-x_2) - \frac{i\lambda}{2}(-i)^3\int d^4x\ G_F(x_1-x)G_F(x-x)G_F(x-x_2)$$

In momentum space:

$$\frac{-i}{k^2+m^2} + \frac{-i}{k^2+m^2}[-i\frac{\lambda}{2}\int\frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2}]\frac{-i}{k^2+m^2}$$

We need to do an integral (we have put back the $i\epsilon$):

$$I = \frac{\lambda}{2} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m^2 - i\epsilon}$$

The location of the poles allows the rotation : $p^0 = ip_E^0$ onto the imaginary axis in an anti clockwise manner, so that p_E^0 goes from $-\infty$ to $+\infty$. So we get

$$i \int \frac{d^4p_E}{(2\pi)^4} \frac{-i}{p_E^2 + m^2}$$

This is called Wick rotation. The integral is divergent and can be cutoff.

$$\int d^4p_E = 2\pi^2 \int p_E^3 dp_E = \pi^2 \int (p_E)^2 d(p_E)^2$$

Let $p_E^2 = \rho$.

$$I = \frac{\lambda}{2} \frac{1}{(4\pi)^2} \int_0^{\Lambda^2} d\rho \ \frac{\rho}{\rho + m^2} = \frac{\lambda}{2} \frac{1}{(4\pi)^2} [\Lambda^2 - m^2 ln \ \frac{\Lambda^2 + m^2}{m^2}] = "\Delta m^2"$$

Our two point function becomes:

$$\frac{-i}{k^2 + m^2} + \frac{-i}{k^2 + m^2}[-i\Delta m^2] \frac{-i}{k^2 + m^2} \approx \frac{-i}{k^2 + m^2 + \Delta m^2}$$

When we include higher order terms we will have an infinite series of such corrections - as shown in the Fig. The geometric series can be summed and then the approximate equality becomes exact. Thus the correction has the effect of changing the mass by an infinite amount.

Thus we have to backtrack: we should not say that m is the mass. Define the physical mass by $m_{phys}^2 = m^2 + \Delta m^2$!.

This is the first step in the programme of "Renormalization". The next step is to show that we can write the final answer (for any sensible experimental question) in terms of m_{phys}^2. Then we can take the experimental value for m_{phys}. But we cannot calculate it from first principles.

5. **Vertex Corrections**

We calculate $\langle 0|T[\phi(1)\phi(2)\phi(3)\phi(4)]|0\rangle$: To leading order we have obtained:

$$-i\lambda \int d^4x\ (-i)G_F(x_1-x)(-i)G_F(x_2-x)(-i)G_F(x_3-x)(-i)G_F(x_4-x)$$

The next order correction is (with the combinatoric factors):

$$\frac{(-i\frac{\lambda}{4!})^2}{2!}4.3.4.3.2.2\int d^4x \int d^4y(-i)G_F(x_1-x)(-i)G_F(x_3-x)$$

$$(-i)G_F(x_2-y)(-i)G_F(x_4-y)[(-i)G_F(x-y)]^2$$

There are also two other diagrams, in which x_1 is paired with x_2 and with x_4. They all lead to the same kind of integrals.

Let us write them in momentum space by writing $G_F(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2+m^2}$

The leading term is (we write $\int_p$ as shorthand for $\int \frac{d^4p}{(2\pi)^4}$)

$$\int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4} e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)}(-i\lambda)$$

$$\int d^4x\ e^{-i(p_1+p_2+p_3+p_4)x}\frac{-i}{p_1^2+m^2}\frac{-i}{p_2^2+m^2}\frac{-i}{p_3^2+m^2}\frac{-i}{p_4^2+m^2}$$

$$= \int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4} e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)}$$

$$\frac{-i}{p_1^2+m^2}\frac{-i}{p_2^2+m^2}\frac{-i}{p_3^2+m^2}\frac{-i}{p_4^2+m^2}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)(-i\lambda) \quad (94)$$

We see the Feynman rules in momentum space at work here: A factor of $\frac{-i}{p^2+m^2}$ for each line, a factor of $-i\lambda$ for each vertex, and a momentum conservation delta function.

The second correction becomes in momentum space:

$$\frac{1}{2!}(\frac{-i\lambda}{4!})^2 4.3.4.3.2.2$$

$$\int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4} e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)}$$

$$\int_{k_1}\int_{k_2}\int d^4x\, e^{-i(p_1+p_3)x}\int d^4y\, e^{-i(p_2+p_4)y}\frac{-i}{p_1^2+m^2}\frac{-i}{p_2^2+m^2}\frac{-i}{p_3^2+m^2}\frac{-i}{p_4^2+m^2}$$
$$e^{ik_1(x-y)}e^{ik_2(x-y)}\frac{-i}{k_1^2+m^2}\frac{-i}{k_2^2+m^2}$$
$$=\int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4} e^{i(p_1x_1+p_2x_2+p_3x_3+p_4x_4)}(2\pi)^4\delta(p_1+p_2+p_3+p_4)$$
$$\frac{-i}{p_1^2+m^2}\frac{-i}{p_2^2+m^2}\frac{-i}{p_3^2+m^2}\frac{-i}{p_4^2+m^2}$$
$$(-i\lambda)^2\frac{1}{2}\int_{k_1}\frac{-i}{k_1^2+m^2}\frac{-i}{(p_1+p_3-k_1)^2+m^2} \tag{95}$$

The same Feynman rules give this expression also, with the additional rule that undetermined momenta are integrated over. Finally we divide by a symmetry factor of 2 for this graph - corresponding to the fact that the two internal lines can be intercghanged. The symmetry factor is $m!$ for interchange of m internal lines between two vertices.

Comparing (94) with (95) we see that the factor $-i\lambda$ is replaced by $(-i\lambda)^2\frac{1}{2}\int\frac{d^4k_1}{(2\pi)^4}\frac{-i}{k_1^2+m^2}\frac{-i}{(p_1+p_3-k_1)^2+m^2}$.

This is called a vertex correction and is a direct correction to the interaction strength λ. The interaction strength is affected by other kinds of correction also indirectly, as will be seen later. This integral is also (logarithmically) divergent as can be seen by counting powers of momenta.

There are two other terms of the same type with p_1+p_3 replaced by p_1+p_2 and p_1+p_4.

6. **Evaluation of the integral: Feynman parameters**

In order to evaluate the integral we use the following parametrization:

$$\frac{1}{A}=\int_0^\infty dt\; e^{-tA}\qquad \frac{1}{B}=\int_0^\infty ds\; e^{-sB}$$
$$\Rightarrow\frac{1}{AB}=\int dt\int ds\; e^{-tA-sB}$$

Let $\tau=t+s$ and $x\tau=t$, $\quad(1-x)\tau=s$ Jacobian $=\tau$

$$\frac{1}{AB} = \int_0^1 dx \int_0^\infty d\tau \ \tau e^{-\tau(xA+(1-x)B} = \int_0^1 dx \ \frac{1}{[xA+(1-x)B]^2}$$

We let $k_1^2 + m^2 = A$ and $p_1 + p_3 - k_1 \equiv P - k_1 = B$. This gives:

$$\int \frac{d^4k_1}{(2\pi)^4} \frac{-i}{k_1^2+m^2} \frac{-i}{(p_1+p_3-k_1)^2+m^2}$$

$$= -\int \frac{d^4k_1}{(2\pi)^4} \int_0^1 dx \ \frac{1}{[(k_1^2+m^2)(1-x)+((P-k_1)^2+m^2)x]^2}$$

Now

$$[(k_1^2+m^2)(1-x)+((P-k_1)^2+m^2)x] = k_1^2 - 2k_1.Px + P^2x + m^2$$

$$= \underbrace{(k_1-Px)^2}_{k'^2} + \underbrace{P^2x(1-x)+m^2}_{M^2} \equiv k'^2 + M^2$$

Plugging this back in and changing integration variables to k' we get:

$$-\int \frac{d^4k'}{(2\pi)^4} \int_0^1 dx \ \frac{1}{[k'^2+M^2]^2}$$

Now performing a Wick rotation, we let $k' = ik'_E$ to get:

$$-i\int_0^1 dx \int \frac{d^4k'_E}{(2\pi)^4} \frac{1}{[k'^2_E+M^2]^2}$$

Letting ${k'_E}^2 = \rho$ as before, we get:

$$-i\int_0^1 dx\, \pi^2 \int \frac{d(k'_E)^2(k'_E)^2}{(2\pi)^4} \frac{1}{[k'^2_E+M^2]^2} = -i\int_0^1 dx \ \frac{\pi^2}{(2\pi)^4} \int_0^{\Lambda^2} d\rho \frac{\rho}{[\rho+M^2]^2}$$

$$= \frac{-i}{(4\pi)^2} \int_0^1 dx \ \{ln \ \frac{\Lambda^2+M^2}{M^2} + M^2[\frac{1}{\Lambda^2+M^2} - \frac{1}{M^2}]\}$$

We can take $\Lambda >> M$ to write

$$\frac{+i}{(4\pi)^2} - \frac{i}{(4\pi)^2} \int_0^1 dx \ ln \ \frac{\Lambda^2}{M^2}$$

Thus the vertex correction is

$$-\frac{i\lambda^2}{2}[\frac{1}{(4\pi)^2} - \frac{1}{(4\pi)^2}\int_0^1 dx\ ln\ \frac{\Lambda^2}{M^2}]$$

with $M^2 = P^2x(1-x) + m^2$. As mentioned earlier we also have two other terms of the same form with $P = p_1 + p_2$ and $P = p_1 + p_4$.

7. **Other Corrections to Scattering:**

Let us consider the other diagrams that contribute to scattering at this order. First there are disconnected graphs. We have already seen that if we evaluate $W[J]$ we need only calculate connected graphs. The disconnected graphs are automatically accounted for when we exponentiate. Similarly vacuum bubble corrections also we are not interested in. The only remaining connected graph that contributes is a propagator correction on one of the legs, for eg.

$$\frac{1}{2}(-i\lambda)^2\int d^4x\ \int d^4y\ (-iG_F(x_1-y))(-iG_F(y-y))(-iG_F(y-x))$$
$$(-iG_F(x_2-x))(-iG_F(x_3-x))(-iG_F(x_4-x))$$

The factor of $\frac{1}{2}$ is the symmetry factor for this graph. This represents a propagator correction on one of the external legs. There are four such graphs - one for each external leg. The evaluation of these graphs does not involve anything new, since it is the same integral encountered earlier. However all these graphs have to be taken into account in a systematic way while calculating scattering amplitudes.

8. **Divergent graphs and power counting:**

We calculate $\langle 0|T[\phi(1)\phi(2)...\phi(N)]|0\rangle$. When we FT to momentum space we get something of the form

$$\prod_{i=1,N}[\int \frac{d^4p_i}{(2\pi)^4}\frac{e^{ip_ix_i}}{p_i^2+m^2}]\mathcal{G}(p_1,p_2,..p_N)\delta^4(p_1+p_2+...+p_N) \qquad (96)$$

We have extracted the external propagators and also an overall mom cons delta function. Let us call $\mathcal{G}$ the amputated Green's function. This is evaluated using the Feynman rules that we have developed. It

has say N_L loop integrals and P propagators. Then "superficial degree of divergence", D = no. of powers of momenta in the numerator - no. of powers in the denominator $= 4N_L - 2P$. By our rules N_L = No. of undetermined momenta = no. of momenta - no. of delta functions. $= P - (V-1)$ where V is the number of vertices in the graph. One of the delta functions imposes overall momentum conservation on the external legs and does not constrain the internal momenta. So

$$D = 4(P - (V-1)) - 2P = 2P - 4(V-1)$$

Now in ϕ^4 theory it is true that $4V = 2P + N$ because each vertex gives 4 lines and out of the total, N are external lines. So $4V - N$ must become internal lines. Two internal lines contract to form a propagator. So $2P = 4V - N$

$$D = 4V - N - 4(V-1) = 4 - N$$

Thus when D is non negative we can expect a divergence: $\int \frac{d^4p}{p^n} \approx \Lambda^{4-n}$. When $n = 4$ we can expect a $ln\ \Lambda$.

Note that this result can be obtained by counting dimensions in (96). Thus the LHS has dimension N (each fld has dim 1). If $[\mathcal{G}]$ denotes the dimension of the amputated green fn we have, counting powers of d^4p and p^2, $N = 2N - 4 + [\mathcal{G}]$. So $[\mathcal{G}] = 4 - N$. Thus amputated green's fn with 0,2,4 external legs can be divergent. 0 corresponds to vacuum bubbles, which we are not interested in for the moment. 2 external legs gives the propgator, which has a quadratic divergence as we explicitly saw. Finally 4 external legs gives the scattering amplitude 2-2. This is logarithmically divergent as we will see explcitly below.

This is only the "superficial" divergence because even a graph with say external legs can be divergent if it contains a subgraph that is divergent. Thus in any graph, we can replace a propagator by the loop corrected propagator, that is quadratically divergent. Similarly one can always have an internal loop corrected $G(1,2,3,4)$ which will be divergent, as we have seen. See Fig. But the point is all divergences are of these two types. This is not trivial to prove, but is intuitively obvious. Such theories are called "renormalizable". Because if we know how to deal with these two divergences, by some clever redefinition (as we did with

mass above) we can get well defined answers for all other scattering amplitudes.

It is interesting to see what would happen to this counting if we add a $\lambda_6\phi^6$ interaction. We still have $D = 4N_L - 2P$, $N_L = P - (V-1)$ and so $D = 2P - 4(V-1)$. But $V = V_4 + V_6$ because we have two types of vertices. Furthermore $4V_4 + 6V_6 = 2P + N$. So we get $D = 2V_6 + 4 - N$. This means the superficial divergence depends not only on the external lines, but also on the number of internal vertices. So a higher order graph with the same external legs will be more divergent. This counting is easy to understand from dimensional arguments. We have, as before $[\mathcal{G}] = 4 - N$. But the graph has dimensional coupling constants λ_6 which has dimensions -2. Thus if we take out the explicit powers of λ_6 in G, we get the contribution to the dimension that comes from the momenta integrals and propagators. Thus $D = [\mathcal{G}] + 2V_6$, which gives the same result for the superficial divergence.

This means that graphs with any number of external legs can be divergent at some high order. This makes the theory "non renormalizable" because it is not enough to redefine two parameters. All scattering amplitudes will continue to be divergent even if we redefine or renormalize a few (finite number) of them. So there is no predictive power.

15 Renormalization

15.1 Philosophy

Having evaluated the integrals and seeing that they are infinite when $\Lambda \to \infty$ we have to figure out a way to make sense of this procedure. Logically one simple solution is to say that Λ is a finite number. The problem is that we don't now what it's value is and since there is no experimental evidence for the discreteness of space-time, Λ had better be very large - much larger than our experimental scales. The question is do we have any predictive power left? If physical quantities depend sensitively on the precise value of Λ, when Λ is large, that means they depend on positive powers of Λ, and then we will not be able to take Λ very large.

So we try the following. Let us pretend that λ, m are just free parameters that have no direct physical or experimental significance. We calculated the one loop corrected mass which is $m^2 + \Delta m^2$ and clalled this the physical mass.

Similarly we calculate the one loop corrected scattering amplitude and call this the physical scattering amplitude. Call this λ_{phys}. These are complicated functions of the original parameter and furthermore involve Λ which should perhaps even be infinite. But if at the end of the day, I can express everything (i.e. various scattering amplitudes that I might want to calculate) in terms of λ_{phys} and m_{phys} without any sign of λ, m, Λ, then we have finite well defined quantities and some predictive power. Actually to require that there should be no sign of Λ is too strong. We don't mind negative powers of Λ. The real requirement is that after we express everything in terms of λ_{phys}, m_{phys} we should be able to take the limit $\Lambda \to \infty$.

For this to work the amplitudes must have the propery that the dependence on the unphysical parameters and Λ is always in some particular form. Otherwise this won't work.

To be precise, let us fix $\Lambda = l_1$ and fix m, λ by computing the mass and 2-2 scattering and checking against experiment. Then we calculate 3-3 scattering and use the particular relation between λ, m and Λ that was fixed by the 2-2 scattering. Repeat the same procedure with $\Lambda = 10l_1$. If the value of 3-3 scattering changes by a large amount, then we say that the theory depends very sensitively on the cutoff. i.e. it doesn't just depend on the physical parameters λ_{phys}, m_{phys} Similarly we can calculate 4-4 scattering and it is likely that this will also depend very sensitively on Λ if the 3-3 scattering does. Typically we will find that *positive* powers of Λ show up in the answers when there is such a sensitive dependence. If there were only negative powers of Λ then the theory would not depend so sensitively on Λ. In particular for instance we would be able to take Λ to infinity and still get finite answers.

Historically the motivation was to take Λ to infinity in the above procedure, because of the belief in the space-time continuum. Theories that allowed that were called renormalizable. That meant that for any given very large Λ the above procedure made sense,i.e. given that mass and 2-2 scattering data is used to fix the two parameters , then all other quantitites could be calculated, and gave sensible finite answers when Λ went to ∞. This is equivalent to saying that they do not depend sensitively on Λ. Of course if the theory had three parameters, then three experimental points are required to fix those parameters for any given Λ and the rest should give results insensitive to Λ.

The $\lambda\phi^4$ theory is renormalizable in this sense. So once we fix λ amd m so that the physical mass is the experimental value and the 2-2 scattering is the experimental value, then all other calculations give well defned results more

or less independent of Λ. That means their dependence on Λ has negative powers only. So in fact we can assume Λ is infinite and get precise predictions.

Note that the mass correction and scattering corrections need not have negative powers of Λ. They can have positive powers and diverge. This is because we can always choose m, λ to make sure we get the experimental numbers no matter what Λ is. We may for instance have to take λ to zero, but since m, λ are free parameters we don't care what values they take.

15.2 Systematic Procedure

The above procedure can be implemented systematically as follows. We start with something we call the "Bare Lagrangian". This is the original (God-given) theory and we have to make sense of this. The parameters of the theory are m_B and λ_B. Thsu

$$\mathcal{L}_B = -\frac{1}{2}(\partial_\mu \phi_B \partial^\mu \phi_B + m_B^2 \phi_B^2) - \frac{\lambda_B}{4!}\phi_B^4 \tag{97}$$

The fields also have a subscript on them. These are clearly not necessary close to the experimental values. We introduce m_R, λ_R as the "Renormalized" values and are (finite and) close to the experimental values and express the bare parameters in terms of these as shown below. We will also define a renormalized field ϕ_R that can differ by a normalization factor from ϕ_B: $\phi_B = \sqrt{Z}\phi_R$.

We write $\mathcal{L}_B$ as the sum of two Lagrangian densities, a renormalized one and whatever remains, called the "counterterm" Lagrangian density.

$$\begin{aligned}
\mathcal{L}_B &= -\frac{1}{2}(\partial_\mu \phi_B \partial^\mu \phi_B + m_B^2 \phi_B^2) - \frac{\lambda_B}{4!}\phi_B^4 \\
&= -\frac{1}{2}Z(\partial_\mu \phi_R \partial^\mu \phi_R + m_B^2 \phi_R^2) - \frac{\lambda_B}{4!}Z^2\phi_R^4 \\
&= \underbrace{-\frac{1}{2}(\partial_\mu \phi_R \partial^\mu \phi_R + m_R^2 \phi_R^2) - \frac{\lambda_R}{4!}\phi_R^4}_{\mathcal{L}_R} + \\
&\underbrace{-\frac{1}{2}(Z-1)(\partial_\mu \phi_R \partial^\mu \phi_R) - \frac{1}{2}\underbrace{(Zm_B^2 - m_R^2)}_{\delta m_R^2}\phi_R^2) - \underbrace{\frac{\lambda_B Z^2 - \lambda_R}{4!}}_{\frac{\delta\lambda_R}{4!}}\phi_R^4}_{\mathcal{L}_{counter\ term}}
\end{aligned}$$

$$= \mathcal{L}_R + \mathcal{L}_{ct} \tag{98}$$

Thus $Zm_B^2 = m_R^2 + \delta m_R^2$ and $Z^2\lambda_B = \lambda_R + \delta\lambda_R$. Calculations are organized as a power series in λ_R. The counterterm parameters δm_R^2, $\delta\lambda_R$ and $Z-1$ are chosen so that physical quantities calculated are finite as $\Lambda \to \infty$ and equal to some experimental numbers. Once the counterterm is fixed (to a given order in λ_R) all physical quantities calculated to that order, will be finite as $\Lambda \to \infty$ since the theory is renormalizable. Furthermore they will depend on m_R, λ_R.

Note that while doing calculations, the quadratic part of $\mathcal{L}_R$ defines the propagator. Everything else is treated as an interaction, to be treated perturbatively. Thus the terms in the counterterm Lagrangian are also to be treated as interaction vertices. Thus $\delta m_R^2\phi^2$, and $(Z-1)\partial_\mu\phi\partial^\mu\phi$ are vertices with two lines coming out, $\lambda_R\phi^4$ and $\delta\lambda_R\phi^4$ have four lines coming out.

15.3 Renormalizing ϕ^4 theory

We apply this procedure now to the ϕ^4 theory for which the one loop graphs have been calculated.

We write $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{ct}$, where $(\delta Z = Z - 1)$,

$$\mathcal{L}_{ct} = -\delta Z \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4$$

These are treated as vertices to be used perturbatively order by order in λ_R.

Thus our equation for mass, including correction is

$$m_R^2 + \frac{\lambda_R}{2}\frac{1}{(4\pi)^2}[\Lambda^2 - m_R^2 ln\ \frac{\Lambda^2 + m_R^2}{m_R^2}] + \delta m^2$$

One possibility is to choose δm^2 so that the m_R^2 is the physical mass (taken from experiment). In that case

$$\delta m^2 = -\frac{\lambda_R}{2}\frac{1}{(4\pi)^2}[\Lambda^2 - m_R^2 ln\ \frac{\Lambda^2 + m_R^2}{m_R^2}]$$

This fixes the counterterm at $O(\lambda_R)$. At higher orders we can expect further modification. There is some freedom in our choice of counterterm. Becuause if we change it by a finite amount then the theory still makes sense. Only the meaning of the parameter m_R^2 changes. The usual choice is

to choose the counterterms δZ and δm^2 in such a way that the propagator has the standard form $\frac{-i}{p^2+m_R^2}$ as $p^2 \to -m_R^2$. In other words it should have a (single) pole at $p^2 = -m_R^2$ and a residue of $-i$.

$$G_F(p) = \frac{-i}{p^2 + m_R^2} + \ terms\ regular\ at\ p^2 = -m_R^2$$

Here m_R is the physical mass of the particle. δZ is non zero starting $O(\lambda_R^2)$.

Thus we have renormalized the propagator at $O(\lambda)$, by adding a counterterm. Now when we calculate any diagram at $O(\lambda^2)$ this counterterm two point vertex will also contribute.

Let us proceed to the 4-point function:

T0 $O(\lambda^2)$ the vertex is

$$-i\lambda_R - \frac{i\lambda_R^2}{2}[\frac{1}{(4\pi)^2} - \frac{1}{(4\pi)^2}\int_0^1 dx\ ln\ \frac{\Lambda^2}{M(1,3)^2}]$$

$$-\frac{i\lambda_R^2}{2}[\frac{1}{(4\pi)^2} - \frac{1}{(4\pi)^2}\int_0^1 dx\ ln\ \frac{\Lambda^2}{M(1,2)^2}]$$

$$-\frac{i\lambda_R^2}{2}[\frac{1}{(4\pi)^2} - \frac{1}{(4\pi)^2}\int_0^1 dx\ ln\ \frac{\Lambda^2}{M(1,4)^2}] - i\delta\lambda$$

with $M(i,j)^2 = (p_i + p_j)^2 x(1-x) + m^2$. We have added the counterterm contribution. Once again while we choose the counterterm to cancel the $ln\ \Lambda$ divergent term, it is possible to add some finite pieces to it. We need some prescription or convention. This is up to us. Note that the correction depends on the invariant momentum in the intermediate state - which is $(p_i + p_j)^2$. So we can adopt the following convention. When the invariant momentum is some fixed value - say $(p_1 + p_2)^2 = (p_1 + p_2)^2 = (p_1 + p_2)^2 = \mu^2$, we let $-i\lambda_R$ be the scattering amplitude. So we choose $\delta\lambda$ to cancel the correction at this value of invariant momentum. Thus

$$\delta\lambda = -3\frac{\lambda_R^2}{2}[\frac{1}{(4\pi)^2} - \frac{1}{(4\pi)^2}\int_0^1 dx\ ln\ \frac{\Lambda^2}{\mu^2 x(1-x) + m^2}]$$

Thus the final result for the renormalized 4-point amplitude is

$$-i\lambda_R - i\frac{\lambda_R^2}{2}\frac{1}{(4\pi)^2}\int_0^1 dx\ \{ln\ \frac{(p_1+p_2)^2 x(1-x) + m^2}{\mu^2 x(1-x) + m^2} +$$

$$ln\ \frac{(p_1+p_3)^2x(1-x)+m^2}{\mu^2x(1-x)+m^2}+\ ln\ \frac{(p_1+p_4)^2x(1-x)+m^2}{\mu^2x(1-x)+m^2}\} \tag{99}$$

We can also call $(p_1+p_2)^2=s$, $(p_1+p_3)^2=t$, and $(p_1+p_4)^2=u$. Thus at $s=t=u=\mu^2$ the scattering amplitude is λ_R. When $s,t,u>\mu^2$, the amplitude increases logarithmically. One can say that the effective λ increases with momentum.

We can also at this point calculate the Bare parameters, although for the purposes of calculating amplitudes in perturbation theory we don't really need them. In the next section we will find a use for them.

At $O(\lambda)$, $Z=1$ so we have $\lambda_B=\lambda_R+\delta\lambda$ and $m_B^2=m_R^2+\delta m^2$. Thus

$$\lambda_B=\lambda_R-3\frac{\lambda_R^2}{2}[\frac{1}{(4\pi)^2}-\frac{1}{(4\pi)^2}\int_0^1 dx\ ln\ \frac{\Lambda^2}{\mu^2x(1-x)+m^2}] \tag{100}$$

$$m_B^2=m_R^2-\frac{\lambda_R}{2}\frac{1}{(4\pi)^2}[\Lambda^2-m_R^2ln\ \frac{\Lambda^2+m_R^2}{m_R^2}] \tag{101}$$

Another scheme

Let us consider a simpler from of the counterterm

$$\delta\lambda=3\frac{\lambda_R^2}{2}\frac{1}{(4\pi)^2}\ ln\ \frac{\Lambda^2}{\mu^2}$$

Here we have chosen the counterterm to just cancel the logarithmic divergence in the amplitude. Since the purpose of adding a counterterm is to make the amplitude finite, we might as well choose a simple form. With this the amplitude takes the form:

$$-i\lambda_R-3i\frac{\lambda_R^2}{2}\frac{1}{(4\pi)^2}-i\frac{\lambda_R^2}{2}\frac{1}{(4\pi)^2}\int_0^1 dx\ \{ln\ \frac{(p_1+p_2)^2x(1-x)+m^2}{\mu^2}$$

$$ln\ \frac{(p_1+p_3)^2x(1-x)+m^2}{\mu^2}+\ ln\ \frac{(p_1+p_4)^2x(1-x)+m^2}{\mu^2}\} \tag{102}$$

The relation between the bare and renormalized parameter becomes:

$$\lambda_B=\lambda_R+3\frac{\lambda_R^2}{2}\frac{1}{(4\pi)^2}\ ln\ \frac{\Lambda^2}{\mu^2} \tag{103}$$

Note that if we set $\mu=\Lambda$, then $\lambda_B=\lambda_R$!

What is λ_B?

Let us go over the sequence of events:

We started out thinking that λ_B was the coupling constant - we didn't have the subscript B then. Then we found that it gives divergent and nonsensical results at one loop. So we decided to abandon the idea of λ_B having any physical significance and decided that the sum of the tree and one loop amplitudes was the physically correct quantity, and this we called λ_R. (Of course the calculation was organized in a different way, but this was what effctively we were trying to do.) Now we see that the difference between λ_B and λ_R is just one of scale. The coupling constant depends on μ and choosing $\mu = \Lambda$ means we choose to define our coupling at the high scale Λ. So that is the meaning of the bare parameter λ_B. It is the amplitude at a very high energy scale. We have already seen that the scattering amplitude grows logarithmically with energy. That is why λ_B is much larger that λ_R. If we take Λ to infinity then in this approximation λ_B is also infinite.

Scheme dependence

We have described two schemes characterized by the choice of counterterms. In one scheme λ_R had a direct physical significance - it was the value of a scattering ampliude at some value of the invariant momenta μ^2. In the other schem λ_R is a parameter that is clearly closely related to the scattering amplitude at μ^2 though not exactly that. This scheme is charaterized by the simplicity of the counterterm - it was just the divergent part of the scattering amplitude. The advantage od this is that the connection with the bare parameter is very direct. The bare and renormalized parameters differ by the value of the renormalization point μ.

We can relate the coupling constants in the two schemes. Let us call λ_{R1} the coupling of the first scheme and λ_{R2} that of the second. The scattering amplitude at any value of p_i have to be equal in both schemes. This gives a relation between the two schemes. Thus if we let $\lambda_{R1} = \lambda_{R2} + a_1\lambda_{R2}^2 + a_2\lambda_{R2}^3 + ...$ then we find (HW problem) (on equating the amplitudes)

$$a_1 = -\frac{1}{2}\frac{1}{(4\pi)^2}[1 + \int_0^1 dx\ \frac{\mu_1^2 x(1-x) + m^2}{\mu_2^2}]$$

Since we only have results upto $O(\lambda_R^2)$ this is sufficient. At higher orders, the other a_i get fixed.

Renormalizability: As we have seen in the section on computing divergences of diagrams, all diagrams with more than 4 external legs are su-

perficially convergent. The only divergences they can have are due to subdivergences, either propagator corrections to the internal lines, or corrections to 4-point vertices. These two become finite (as we have just seen above) on adding the counterterms, that we have calculated (to leading order). So all diagrams with more than 4 external legs are finite once we have made finite the diagrams with 2 and 4 legs. This is the content of the statement that $\lambda\phi^4$ is a renormalizable theory. The proof of this is not trivial however and involves classifying all types of subdivergences, and proving it in a recursive fashion.

15.4 What does "renormalization" mean?

1. Our starting point is the God given Bare Lagrangian:

$$\mathcal{L}_B = -\frac{1}{2}(\partial_\mu \phi_B \partial^\mu \phi_B + m_B^2 \phi_B^2) - \frac{\lambda_B}{4!}\phi_B^4$$

2. Using this Lagrangian we can calculate $\Gamma[\phi_B]$ by evaluating 1PI graphs. This has all the information we need about the theory. Since the momentum integrals are divergent they have to be cutoff with a cutoff Λ. The result is

$$\Gamma[\phi_B] = \sum_n \int_{p_1}\int_{p_2}...\int_{p_n} \Gamma_B^n[p_1, p_2, ..., p_n; m_B, \lambda_B, \Lambda]\phi_B(p_1)\phi_B(p_2)...\phi_B(p_n) \tag{104}$$

   We have indicated the dependence on the bare parameters and the cutoff. What we would like is a finite Γ.

3. Then we reorganize the calculation in terms of a "renormalized" Lagrangian and a "counterterm" Lagrangian

$$\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{ct} \tag{105}$$

   so that order by order the 1PI graphs are finite. First we define

$$\phi_B = \sqrt{Z}\phi_R \tag{106}$$

   Z is chosen to normalize the kinetic term in $\mathcal{L}_R$ correctly. Field normalization change is allowed because the field is a variable of integration. Thus

$$\mathcal{L}_R = -\frac{1}{2}(\partial_\mu \phi_R \partial^\mu \phi_R + m_R^2 \phi_R^2) - \frac{\lambda_R}{4!}\phi_R^4$$

and

$$\mathcal{L}_{ct} = -\frac{1}{2}(Z-1)(\partial_\mu \phi_R \partial^\mu \phi_R) - \frac{1}{2}\delta m^2 \phi_R^2 - \frac{\delta\lambda_R}{4!}\phi_R^4$$

4. Using this Lagrangian we calculate the same $\Gamma[\phi_B]$ again, but now in terms of ϕ_R and λ_R, m_R:

$$\Gamma[\phi_B] = \Gamma[\phi_R] = \sum_n \int_{p_1}\int_{p_2}...\int_{p_n} \Gamma_R^n[p_1, p_2, ..., p_n; m_R, \lambda_R, \mu]\phi_R(p_1)\phi_R(p_2)...\phi_R(p_n) \tag{107}$$

An example of this is (102) where the leading correction to $\Gamma^4[p_1, p_2, p_3, p_4; \lambda_R, m_R, \mu]$ has been calculated.

Comparing (104) and (107), and using (106), we see that

$$\boxed{\boldsymbol{\Gamma_B^n[p_1, p_2, ..., p_n; \lambda_B, m_B, \Lambda] Z^{\frac{n}{2}} = \Gamma_R^n[p_1, p_2, ..., p_n; \lambda_R, m_R, \mu]}} \tag{108}$$

The RHS is finite (if the theory is renormalizable). Thus we have a well defined finite object from which experimental numbers can be predicted - input from experiment being used to first fix the two parameters λ_R, m_R. This summarizes the effect of renormalization.

16 Wilson's Interpretation

1. Why are λ_B and λ_R so different? In fact the difference is infinity if we take $\Lambda = \infty$. The source of the large difference is the sum over intermediate states that one has to do in pertrubation theory in quantum mechanics of the form: $\Delta E \approx \sum_I \frac{\langle f|H_{int}|I\rangle\langle I|H_{int}|i\rangle}{E_I - E_i}$. In this case there are a large number of states that contribute because we are dealing with a field, which always has a large (infinite) number of degrees of freedom. The number of modes per unit volume is $\int \frac{d^3p}{(2\pi)^3}$. This is infinity if there is no cutoff on mode number. The contribution of higher modes is suppressed but not sufficiently. Thus we typically get an integral of the form $\int_0^\infty \frac{dp}{p}$. Note that each decade of mode number contributes the same amount: $\int_\mu^{10\mu} \frac{dp}{p} = \int_{10\mu}^{100\mu} \frac{dp}{p}$. This is the source of the logarithmic divergence.

This explains why λ_R is so different from λ_B. It has to describe the scattering amplitude at a low energy μ but this gets contributions from very high modes also. Note that if the momenta of external particles is of $O(\Lambda)$ also, then we do not get a divergence. We get $O(ln\frac{\Lambda}{p}) \approx ln1 \approx 0$ as the loop correction term.

2. **Wilson's** idea was as follows: Suppose we could construct an "effective action" that has the property that it can describe only low energy phenomena, but can describe these as well as the orginal action. Then, working with this action would be a lot easier because we do not get any log divergences. There are only a few modes in this theory. The reason it is plausible that a theory with fewer degree of freedom can be equivalent to the full theory is because the smaller theory has a very limited range of validity. So it only has the modes that are essential for the problem. On the other hand, even though the theory has fewer degrees of freedom, if it is a very complicated action, then it doesn't help much. What Wilson showed is that renormalizable theories are such that their effective action retains the simple form.

3. His procedure to construct the action was as follows:

   Integrating Out

   In the functional form we can integrate out some degrees of freedom.

   Thus consider a quantum mechanical system described by two variables $x(t), X(t)$

$$K(x_f, X_f, T; x_i, X_i, 0) = \int \mathcal{D}x \ \mathcal{D}X \ e^{iS[x,X]}$$

   If do a Wick rotation $t = i\tau$ we can write this as the density matrix $\rho(x, X; x', X', \beta)$ with $\beta = \frac{1}{kT} = iT$.

   The expectation value of any operator O is given by

$$Tr\rho O = \int dx \ dx' \ dX \ dX' \rho(x, X; x', X') O(x, x'; X, X')$$

   here $O(x, x'; X, X') = \langle x', X'|O|x, X\rangle$. Suppose we are interested in expectation value of some quantities O that depend only on x. Then $O(x, x'; X, X') = O(x, x')\delta(X - X')$. Thus

$$\langle O\rangle = Tr\rho O = \int dx \ dx' \ dX \ dX' \rho(x, X; x', X') O(x, x')\delta(X - X')$$

We can define $\rho_{eff}(x,x') = \int dX \rho(x,X;x',X)$ and write

$$\langle O \rangle = \int dx\ dx' \rho_{eff}(x,x') O(x,x')$$

This is the idea of "integrating out" degrees of freedom that we are not interested in observing. We illustrated this with the density matrix, but it can also be done in the original quantum mechanical problem. Thus in the original problem we know that X wave function is in the ground state at initial and final time, and we are not interested in expectation value of any X dependent quantity we can do the following: Define

$$K_{eff}(x_f,T;x_i,0) = \int dX_f \int dX_i\, \psi_0^*(X_f) K(x_f,X_f,T;x_i,X_i,0)\psi_0(X_i)$$

All quantum mechanical averages involving only x can be done with this K_{eff}. The point is that the integration over X can be done once and for all, if we are not interested in expectation value of X dependent quantities. We need not do it over and over again for each calculation.

This can be represented at the level of the action itself as follows: We are using fld theory notation and suppressing all the wave function infmn. In fld theory the $i\epsilon$ prescription takes care of this.

$$Z[j] = \int \mathcal{D}x\ \mathcal{D}X\ e^{iS[x,X]+jx} = \int \mathcal{D}x e^{iS_{eff}[x]+jx}$$

where:

$$\int \mathcal{D}X e^{iS[x,X]} \equiv e^{iS_{eff}[x]}$$

4. **Field theory** We now apply these ideas to a field theory such as the ϕ^4 theory, with action $S[\phi]$.

   If we want to study scattering of particles with momenta of $O(\mu)$ we need to calculate $\langle 0|T[\phi(p_1)\phi(p_2)....\phi(p_N)]|0\rangle$, with $p_i \approx \mu$. So we can in principle "integrate out" (in the sense explained above) all modes with momenta significantly larger than μ. However it is not possible to integrate them out exactly, so we have to rely on perturbation theory. Furthermore, in perturbation theory, if we integrate out a large region of momenta, say between μ and Λ this will intoduce large logarithms

of the form $ln\ \frac{\Lambda}{\mu}$. Thus we need some other method. Wilson's (and Kadanoff's) idea is to integrate them out in many stages. Thus we integrate out first all modes $\Lambda_1 < |p| < \Lambda$. Get an effective action -say $S^1[\phi]$. Then integrate out modes $\Lambda_2 < |p| < \Lambda_1$. This gives $S^2[\phi]$. Iterate the process:

$$S[\phi] \ \underrightarrow{R}\ S^1[\phi] \ \underrightarrow{R}\ S^2[\phi] \ \underrightarrow{R} .... \underrightarrow{R}\ S^n[\phi]$$

The idea then is to find out what the changes are after one iteration, and then infer from this what the changes are going to be after n iterations, without actually repeating the calculation.

So let us integrate out all modes between Λ and say $\frac{\Lambda}{2}$. This reduces the number of degrees of freedom to half. Thus $\Lambda_1 = \frac{\Lambda}{2}$ and $\Lambda_n = \frac{\Lambda}{2^n}$, at which point the number of DOF is 2^{-n} of the starting number.

In order to do this the first step is to write the action in terms of the high and low momentum modes.

5. **Position Space Interpretation** To get a physical picture of the above let us consider an Ising model on a lattice of spacing $a = \frac{1}{\Lambda}$. We want to reduce the number of degrees of freedom to half. One way is to combine two lattice points into one and put a spin whose value is the sum of the two spins that are being replaced. Thus if the spin values are $1, -1$ to begin with, the new lattice with a spacing of $2a$ will have spins whose values can be $2, 0, -2$. At the next step it will take values $4, 2, 0, -2, -4$ and the spacing will be $4a$. This process can be iterated. If we continue this we get a coarse lattice with a field that can essentially take an infinite discrete number of values - so it can be approximated by a scalar field of the type we have been dealing. This is the order parameter field of Landau. Alternately, we can average three spins at each step and after each step we can specify that we set the spins to be either +1 or -1 depending on whether it is greater than zero or less than zero. This way after each step we get back an Ising model with fewer degrees of freedom and also with a different set of parameters. The meaning of integrating out high modes is physically more transparent here - we clearly see that it corresponds to an averaging or "coarse graining". We start with a fine grained description and progress towards a coarse grained description.

There are standard techniques for obtaining the new set of parameters in terms of the old. We will not do this in this position space formulation. Let us go back to the momentum space formulation.

6. **Back to the field theory**

Let us write $\phi(p) = \phi_1(p) + \chi(p)$ with $\phi_1(p) = \phi(p),\ |p| < \Lambda_1; \phi_1(p) = 0,\ \Lambda_1 < |p| < \Lambda$. And $\chi(p) = \phi(p),\ \Lambda_1 < |p| < \Lambda; \chi(p) = 0,\ |p| < \Lambda_1$.

Thus in the functional integral we integrate over $\chi(p)$ and get an action functional for $\phi_1(p)$. ϕ_1 has fewer modes than ϕ. This process is iterated until we get a field ϕ_n that has only low momentum modes $|p| < \Lambda_n \approx \mu$.

So we write in general: $S[\phi] = S[\phi_1 + \chi] = S[\phi_1] + S[\chi] + \Delta S[\phi_1, \chi]$.

$$\int \mathcal{D}\phi e^{iS[\phi]} = \int \mathcal{D}\phi_1] e^{iS[\phi_1]} \int \mathcal{D}\chi e^{iS[\chi] + \Delta S[\phi_1, \chi]} \equiv \int \mathcal{D}\phi_1 e^{iS^1[\phi_1]}$$

The crucial step is the integration over χ, which of course cannot be done exactly. It has to be done perturbatively - often diagrammatically.

Note that we are primarily interested in the ϕ_1 dependence of S^1. Constant terms are only overall normalizations. They do not affect correlation functions. Consider

$$Z[J; \phi_1, \lambda, m^2] = \int \mathcal{D}\chi e^{iS[\chi] + \Delta S[\phi_1, \chi] + \int d^4x\ J\chi}$$

The action ΔS contains terms with different numbers of χ. The vertices have ϕ_1 in them - in the form of coupling constants (except that they can carry momenta). We can evaluate $Z[J, \phi_1, \lambda, m^2]$ in the usual manner by writing $\chi(x) = \frac{-i\delta}{\delta J(x)}$ acting on $Z_0[J]$ etc as we did earlier. But we don't really need $Z[J, \phi_1, \lambda, m^2]$ - we only need $Z[0, \phi_1, \lambda, m^2]$ because we are not interested in correlators involving χ. Diagrammatically these are diagrams with no external χ's- this means we just evaluate vacuum bubbles. The ϕ_1 can be thought of as coupling constants, but since they represent varying fields, we can just think of these as external background fields, that are fixed - i.e. not dynamical. So our vacuum bubbles will have ϕ_1 fields attached to them.

Furthermore we are interested in writing $Z[0, \phi_1, \lambda, m^2]$ as $e^{i\delta S[\phi_1]}$ to be thought of as a correction to $S[\phi_1]$. Thus we need only the *connected* diagrams.

To be concrete

$$\Delta S = \int d^4x \ [4\phi_1^3\chi + 6\phi_1^2\chi^2 + 4\phi_1\chi^3]$$

Diagrammatically we can classify the vacuum diagrams in terms of the number of external ϕ_1's that are present. If no ϕ_1 are present these are the constant terms of S^1. Thus if only vertices from $S[\chi]$ are used, these have no ϕ_1 in them, so these contribute to the constant term. If there are two ϕ_1's (from ΔS), these contribute to the kinetic term for ϕ_1 in S^1. If there are four ϕ_1's then it contributes to the quartic interaction in S^1.

Let us see what the vertices are and what the diagrams are to leading order Fig:

Fig 8 is a mass correction and is clearly $O(\lambda^2\Lambda^2)$. Fig 9 is the correction to the quartic coupling. The numerical factor is $-\frac{1}{2!}(\frac{\lambda}{4})^2.2.4! = -\frac{3\lambda^2}{2}$. The 2! in the denominator is because it comes from two powers of the $\phi_1^2\chi^2$ interaction term brought down from the exponent. The coefficient of this term is $\frac{6\lambda}{4!} = \frac{\lambda}{4}$. The 2 comes from the two ways of contracting. Finally the 4! is because this corrects $\frac{\lambda}{4!}$ and we want the correction to λ. The integral is the old one:

$$-i\int_0^1 dx \ \frac{\pi^2}{(2\pi)^4}\int_{\Lambda'^2}^{\Lambda^2} d\rho \frac{\rho}{[\rho^2 + M^2]^2}$$

where $M^2 = P^2x(1-x) + m^2$ and $\rho = (k_1 - Px)^2$. $|k_1| > \Lambda$and $|k_1 - Px| > \Lambda$. This gives including the numerical factors, as correction to $-i\lambda$:

$$i\frac{3\lambda^2}{2}\frac{1}{(4\pi)^2}\int_0^1 dx \ ln\frac{\Lambda^2}{\Lambda_1^2}$$

So

$$\lambda \to \lambda - \frac{3\lambda^2}{2}\frac{1}{(4\pi)^2}\int_0^1 dx \ ln\frac{\Lambda^2}{\Lambda_1^2}$$

As expected from the behaviour of the scattering amplitude, λ decreases with decreasing Λ_1.

Notice that the connection between the two couplings is pretty much the same as the connection between the bare and renormalized couplings obtained earlier in the second scheme, if we take $\mu = \Lambda_1$. This establishes the point made about the connection between the Wilsonian way of thinking and the earlier one where we hade a bare coupling and a renormalized coupling. That is to say we can think of the renormalized coupling as the effective coupling at a lower scale.

We can also calculate $\frac{d\lambda}{dln\ \Lambda_1}|_{\Lambda=\Lambda_1}$ to get $\frac{3\lambda^2}{(4\pi)^2}$ which is called the beta function. It tells you the rate of change. More about this below.

17 Renormalization Group

17.1 β function and evolution of coupling

We have seen that scattering amplitudes depend on momentum. This is not surprising at all. We have also seen in the Wilsonian formulation that λ defined as the effective four point coupling chnages with the cutoff. Let us now study this scale dependence in in yet another way. In fact this was how these facts were originally discovered. This come from our definition of the renormalized coupling that had a parameter μ.

What is the meaning of this μ dependence? μ was an arbitrary parameter - that was used to define what we meant by λ_R. Why should a physical amplitude depend on that? The answer is that it shouldn't. But if we change μ the only way the amplitude can remain the same is if λ_R changes! Thus if we change μ, λ_R must change in just such a way that the actual physical scattering amplitude is the same. In this way λ_R becomes an implicit function of μ. This is an important result. The coupling constant becomes scale dependent and this is called the "running of the coupling".

Consider the relation between the bare and renormalized couplings obtained in the second scheme. One way of finding the μ dependence of λ_R is to note that λ_B cannot possibly depend on μ, since it is the original bare parameter. So we can find the μ-dependence by acting on both sides of the equation by $\frac{d}{d\ ln\ \mu}$ and saying that $\frac{d\lambda_B}{dln\ \mu} = 0$. Thus

$$\frac{d\lambda_B}{dln\ \mu} = 0 = \frac{d\lambda_R}{dln\ \mu} + 3\lambda_R \frac{d\lambda_R}{dln\ \mu} \frac{1}{(4\pi)^2}\ ln\ \frac{\Lambda^2}{\mu^2} - 3\lambda_R^2 \frac{1}{(4\pi)^2}$$

$$\Rightarrow \frac{d\lambda_R}{dln\ \mu}[1 + 3\lambda_R \frac{1}{(4\pi)^2}\ ln\ \frac{\Lambda^2}{\mu^2}] = 3\lambda_R^2 \frac{1}{(4\pi)^2}$$

$$\Rightarrow \frac{d\lambda_R}{dln\ \mu} = 3\lambda_R^2 \frac{1}{(4\pi)^2} \frac{1}{[1 + 3\lambda_R \frac{1}{(4\pi)^2}\ ln\ \frac{\Lambda^2}{\mu^2}]}$$

Since the calculation is only accurate to $O(\lambda_R)^2$ we get

$$\beta_\lambda \equiv \frac{d\lambda_R}{dln\ \mu} = 3\lambda_R^2 \frac{1}{(4\pi)^2} \tag{109}$$

This is called the β function. It tells us how the λ_R should change as we change the renormalization point μ. Notice that because $\beta > 0$, λ_R increases with μ. Since λ_R is a measure of the strength of interaction at $s \approx t \approx u \approx \mu$, it means that the theory becomes strongly interacting at high energies. This is also clear from the 4 -point scattering vertex calculation. We see from the one loop correction increases with p^2.

The fact that λ_R increases with μ can be made more dramatic by solving the differential equation

$$\frac{d\lambda_R}{dln\ \mu} = 3\lambda_R^2 \frac{1}{(4\pi)^2}$$

$$\Rightarrow \frac{d\lambda_R}{\lambda_R^2} = \frac{3}{(4\pi)^2} d\ ln\ \mu$$

$$\frac{1}{\lambda_{R1}} - \frac{1}{\lambda_{R2}} = \frac{3}{(4\pi)^2} ln\ \frac{\mu_2}{\mu_1}$$

where $\lambda_{R1} = \lambda_R(\mu_1)$. This gives the renormalization group (RG) trajectory of λ_R as a function of μ. One can also introduce a mass scale M defined by

$$\frac{1}{\lambda_{R1}} + \frac{3}{(4\pi)^2} ln\ \mu_1 = \frac{1}{\lambda_{R2}} + \frac{3}{(4\pi)^2} ln\ \mu_2 = \frac{3}{(4\pi)^2} ln\ M$$

$$M = \mu_1 e^{\frac{(4\pi)^2}{3\lambda_{R1}}} = \mu_2 e^{\frac{(4\pi)^2}{3\lambda_{R2}}}$$

Clearly M is a mass that characterizes the (RG) trajectory. In fact it can be defined as the scale μ at which $\lambda_R = \infty$. It is called the "Landau pole", after Landau who argued that the theory doesn't make sense as a field theory because of this. (Quantum Electrodynamics also has a similar problem.) At the other end of the trajectory, at $\mu = 0$, $\lambda = 0$ and we get a free non-interacting theory.

The scale M characterizes this theory. One should remember that classically the theory was scale invariant (except for a mass term). Even with a mass term it should have become scale invariant at high energies where the mass is negligible. Instead we have a situation where the coupling constant changes with scale, and in fact blows up at some high scale M. $M = \mu_1 e^{\frac{(4\pi)^2}{3\lambda_{R1}}} >> \mu_1$ is a large number if $\lambda_{R1} << 1$. This scale M has nothing to do with the physical mass m_R and would have more or less this value even if $m_R = 0$ (m_R does not enter the formula for M.)

Furthermore, we see that λ_R is not really a free parameter in this theory. Two theories with different λ_R are simply related by an overall change of scale! The only difference between the two theories being an overall change of scale, we can say that the theory is characterized by a scale M rather than by the parameter λ_R! Thus a dimensionless free parameter is replaced by a dimensionful parameter - a mass. This is called "dimensional transmutation". It can be traced to the fact that the quantum theory violates scale invariance, even though the classical theory had scale invariance. This in turn can be traced to the UV divergences that necessitated the introduction of a UV cutoff Λ.

When quantum effects violate a classical symmetry, it is called an "anomaly". In this case it is an anomaly in scale invariance.

17.2 Wilson's Renormalization group, Universality in Critical Phenomena

1. Near the critical temperature (of, say, a ferromagnet) the correlation length becomes very large. This means that a large number of DOF (spins) interact with and influence each other. So we have a problem tailor made for a field theory. Furthermore what is found is that many results (eg critical exponents) do not depend on details of the microscopic Hamiltonian and are the same for a large class of systems - these systems are said to belong to the same universality class. This independence of the microscopic details is called **universality**.

   As explained above, in understanding the dynamics of a field theory at energies much lower than the cutoff, the RG helps. The idea is integrate out degrees of freedom and get a theory with fewer number of DOF. We should integrate out all modes with wave number much larger than the inverse correlation length. In other words we have to bring the cutoff

down far enough that only a few DOF are left in a volume of size one correlation length. Once we have such a Hamiltonian calculations are easy and perturbation theory will converge quite fast.

It is very useful to do the RG transformation in some detail to see what exactly the steps are. What we have done in the previous section is a schematic calculation that indicates how the coupling constant changes. We have also related this to the earlier renormalization program involving counterterms etc.

We describe the procedure in some detail. But for simplicity we do it for the free theory first. We then do the same for the ϕ^4 term, but again we neglect the coupling between the high and low modes. This simplified setup is very instructive. We then incorporate the effects of the cross coupling between the high and low modes. For this we use the results of the previous section where we diagrammatically calculated the change in λ.

2. We consider a scalar field on a lattice exactly like the one at the beginning of these lectures. [3] We will work in Euclidean space. So we can take over the results from the first chapter of these notes, but discard the time dependence. We started with a Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N}(q_{i+1} - q_i)^2$$

After a mode expansion $q_m = \sum_K (Q_K e^{imK} + cc)$, we get (upto some numerical factors):

$$H \approx N \sum_K \omega_K^2 Q_K Q_{-K}$$

where $\omega_K = 2sin\frac{K}{2}$ and $K = \frac{2\pi n}{N},\ n = 1, 2.., N$. Note that $Q_{-K} = Q_K^*$. Also if N is large and $n << N$, we have $\omega_K = K$. (Note that $-\pi < K < \pi$.) Motivated by this we work with

$$H \approx N \sum_K K^2 Q_K Q_{-K}$$

Furthermore we add a term $r^2 \sum_m q_m^2$ which makes H:

$$H \approx N \sum_K (K^2 + r^2) Q_K Q_{-K}$$

[3] The best reference for this is the Phys Reports by Kogut and Wilson

This is the crystal lattice version of a free massive scalar. Finally we go over to continuum notation and write

$$H \approx N^2 \int_0^1 d^D K \ (K^2 + r^2) Q_K Q_{-K} \approx \int_0^1 d^D K \ (K^2 + r^2) \phi(K) \phi(-K)$$

We have kept the upper limit as 1 rather than π - this is a matter of rescaling. Finally note that $K = \frac{2\pi n}{N} = \frac{2\pi n a}{L} = ka = \frac{k}{\Lambda}$. Here k is the physical momentum, a is the lattice spacing and L is the size of the crystal and we have set $a = \frac{1}{\Lambda}$. Thus K is the physical momentum measured in units of the cutoff-which is why the max value is 1. This being the case we can set $m = r\Lambda$ and think of m as the physical mass. Thus r is the mass measured in units of the cutoff.

3. Now we perform the integrating out - we integrate all modes with momenta between $\frac{1}{2}$ and 1. In physical units, from $\frac{\Lambda}{2}$ to Λ. Since we are considering the free theory there are no cross terms - so $\Delta S[\phi_1, \chi] = 0$. Integrating χ thus only gives an overall normalization factor, and $S[\phi_1]$ is unmodified. Thus we get

$$Z = \int \mathcal{D}\phi_1(K) e^{i \int_0^{\frac{1}{2}} d^D K \ (K^2 + r^2) \phi_1(K) \phi_1(-K)}$$

In order to figure out how the parameters of the new action [4]are related to those of the old one. In order to do that we must rescale the momentum so that the range of integration is $0 - 1$. This will also entail a rescaling of the field so that the k^2 part is unchanged. Thus let $k' = 2k$ and $\phi_1 = \zeta\phi$. We get

$$S^1[\phi] = 2^{-D}\zeta^2 \int_0^1 d^D K' \ (\frac{K'^2}{4} + r^2) \phi(K') \phi(-K')$$

$$= \frac{2^{-D}\zeta^2}{4} \int_0^1 d^D K' \ (K'^2 + 4r^2) \phi(K') \phi(-K')$$

We now set $\zeta = 2^{1+\frac{D}{2}}$ so that

$$S^1[\phi] = \int_0^1 d^D K' \ (K'^2 + 4r^2) \phi(K') \phi(-K')$$

[4]We sometimes refer to it as the Hamiltonian. It can equally well be considered to be the (Euclidean) action.

Thus we get back the original action with a parameter $r' = 2r$. This is the only effect of the transformation. This is easy to understand. $r = \frac{m}{\Lambda}$. We have rescaled the momenta so that now it is expressed in terms of the new cutoff $\Lambda' = \frac{\Lambda}{2}$. Thus $r' = \frac{m}{\Lambda'} = 2r$. Thus it just follows from dimensional analysis. This naive dimensional works here because we are not considering interactions. Interactions can modify the "engineering" dimensions (naive dimensions) of operators.

4. In the context of critical phenomena of say spin systems, the "mass" m, should be thought of as $\frac{1}{\xi_{phys}}$ - the inverse physical correlation length. r can be thought of as the inverse correlation length in units of the lattice spacing. Thus we can expect that after blocking together spins to get a lattice of spacing $2a$, the correlation length becomes half (measured in units of the new lattice spacing.) This is of course what happens.

   Parameters such as r that grow bigger after an RG transformation are termed "**relevant**" parameters. They become more important after a few RG transfromations. Note that r^2 in the Landau Ginzburg free energy was $T - T_c$. Thus after some blocking the theory looks less and less critical - because th effective $T - T_c$ gets bigger - in umits of the cutoff momentum. After some blocking the correlation length comes down to around 1 (in units of the lattice). At this point the theory has become quite simple to handle because there are few degrees of freedom that one needs to worry about in any calculation.

5. One can do the same analysis for the ϕ^4 term. In momentum space it is

$$\frac{\lambda}{4!}\int_0^1 d^D p_1 \int_0^1 d^D p_2 \int_0^1 d^D p_3 \int_0^1 d^D p_4 \, \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\delta(p_1+p_2+p_3+p_4)$$

   We split this into high and low momentum fields as before. Let us concentrate on the low momentum piece and do the rescalings. The low momentum piece is

$$S[\phi_1] = \frac{\lambda}{4!}\int_0^{\frac{1}{2}} d^D p_1 \int_0^{\frac{1}{2}} d^D p_2 \int_0^{\frac{1}{2}} d^D p_3 \int_0^{\frac{1}{2}} d^D p_4$$

$$\phi_1(p_1)\phi_1(p_2)\phi_1(p_3)\phi_1(p_4)\delta(p_1+p_2+p_3+p_4)$$

We rescale and write $p' = 2p$ as before and $\phi_1 = \zeta\phi$.

$$S^1[\phi] = \zeta^4 2^{-3D}\frac{\lambda}{4!}\int_0^1 d^D p'_1 \int_0^1 d^D p'_2 \int_0^1 d^D p'_3 \int_0^1 d^D p'_4$$
$$\phi(p'_1)\phi(p'_2)\phi(p'_3)\phi(p'_4)\delta(p'_1 + p'_2 + p'_3 + p'_4)$$

The momentum conservation delta function soaks up one of the momentum integrals. Substituting $\zeta = 2^{1+\frac{D}{2}}$ we get

$$S^1[\phi] = 2^{4-D}\frac{\lambda}{4!}\int_0^1 d^D p'_1 \int_0^1 d^D p'_2 \int_0^1 d^D p'_3 \int_0^1 d^D p'_4$$
$$\phi(p'_1)\phi(p'_2)\phi(p'_3)\phi(p'_4)\delta(p'_1 + p'_2 + p'_3 + p'_4)$$

This is of course only part of the contribution to $S^1[\phi]$. The effect of the interactions we have already seen earlier and has to be included. We will do this later. What we see is that $\lambda' = 2^{4-D}\lambda$. This when $D < 4$ it is a relevant parameter just like r. On the other hand for $D > 4$ it gets smaller with each RG transformation. These are called "**irrelevant**" parameters - precisely because they get smaller and smaller, so in the low energy action it has no effect. Precisely at $D = 4$ it doesn't change and is called a "**marginal**" parameter. This analysis is again just a restatement of dimensional analysis. We know that in 4 dimensions λ is dimensionless, which means it doesn't scale. In three dimensions it has dimensions of mass. So clearly it gets bigger when measured in units of the cutoff. In nore than 4 dimensions it has length dimensions, so it gets smaller (measured in units of the cutoff).

The same result can be obtained more simply by taking the ϕ^4 theory in x-space and scaling $x = bx'$ (length specified in units of the new lattice spacing, is x' and is smaller), $\phi = \frac{\phi'}{b}$. The mass parameter m^2 becomes b^2m^2 and λ does not scale.

Once again we caution the reader that naive dimensional analysis works because we are not including the effect of interactions. Below we will see some effects of interactions.

6. Thus we see that some coefficients get smaller and smaller. In fact if we consider a general Hamiltonian for an order parameter field, in any dimension, only the first few powers are relevant. All higher powers are irrelevant. Thus in 4 dimensions, (if we neglect intarctions, or if λ

is very small) all ϕ^n for $n > 4$ are irrelevant. So that means the low energy dynamics is governed by one or two parameters, no matter what the initial Hamiltonian. (The coefficient of the ϕ^2 term is proportional to $T - T_c$ and is tuned by the experimenter - it does not depend on the microscopics). This is the basis of **universality** in critical phenomena. Near the critical point because the correlation function is large we need to work with the long distance effective Hamiltonian that is obtained by this RG process. The details of the microscopic theory determine the starting Hamiltonian. But only one or two of the lowest order operators of the original Hamiltonian are important in determining the form of the final Hamiltonian. Two Hamiltonians differing in all other higher order terms will have identical long distance behaviour - i.e. near the critical point. This is the universality that is observed in critical phenomena.

7. In the above discussion we have neglected the effect of the interactions. Let us indicate qualitatively what happens when we include the effect of interactions. We saw above that $\lambda' = 2^{4-D}\lambda$. When $4 - D \approx 0$ [5] We have:

$$\delta\lambda = (2^{4-D} - 1)\lambda \approx ln\ 2(4 - D)\lambda$$

This does not include the effect of the interactions. But we have seen that the change due to the interactions (from the one loop graph) is $\delta\lambda \approx -\frac{3\lambda^2}{(4\pi)^2} ln \frac{\Lambda}{\Lambda'}$. (The sign is negative - the coupling decreases at low energies.)This calculation was done in 4 dimensions. So we assume that D is close to 4. In the above calculation $\frac{\Lambda}{\Lambda'} = 2$ and including this effect we have

$$\delta\lambda \approx ln\ 2(4 - D)\lambda - \frac{3\lambda^2}{(4\pi)^2} ln\ 2$$

This modifies the situation somewhat. Earlier we had concluded that for $D < 4$, λ is a relevant parameter. Now we see that this depends on the value of λ. In fact there is some value of λ that makes $\delta\lambda = 0$. This satisfies

$$(4 - D)\lambda^* - \frac{3\lambda^{*2}}{(4\pi)^2} = 0 \Rightarrow \lambda^* = 0\ or\ \lambda^* \approx \epsilon \frac{(4\pi)^2}{3}$$

[5] D is the space dimensionality here and is actually an integer- usually 3. However for mathematical purposes we assume that it is a real number close to 4. This is called the ϵ ($\epsilon = 4 - D$) expansion. At the end of the calculation we can set $\epsilon = 1$.

The above calculation is approximate. We have used the continuum one loop calculation results. This is not quite the same as integrating out from Λ' to Λ. Also the effect of the mass term is not taken into account. See the original papers (such as Phys Rep by Kogut and Wislon) for a more detailed calculation.

These values are called fixed points. So when $D < 4$ there is a non trivial fixed point. Once the coupling has that value it doesn't change and is exactly marginal. In the above example if $\lambda > \lambda^*$, $\delta\lambda < 0$, so λ decreases and approaches the fixed point. Similarly if $\lambda < \lambda^*$, $\delta\lambda > 0$ and once again it approaches the fixed point. So either way at low energies $\lambda \to \lambda^*$.

The other fixed point $\lambda = 0$ is called the trivial fixed point (or the Gaussian fixed point). If $\lambda > 0$ it flows to λ^* under RG transformations. This analysis is for $D < 4$. For $D \geq 4$ the only known fixed point is the Gaussian one.

This fact makes universality even stronger. Not only are the irrelevant couplings unimportant, but even the consequences of the $\lambda\phi^4$ term in the microscopic theory is independent of λ, because it always ends up at the fixed point value λ^*! This also explains the scaling hypothesis described in section 14 that says that the only scale in the problem near the critical temperature is ξ and there is some universal function $f(\frac{r}{\xi})$ that describes correlation functions. This means that there is no free parameter such as λ that the function depends on. This is now explained by saying that $\lambda \approx \lambda^*$ near the critical temperature and is some fixed number - not a free parameter.

17.3 Callan-Symanzik Equation

17.3.1 Renormalization Group eqn

The Callan Symanzik equation is a useful way to extract the consequences of RG for correlation functions. Our starting point is this equation derived earlier:

$$\Gamma^n_B[p_1, p_2, ..., p_n; \lambda_B, m_B, \Lambda] Z^{\frac{n}{2}} = \Gamma^n_R[p_1, p_2, ..., p_n; \lambda_R, m_R, \mu] \tag{110}$$

Write this as:

$$\Gamma^n_B[p_1, p_2, ..., p_n; \lambda_B, m_B, \Lambda] = Z^{-\frac{n}{2}} \Gamma^n_R[p_1, p_2, ..., p_n; \lambda_R, m_R, \mu] \tag{111}$$

and observe that the LHS does not depend on μ, it only depends on Λ.

It follows that

$$\mu\frac{d}{d\mu}\Gamma_B^n[p_1,p_2,...,p_n;\lambda_B,m_B,\Lambda]=0$$

$$\implies \mu\frac{d}{d\mu}[Z^{-\frac{n}{2}}\Gamma_R^n[p_1,p_2,...,p_n;\lambda_R,m_R,\mu]]=0$$

This gives a constraint on Γ^R. Now define

$$\mu\frac{d}{d\mu}\ln\sqrt{Z}=\gamma_\phi \tag{112}$$

This is called the anomalous dimension of the field ϕ.

$$\mu\frac{d}{d\mu}m_R=m_R\gamma_m \tag{113}$$

This is the anomalous dimension of the operator ϕ^2.

Then using the chain rule

$$\mu\frac{d}{d\mu}[Z^{-\frac{n}{2}}\Gamma_R^n[p_1,p_2,...,p_n;\lambda_R,m_R,\mu]]=$$

$$Z^{-\frac{n}{2}}[-n\gamma_\phi+\mu\frac{d\lambda}{d\mu}\frac{\partial}{\partial\lambda}+\mu\frac{dm_R}{d\mu}\frac{\partial}{\partial m_R}+\mu\frac{\partial}{\partial\mu}][\Gamma_R^n[p_1,p_2,...,p_n;\lambda_R,m_R,\mu]]=0$$

$$\implies [-n\gamma_\phi+\beta_\lambda\frac{\partial}{\partial\lambda}+m_R\gamma_m\frac{\partial}{\partial m_R}+\mu\frac{\partial}{\partial\mu}][\Gamma_R^n[p_1,p_2,...,p_n;\lambda_R,m_R,\mu]]=0 \tag{114}$$

This is a **Renormalization Group** equation for Γ^R

17.3.2 Callan Symanzik equation

Now combine with **simple dimensional analysis** to get it in a different form: Let $[x]$ denote the mass dimension of x. Then

$$[p]=1,[m]=1,[\mu]=1$$

From the form of the kinetic term in the action which is dimensionless: $\int d^Dx\ \phi(x)\Box\phi(x)$, we conclude that

$$[\phi(x)]=\frac{D-2}{2}$$

and from $\phi(p) = \int d^D x\ \phi(x) e^{-ipx}$ we conclude that

$$[\phi(p)] = [\phi(x)] - D = -\frac{D+2}{2}$$

Let dimension of $\Gamma(p_1, ...p_n)$ be b:

$$\Gamma[\phi] = \sum_n \int d^D p_1 \int d^D p_2 .... \int d^D p_{n-1} \Gamma(p_1, ...p_n; m, \lambda, \mu)\phi(p_1)....\phi(p_n)$$

Since $\Gamma[\phi]$ is dimensionless we have:

$$b + (n-1)D + n(-\frac{D+2}{2}) = 0 \implies b = D + n - \frac{nD}{2}$$

For our case $D = 4$ so $b = 4 - n$. Then dimensional analysis tells us that if we perform the scaling:

$$p \to tp, m \to tm, \mu \to t\mu \qquad \implies \Gamma \to t^b \Gamma$$

i.e.

$$[p_i \frac{\partial}{\partial p_i} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu}]\Gamma(p_1, .., p_n; m, \lambda, \mu) = b\Gamma(p_1, .., p_n; m, \lambda, \mu)$$

It can also be written as

$$[t\frac{\partial}{\partial t} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} - b]\Gamma(tp_1, .., tp_n; m, \lambda, \mu) = 0 \tag{115}$$

Combining (115) with (114) and eliminating $\mu\frac{\partial}{\partial \mu}$ gives:

$$\boxed{[-\boldsymbol{t\frac{\partial}{\partial t}} + (\boldsymbol{\gamma_m - 1})\boldsymbol{m\frac{\partial}{\partial m}} + \boldsymbol{\beta_\lambda \frac{\partial}{\partial \lambda}} + \boldsymbol{b - n\gamma_\phi}]\boldsymbol{\Gamma^n_R(tp_1, .., tp_n; m_R, \lambda_R, \mu) = 0}} \tag{116}$$

This is the Callan-Symanzik equation and relates $\Gamma(tp_1, .., tp_n; m, \lambda, \mu)$ at different momenta rather than at different μ.

17.3.3 Solution to C-S eqn

Since the equation relates changes in t to changes in λ, m and overall normalization one can expect the solution to be of the form

$$\Gamma(tp_1, .., tp_n; m, \lambda, \mu) = f(t)\Gamma_0(p_1, .., p_n; m(t), \lambda(t), \mu)$$

Applying $t\frac{\partial}{\partial t}$ and using the chain rule one gets

$$\frac{\partial \Gamma}{\partial t} = \frac{df}{dt}\Gamma_0 + f(\frac{dm}{dt}\frac{\partial \Gamma_0}{\partial m} + t\frac{d\lambda}{dt}\frac{\partial \Gamma_0}{\partial \lambda})$$

$$t\frac{\partial \Gamma}{\partial t} = t\frac{1}{f}[\frac{df}{dt}\Gamma + f(\frac{dm}{dt}\frac{\partial \Gamma}{\partial m} + t\frac{d\lambda}{dt}\frac{\partial \Gamma}{\partial \lambda})]$$

$$[-t\frac{\partial}{\partial t} + t\frac{1}{f}\frac{df}{dt} + t\frac{dm}{dt}\frac{\partial}{\partial m} + t\frac{d\lambda}{dt}\frac{\partial}{\partial \lambda}]\Gamma(tp_1, .., tp_n; m, \lambda, \mu) = 0 \tag{117}$$

Comparing (117) with (116) we see that they are the same if we identify

$$t\frac{d\lambda}{dt} = \beta_\lambda(\lambda, m), \quad t\frac{dm}{dt} = m(\gamma_m(\lambda, m) - 1), \quad t\frac{1}{f}\frac{df}{dt} = b - n\gamma_\phi(t) \tag{118}$$

In other words we need to solve the above and obtain, $\lambda(t), m(t)$ and then solve for $f(t)$. The solution is thus:

$$\boxed{\boldsymbol{\Gamma(tp_1, .., tp_n; m, \lambda, \mu) = t^b e^{-\int_0^t \frac{n\gamma_\phi}{t} dt} \Gamma(p_1, p_2, ..., p_n; m(t), \lambda(t), \mu)}} \tag{119}$$

b is the engineering dimension and $-n\gamma_\phi$ is the anomalous dimension.

18 Quantum Electrodynamics

The EM field is described by the four-vector A^μ. This has 4 components. But we know that the em field has only 2 DOF's - two polarizations of the em wave. So the question is how does this happen. We are forced to have 4 components because that is the smallest representation of the Lorentz group after the scalar - which has one, and we know the structure of the theory from classical em. Somehow two DOF's have to drop out. This is best seen in the canonical analysis. But actual quantization is easier in the functional formalism.

We start with the action

$$S = -\frac{1}{4}\int d^4x \; F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu$$

To make contact with our standard definitions in electrostatics where $\vec{E} = -\vec{\nabla}\phi$, we let $A^0 = \phi = -A_0$ so $E_i = -\partial_i A^0 = +\partial_i A_0$. In general then $E_i = F_{io} = \partial_i A_0 - \partial_0 A_i$. $J^0 = \rho$ is the charge density.

18.1 Canonical Analysis

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu \\
&= -\frac{1}{2}F_{0i}F^{0i} - \frac{1}{4}F_{ij}F^{ij} + [J_0A^0 + J_iA^i] \\
&= \frac{1}{2}F_{0i}F_{0i} - \frac{1}{4}F_{ij}F^{ij} + [J_0A^0 + J_iA^i] \\
&= \frac{1}{2}(\partial_0 A_i)^2 + \frac{1}{2}(\partial_i A_0)^2 - \partial_i A_0 \partial_0 A_i - \frac{1}{4}F_{ij}F_{ij} + [J_0A^0 + J_iA^i] \qquad (120)
\end{aligned}$$

canonical momenta are:

$$\begin{aligned}
\Pi_i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial_0 A_i - \partial_i A_0 = F_{0i} = -E_i \\
\Pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \qquad (121)
\end{aligned}$$

The above is a ("primary") constraint - there is no momentum conjugate to A_0.

The Hamiltonian is:

$$\mathcal{H} = \Pi_i \dot{A}_i - \mathcal{L}$$

Use $\dot{A}_i = \Pi_i + \partial_i A_0$ to get

$$\mathcal{H} = \frac{1}{2}\Pi_i^2 + \Pi_i \partial_i A_0 + \frac{1}{4} F_{ij} F_{ij} - [J_0 A^0 + J_i A^i]$$

$$= \frac{1}{2}\Pi_i^2 - A_0 \partial_i \Pi_i + \frac{1}{4} F_{ij} F_{ij} + [J^0 A^0 - J_i A^i]$$

We have integrated by parts and dropped a total divergence. Using $\{A^0, \Pi_0\}_{PB} = 1$ If we now calculate

$$\{\mathcal{H}, \Pi_0\}_{PB} = \partial_i \Pi_i + J^0 = 0$$

$$\Rightarrow \partial_i E_i = J^0$$

This must also be satisfied. This is another ("secondary") constraint. In fact this is just Gauss Law.

Once both constraints are obtained using equations of motion, then the constraints must be consistent with EOM:

$$\dot{A}_i = [A_i, H]_P$$

This means, the constraints χ_N must satisfy:

$$[\chi_N, H] = 0$$

A constraint is **first class** if they all commute with each other $[\chi_N, \chi_M]_P = 0$ (after calculating PB and then imposing constraints, the PB should vanish).

These can be understood as symmetries of H under (gauge) transformations

$$\delta_N A = \epsilon_N [\chi_N, A]$$

$\Pi_0, \partial_i \Pi_i + J_0$ are two first class constraints. If we could solve for A_0 by using $\partial_i \Pi_i + J^0 = 0$ we could reduce the variables. But this equation has no time derivatives - if satisfied at $t = 0$ it is always satisfied identically:

$$\partial_0 [\partial_i \frac{\partial L}{\partial F_{i0}} - J^0] = -\partial_0 [\partial_i \frac{\partial L}{\partial F_{0i}} + J^0] = \partial_i \partial_j \frac{\partial L}{\partial F_{ij}} - \partial_i J_i - \partial_0 J^0$$

$$= 0 - \partial_\mu J^\mu = 0$$

(Using current conservation.) This is of course due to gauge invariance. Soln is to impose a gauge condition eg $\partial_i A_i = 0$ Coulomb gauge is one choice. Now we can solve for A_0

$$-\partial_i \partial^i A^0 = J^0 \implies A^0 = \int d^3y \frac{J^0}{4\pi|x-y|}$$

Now the constraints are $\partial_i \partial^i A^0$ and $\partial_i \Pi_i + J^0$. These do not commute : **Second Class**. Use Dirac brackets. Or eliminate extra degree of freedom.

We will come back to canonical quantization later.

18.2 Green's Function and Functional Formalism

The EOM is

$$\partial^\mu F_{\mu\nu} = \partial^\mu[\partial_\mu A_\nu - \partial_\nu A_\mu] = -J_\nu$$
$$\Rightarrow [\partial^\mu \partial_\mu g_{\nu\rho} - \partial_\rho \partial_\nu] A^\rho = -J_\nu$$

This wave operator has no inverse, hence the Green' function does not exist. To see that it has no inverse, we observe that it has a zero mode (i.e. there is a solution to the homogeneous equation) so the determinant vanishes. $A_\mu = \partial_\mu \Lambda$ solves the homogeneous eqn for any function $\Lambda(x,t)$. This means that given J_μ , A_μ is not uniquely fixed. $A_\mu + \partial_\mu \Lambda$ is also a solution. These are "gauge transformations". $F_{\mu\nu}$ and hence the action and EOM are invariant under this gauge transformation.

The same fact shows up in a different way in the fnl integral

$$Z[0] = \int \mathcal{D}A^\mu(x) e^{iS[A]}$$

Since any gauge tranformed configuration gives the same value for the action, There is an infinite factor in Z corresponding to the volume of the gauge group - *at each space time point.*

Gauge fixing illustration:

To illustrate this consider a simpler problem involving two variables:

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; f(x_1, x_2)$$

Where f is a fn that is symmetric under translations of x. $f(x_1, x_2) = f(x_1 + a, x_2 + a)$. Clearly it depends on the difference $x_1 - x_2$ only.[6] This can

[6] An example to keep in mind is $f = e^{-(x_1 - x_2)^2}$.

be written more compactly as $f(x) = f(x^a)$. We assume that $x^a \equiv x+a$, and $x^{aa_1} \equiv (x^{a_1})^a$. Clearly the above integral is of the form $\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy f(y) = L \times \int_{-\infty}^{\infty} dy f(y)$, where $y = x_1 - x_2$ and L is the regularized size of the box and is actually ∞.

This can be done in a more general way as follows: Define a "gauge fixing" function $\chi(x)$ that is **not** invariant under the symmetry, i.e. $\chi(x^a) \neq \chi(x)$. Define

$$\int da\ \delta[\chi(x^a)] = \Delta^{-1}(x)$$

An expression for $\Delta(x)$ is obtained from:

$$\int d\chi \delta[\chi(x^a)] = 1 = \int da ||\frac{\partial \chi(x^a)}{\partial a}|| \delta[\chi(x^a)]$$

So $\Delta(x) = ||\frac{\partial \chi(x^a)}{\partial a}||$ is the Jacobian for the change of variables.

Here da is the Haar measure for the translation group and satisfies $da = d(aa_1) = d(a_1 a)$. Also the measure satisfies $dx = dx^a$. Now consider inserting 1 ("unity") in the following way inside the integral:

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \underbrace{\int da\ \delta[\chi(x^a)]\Delta(x)}_{=1} f(x_1, x_2)$$

$$= \int_{-\infty}^{\infty} dx_1^a \int_{-\infty}^{\infty} dx_2^a \int da\ \delta[\chi(x^a)]\Delta(x^a) f(x_1^a, x_2^a)$$

All the quantities where x was replaced by x^a are invariant, so nothing has been changed. Now x^a is an integration variable and can be replaced by x:

$$I = \int da \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\ \ \delta[\chi(x)]\Delta(x) f(x_1, x_2)$$

The integral over the group volume, which is infinite has been factored out, and we can divide by it to get:

$$I' \equiv \frac{I}{\int da} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\ \ \delta[\chi(x)]\Delta(x) f(x_1, x_2) \qquad (122)$$

As an example let $\chi(x^a) = x_1^a$. Then $\Delta(x) = 1$. So

$$I' = \int_{-\infty}^{\infty} dx_2 f(0, x_2)$$

Gauge fixing in QED

The above example can be applied directly to the full problem with the changes:

$$x \to A_\mu(x), \quad dx \to \mathcal{D}A_\mu(x), \quad x^a \to A^g_\mu(x), \quad da \to \mathcal{D}g,$$

$$\chi(x^a) \to \chi(A^g(x)), \quad \Delta(x) \to \Delta[A], \quad f(x) \to e^{iS[A]}$$

We get

$$Z' = \frac{Z}{\int \mathcal{D}g} = \int \mathcal{D}A_\mu(x)\ \delta[\chi(A(x))]\Delta[A]e^{iS[A]}$$

Z'is finite. Here $\Delta[A] = ||\frac{\delta\chi}{\delta g}||$ is known as the Fadeev-Popov determinant.

The LHS Z' does not depend on χ (since neither Z nor $\int \mathcal{D}g$ do). So we can consider $\delta[\chi - c]$ for arbitrary function c instead, and we can even average over such functions with a weight $e^{-\frac{i}{2\alpha}\int d^4x\ c^2}$, to get

$$Z' = \int \mathcal{D}A_\mu(x)\ e^{-i\frac{1}{2\alpha}[\chi(A(x))]^2}\Delta[A]e^{iS[A]}$$

This form has the advantage that it is easy to incorporate χ^2 as part of the action.

Furthermore let us choose $\chi(A) = \partial_\mu A^\mu$. Then $\Delta[A] = Det[\partial^2\delta(x-y)]$ (using $\delta\chi = \partial^2\Lambda$). This does not depend on A so it is just a constant normalization factor which we henceforth drop. (When we calculate correlation function we always divide by $Z[0]$, so the normalizations cancel out anyway.)

Finally we can calculate the Green function for the modified wave equation:

$$(\Box g_{\nu\rho} - (1 - \frac{1}{\alpha})\partial_\nu\partial_\rho)A^\rho = -J_\nu$$

$$(\Box g_{\nu\rho} - (1 - \frac{1}{\alpha})\partial_\nu\partial_\rho)G^{\rho\mu}(x,x') = -\delta^\rho_\nu\delta(x-x')$$

In momentum space we can solve

$$[-k^2 g_{\nu\rho} + k_\nu k_\rho(1 - \frac{1}{\alpha})][Ag^{\rho\mu} + Bk^\rho k^\mu] = -\delta^\rho_\nu$$

where we have made a general ansatz for the Green's function.

This gives for the photon propgator:

$$G^{\mu\nu}(k) = \frac{i}{k^2}[g^{\mu\nu} - (1-\alpha)\frac{k^\mu k^\nu}{k^2}] = -\langle 0|T(A^\mu(k)A^\nu(-k)|0\rangle$$

Where we have inserted the factor of i which is required just as in the scalar case. $\alpha = 1$ is called Feynman gauge and $\alpha = 0$ is called Landau gauge.

18.3 Canonical Quantization

Since we need to get rid of two coordinates, so that we have the same number of independent coordinates and momenta, we can impose a gauge condition. $\partial_i A_i = 0$ called the Coulomb gauge, is quite convenient. Once we impose this the EOM

$$\partial_i F^{i0} + J^0 = 0 \Rightarrow \partial_i(\partial_i A_0 - \partial_0 A_i) = J^0$$

$$\Rightarrow -\nabla^2 A^0 = J^0$$

This can be solved explicitly for A^0. (Note that Gauss Law by itself, without the gauge condition, cannot be used to solve for A^0 for all time, because if it is satisfied at some initial time, the EOM guarantees that it is satisfied for all time. This has to do with gauge invariance - A^0 as long as it obeys the EOM will satisfy Gauss law. We have already seen that EOM cannot be used to solve for the A's because of gauge invariance.) The solution is well known:

$$A^0(x,t) = \int d^3y \ \frac{J^0(y,t)}{4\pi|\vec{x}-\vec{y}|}$$

Now that we have two DOF we can define $\vec{\Pi} = \vec{\Pi}_\perp + \vec{\Pi}_L$ the longitudinal and transverse components, which satisfy $\partial_i \Pi_{\perp i} = 0, \ \ \partial_i \Pi_{Li} = -J^0$. In fact $\Pi_i = F_{0i} = \underbrace{\partial_0 A_i}_{\Pi_\perp} - \underbrace{\partial_i A_0}_{\Pi_L}$. Thus $\Pi_{\perp i} = \partial_0 A_i$ and $\Pi_{Li} = \partial_i A^0$.

We can write the Hamiltonian in terms of the independent unconstrained momenta:

$$\mathcal{H} = \frac{1}{2}(\Pi_{\perp i} + \partial_i A^0)^2 - A_0 \partial_i(\partial_i A^0) + \frac{1}{4}F_{ij}F_{ij} + J^0 A^0 - J_i A^i$$

$$= \frac{1}{2}(\Pi_{\perp i})^2 + \frac{1}{2}(\partial_i A^0)^2 - A_0 \partial_i(\partial_i A^0) + \frac{1}{4}F_{ij}F_{ij} + J^0 A^0 - J_i A^i$$

Now $-A_0 \nabla^2 A^0 = +A_0 J^0$. The second term also after an integration by parts, is of this form. Combining all the $J^0 A^0$ terms we get:

$$\mathcal{H} = \frac{1}{2}(\Pi_{\perp i})^2 + \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}J^0 A^0 - J_i A^i \tag{123}$$

Here $\partial_i A_i = 0$ is understood. Note that

$$\int d^3x \ \frac{1}{2} J^0(x,t) A^0(x,t) = \frac{1}{2} \int d^3x \ \int d^3y \ \frac{J^0(x,t) J^0(y,t)}{4\pi |x-y|}$$

is just the electrostatic energy of the charges. Note also that this expression is *non-local* i.e. it involves quantities at different points in space. This is the price we pay for expressing the hamiltonian in terms of the physical variables only. The gauge invariant description was local but the price we paid was the redundancy of variables. This is always the trade off in gauge theories.

Poisson Brackets: We can now calculate the PB of the independent momenta and then replace them by commutators.

$$\{A^i(x), \Pi^i_\perp(y)\}_{PB} = \{A^i(x), \Pi^i(y) - \Pi^i_L(y)\}_{PB} = \delta^{ij}\delta^3(x-y) - \{A^i(x), \Pi^i_L(y)\}_{PB}$$

We need to express Π_L in terms of Π^i.

$$\Pi_{Lj}(y) = \partial_j A^0(y) = \partial_{yj} \int d^3z \ \frac{J^0(z)}{4\pi|y-z|} = \partial_{yj} \int d^3z \ \frac{-\partial_{zk}\Pi_k(z)}{4\pi|y-z|}$$

$$= \partial_{yj} \int d^3z \ \partial_{zk} \frac{1}{4\pi|y-z|} \Pi_k(z) =$$

substituing in the PB we get

$$\{A^i(x), \Pi^i_\perp(y)\}_{PB} = \delta^{ij}\delta^3(x-y) + \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{4\pi|x-y|} \tag{124}$$

On multiplying by i we get the commutators.

The same result is obtained for the Dirac brackets between Π and A. See Weinberg's book for the Dirac bracket treatment.

18.4 Generating Functional for QED

The Lagrangian density with source is:

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + (\bar\psi (i\gamma^\mu \partial_\mu + \underbrace{e\gamma^\mu A_\mu}_{\mathcal{L}_{int}} - m)\psi + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta$$

The generating functional:

$$Z[J^\mu, \bar\eta, \eta] = \int \mathcal{D}A^\mu \mathcal{D}\bar\psi \mathcal{D}\psi e^{i\int d^4x [\mathcal{L} + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta]} = e^{i\int d^4x \ e(-\frac{1}{i}\frac{\delta}{\delta\eta_a(x)})(\gamma^\mu)^{ab}(\frac{1}{i}\frac{\delta}{\delta\bar\eta_b(x)})(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)})} Z_0[J^\mu, \bar\eta, \eta]$$

with

$$Z_0[J^\mu,\bar\eta,\eta]=Z_0[0]e^{-\int\int[\bar\eta S_F\eta+\frac{1}{2}J_\mu D_F^{\mu\nu}J_\nu]}$$

where $D_F=\frac{-ig_{\mu\nu}}{p^2+m^2}$ -in the Feynman gauge and in momentum space.

Let us now calculate a term in $Z[J]$ with two J's to lowest non trivial order. The leading term of course is $-\frac{1}{2}\int d^4x\ \int d^4yJ_\mu D_F^{\mu\nu}J_\nu$ and represents the propgation of a photon between two currrents (see Fig).

The next such term comes at second order in perturbation theory:

$$\frac{1}{2!}i\int d^4x\ e(-\frac{1}{i}\frac{\delta}{\delta\eta_a(x)})(\gamma^\mu)^{ab}(\frac{1}{i}\frac{\delta}{\delta\bar\eta_b(x)})(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)})$$

$$i\int d^4y\ e(-\frac{1}{i}\frac{\delta}{\delta\eta_c(y)})(\gamma^\nu)^{cd}(\frac{1}{i}\frac{\delta}{\delta\bar\eta_d(y)})(\frac{1}{i}\frac{\delta}{\delta J^\nu(y)})Z_0[J,\bar\eta,\eta]$$

$$=(ie)^2\int d^4x\int d^4y(-\frac{1}{i}\frac{\delta}{\delta\eta_a(x)})(\gamma^\mu)^{ab}(\frac{1}{i}\frac{\delta}{\delta\bar\eta_b(x)})(-\frac{1}{i}\frac{\delta}{\delta\eta_c(y)})(\gamma^\nu)^{cd}(\frac{1}{i}\frac{\delta}{\delta\bar\eta_d(y)})$$

$$(-1)\int d^4x_1J^\rho(x_1)D_{F\rho\mu}(x_1-x)\int d^4x_2J^\sigma(x_2)D_{F\sigma\nu}(x_2-y)Z_0[J=0,\bar\eta,\eta]$$

where we have first extracted the two J's so we can now set $J=0$ inside Z_0. We now do the fermionic part. Since we don not want any η's in the final answer, the η and $\bar\eta$ derivatives have to contract in pairs. We are interested in connected diagrams, so we make sure that acting on $\bar\eta(y_1)S_F(y_1-y_2)\eta(y_2)$ $x=y_1$ and $y=y_2$ or vice versa. (This gives a factor of 2, to cancel the 2! in the denominator.)

Thus one term comes from $\frac{\delta}{\delta\eta_a(x)}\frac{\delta}{\delta\bar\eta_d(y)})\int\int(-\bar\eta(y_1)S_F(y_1-y_2)\eta(y_2))=-S_F^{da}(y-x)$. This is multiplied by $-\frac{\delta}{\delta\eta_c(y)}\frac{\delta}{\delta\bar\eta_b(x)})\int\int(-\bar\eta(y_1)S_F(y_1-y_2)\eta(y_2))=S_F^{bc}(x-y)$ where the $(-)$ sign is because we have interchanged the order of the derivatives. The factors of $\frac{1}{i}$ and $\frac{-1}{i}$ all combine to give 1. Thus we get

$$(ie)^2(\gamma^\mu)^{ab}S_F^{bc}(x-y)(\gamma^\nu)^{cd}(-S_F^{da}(y-x))=e^2Tr[\gamma^\mu S_F(x-y)(\gamma^\nu)S_F(y-x)]$$

So the final result is

$$-\int d^4x\int d^4y\int d^4x_1J^\rho(x_1)D_{F\rho\mu}(x_1-x)\int d^4x_2J^\sigma(x_2)D_{F\sigma\nu}(x_2-y)$$

$$e^2Tr[\gamma^\mu S_F(x-y)(\gamma^\nu)S_F(y-x)]$$

The correction to the propagator is

$$-\frac{\delta^2 Z}{\delta J^\mu(x')\delta J^\nu(y')} = \int d^4x \int d^4y D_{F\rho\mu}(x'-x) D_{F\sigma\nu}(y'-y) e^2 Tr[\gamma^\mu S_F(x-y)(\gamma^\nu) S_F(y-x)]$$

Let us work in momentum space and calculate the amputated Green function (i.e. without the two factors of D_F):

$$e^2 \int \frac{d^4k}{(2\pi)^4} Tr[\gamma^\mu \frac{-i}{(p+k)_\rho \gamma^\rho + m} \gamma^\nu \frac{-i}{k_\sigma \gamma^\sigma + m}]$$

This can be summarized by the Feynman rules: i) $ie\gamma^\mu$ at each vertex. ii) S_F for each fermion propagator and D_F for each photon propgagtor iii) (-1) for each fermion loop and iv) trace over Dirac indices for each loop.

The evaluation of this integral will be done later.

18.5 Tree Level Processes

DONE IN PARTICLE PHYSICS COURSE

18.6 One Loop Diagrams, Renormalization, beta-function

We start with the Bare Lagrangian (density):

$$\mathcal{L}_B = -\frac{1}{4} F_{B\mu\nu} F^{B\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A_B^\mu)^2 + \bar{\psi}_B(i\gamma^\mu \partial_\mu - m_B)\psi_B + e_B \bar{\psi}_B \gamma^\mu A_{B\mu} \psi_B$$

and write it as a sum of the Renormalized Lagrangian and a counterterm Lagrangian:

$$\mathcal{L}_R = -\frac{1}{4} F_{R\mu\nu} F^{R\mu\nu} - \frac{1}{2}(\partial_\mu A_R^\mu)^2 + \bar{\psi}_R(i\gamma^\mu \partial_\mu - m_R)\psi_R + e_R \bar{\psi}_R \gamma^\mu A_{R\mu} \psi_R$$

$$\mathcal{L}_{ct} = -\frac{(Z_A - 1)}{4} F_{R\mu\nu} F^{R\mu\nu} - \frac{1}{2}(\frac{Z_A}{\alpha} - 1)(\partial_\mu A_R^\mu)^2 + (Z_\psi - 1)\bar{\psi}_R(i\gamma^\mu \partial_\mu)\psi_R$$
$$-(Z_\psi m_B - m_R)\bar{\psi}_R \psi_R + (Z_\psi \sqrt{Z}_A e_B - e_R)\bar{\psi}_R \gamma^\mu A_{R\mu} \psi_R$$

1. Note that the bare coupling to the EM field can be written as $Z_\psi \sqrt{Z}_A e_B$. So the covariant derivative in the bare theory is $(i\partial_\mu + \sqrt{Z}_A e_B A_{\mu R})$. In the renormalized theory it is $(i\partial_\mu + e_R A_{\mu R})$. If these are to be equal then

$\sqrt{Z}_A e_B = e_R$. The coupling in the bare theory $Z_\psi \sqrt{Z}_A e_B \bar{\psi}_R \gamma^\mu \psi_R A_{R\mu}$ is conventionally written as $Z_1 e_R \bar{\psi}_R \gamma^\mu \psi_R A_{R\mu}$, (i.e.$Z_\psi \sqrt{Z}_A e_B = Z_1 e_R$). Thus **equality of covariant derivatives implies** $Z_1 = Z_\psi$. In conventional notation $Z_\psi \equiv Z_2$ and $Z_A \equiv Z_3$. So in conventional textbooks one finds the statement $Z_1 = Z_2$. This is a Ward Identity. More about this in the next subsection.

2. Another Ward Identity states that the longitudinal part of the propagator is fixed to be whatever it is in the renormalized Lagrangian. We have chosen the renormalized Lagrangian to be in the Feynman gauge. There is no correction to this. Thus there is no need to modify the coefficient of $(\partial_\mu A^\mu)^2$ - it remains as $\frac{1}{2}$. So we can choose $\alpha = Z_A$.

18.6.1 One Loop Graphs

We already have the one loop vacuum polarization graph. Let us evaluate it.

We will use dimensional regularization: Work in $D = 4 - 2\epsilon$.

18.6.2 Mathematical Digression: Volume element

Let us first of all work out the expression for a volume element in D dimensions using spherical coordinates. Do this in a recursive way.

1. S_1 Start with a circle of radius r_1 in the $x - y$ plane. The length is
$$dA_1 = r_1 d\theta_1$$

2. Now we go to three dimensions. The angle in the $x-y$ plane is usually called ϕ. The angle wrt the z- axis is usually called θ, here we call it θ_1. The radius vector in three dimensions will be called r_2. Then $r_1 = r_2 sin\ \theta_1$. The area element on S^2 embedded in R^3 is
$$dA_2 = r_2 d\theta_1 r_1 d\phi$$

3. Similarly if A_{n-1} is the area element of S^{n-1} embedded in R^n
$$\begin{aligned} dA_3 &= r_3 d\theta_2 r_2 d\theta_1 r_1 d\phi \\ dA_4 &= r_4 d\theta_3 r_3 d\theta_2 r_2 d\theta_1 r_1 d\phi \\ dA_{n-1} &= \prod_{i=1}^{n-1} r_i d\theta_{i-1} d\phi \end{aligned} \tag{125}$$
(126)

4. Furthermore just as in three dimensions:

$$\begin{aligned} r_1 &= r_2 sin\ \theta_1 \\ r_2 &= r_3 sin\theta_2 \\ r_{n-2} &= r_{n-1} sin\theta_{n-2} \end{aligned} \tag{127}$$

5. In n dimensions, the relevant radius is $r_{n-1} \equiv R$ and plugging the expressions recursively we get

$$dA_{n-1} = R^{n-1}(sin^{n-2}\theta_{n-2}d\theta_{n-2})\ (sin^{n-3}\theta_{n-3}d\theta_{n-3})....(sin\theta_1 d\theta_1)d\phi \tag{128}$$

All the angles range from 0 to π except ϕ which ranges over 2π.

6. Finally we write

$$dV_n = dA_{n-1}dR$$

The final integral of a function that depends only on R is thus:

$$I_n = 2\pi \prod_{k=1}^{n-2} \int sin^k\theta_k d\theta_k \int_0^\infty R^{n-1}dR\ f(R) \tag{129}$$

Now we can use

$$\int_0^{\frac{\pi}{2}} sin^{2a-1}x\ cos^{2b-1}x\ dx = \frac{1}{2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Set $b = \frac{1}{2}$ and $a = \frac{k+1}{2}$ to get

$$\int_0^{\pi} sin^k x\ dx = 2\int_0^{\frac{\pi}{2}} sin^k x\ dx = \sqrt{\pi}\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}$$

Plugging this into I_n we get

$$I_n = 2\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^\infty R^{n-1}dR\ f(R) \tag{130}$$

18.6.3 Mathematical Digression: Some useful Integrals

Using the above we obtain

$$\int \frac{d^n p}{(p^2 + M^2)^a} = \pi^{\frac{n}{2}} \frac{\Gamma(a - \frac{n}{2})}{\Gamma(a)} \frac{1}{(M^2)^{a-n/2}} \tag{131}$$

Also replacing p by $p + k$ we can get results of the form

$$\int \frac{d^n p}{(p^2 + 2k \cdot p + M^2)^a} = \pi^{\frac{n}{2}} \frac{\Gamma(a - \frac{n}{2})}{\Gamma(a)} \frac{1}{(M^2 - k^2)^{a-n/2}}$$

Differentiating wrt k we can get

$$\int d^n p \ \frac{p^\mu}{(p^2 + 2k \cdot p + M^2)^a} = \pi^{\frac{n}{2}} \frac{\Gamma(a - \frac{n}{2})}{\Gamma(a)} \frac{(-k^\mu)}{(M^2 - k^2)^{a-n/2}}$$

18.6.4 One loop graphs

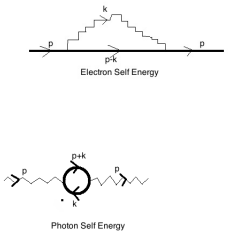


Figure 4: Electron and Photon Self Energy

18.7 Ward Identities

19 Yang-Mills Theory

Yang-Mills is the natural generalization of electromagnetism to a non-Abelian group.

19.1 Lie Groups Basics

Lie Groups are continuous groups. The group is also a differentiable manifold. (eg $SU(2) = S^3$). At any point one can define a tangent space and the tangent vectors define generators of the group. They form a vector space - the tangent space. The generators can thus be added and multiplied so they form an algebra - called the Lie Algebra. These are defined by commutation relations:

$$[T_a, T_b] = C_{ab}^{\ c} T_c \tag{132}$$

$C_{ab}^{\ c}$ are called the structure constants of the group. They also obey the Jacobi identity

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0$$

A general element of the group is then

$$g = e^{\Lambda^a T_a} \equiv e^{\Lambda}$$

If we want them to be represented by unitary matrices then we choose T_a to be anti-Hermitian, and the structure constants are real. (Some people prefer Hermitian generators, in which case one has an extra factor of i in the exponent.)

The simplest example is the $SU(2)$ group. The Pauli matrices define Hermitian generators obeying

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

If we prefer to work with Anti Hermitian generators we can define: $T_i = -\frac{i}{2}\sigma_i$ so that

$$[T_i, T_j] = \epsilon_{ijk} T_k$$

Further the structure constants define a normalization of the T's. So with anti-Hermitian generators, we can choose for instance:

$$Tr(T_a T_b) = -\frac{1}{2}\delta_{ab} \tag{133}$$

The same algebra can be represented by different matrices $(T_a)^k_l$, of different rank. Thus the Pauli matrices define a 2×2 representation of SU(2). One also has $(2j+1) \times (2j+1)$ dimensional matrices, as we know. $j = \frac{1}{2}$ gives the Pauli matrix.

The number of generators is the dimension of the group. If the dimension is n, then a special representation is the one by $n \times n$ matrices. This is called the adjoint representation. In this case the matrix looks like $(T_a)^c_b$. In fact one can show, using the Jacobi identity, that these are just the structure constants:

$$(T_a)^c_b = +C_{ab}{}^c \tag{134}$$

For $SU(2)$ the adjoint representation is three dimensional i.e. by 3×3 matrices. This is the well known matrix representation of rotations of a three vector.

19.2 Covariant Derivative

As in EM we define a covariant derivative:

$$D_\mu = \partial_\mu + A_\mu \quad ; \quad A_\mu = A^a_\mu T_a \tag{135}$$

$(T_a)^j_i$ is a matrix representation of the generator, chosen according to the representation of the field it is acting on. Thus if ϕ_i is the field then

$$(D_\mu \phi)_i \equiv \partial_\mu \phi_i + A^a_\mu (T_a)^j_i \phi_j$$

Thus in $SU(2)$ if ϕ is a spin half field then it has two components: $i = 1, 2$ and then $T_a = -\frac{i}{2}\sigma_a$.

Under group rotations

$$\phi \to g\phi g^\dagger$$

If D_μ is a covariant derivative then

$$D_\mu \phi \to g D_\mu \phi g^\dagger = (g D_\mu g^\dagger) g \phi g^\dagger$$

Letting $g = e^{-\Lambda}$, we conclude that

$$\partial_\mu + A_\mu \to g\partial_\mu g^\dagger + g A_\mu g^\dagger = \partial_\mu + \partial_\mu \Lambda + g A_\mu g^\dagger$$

Thus we require that the gauge field should transform as

$$A_\mu \to \partial_\mu \Lambda + g A_\mu g^\dagger \tag{136}$$

It is often easier to specify the infinitesimal form of the transformation: $g = I - \Lambda$

$$\delta A_\mu = \partial_\mu \Lambda - [\Lambda, A_\mu] = \partial_\mu + [A_\mu, \Lambda]$$

In component notation:

$$\delta A^a_\mu = \partial_\mu \Lambda^a + A^b_\mu \Lambda^c C_{bc}{}^a = \partial_\mu \Lambda^a + A^b_\mu C_{bc}{}^a \Lambda^c = \partial_\mu \Lambda^a + A^b_\mu (T_b)^a_c \Lambda^c = ((\partial_\mu - + A_\mu)\Lambda)^a = (D_\mu \Lambda)^a$$

where we have used (134).

19.3 Action

Given the definition of the covariant derivative it is easy to see that

$$[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv F_{\mu\nu}$$

is a gauge *covariant* field strength, in the sense that under gauge transformations

$$F_{\mu\nu} \to g F_{\mu\nu} g^\dagger$$

and therefore an invariant (because of cyclicity of the trace) action is

$$\frac{1}{2} \int d^4x \; Tr(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} \int d^4x \; F^a_{\mu\nu} F^{\mu\nu a} \tag{137}$$

where the orthonormality (133) of the generators have been used.

Finally a coupling constant can be introduced by rescaling $A \to g_{YM} A$ so

$$F_{\mu\nu} \to g_{YM} F_{\mu\nu}$$

where the new field strength is:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM} [A_\mu, A_\nu] \equiv F_{\mu\nu}$$

Similarly the covariant derivative becomes

$$D_\mu = \partial_\mu + g_{YM} A_\mu$$

Thus we can write the YM action as,

$$\frac{1}{2 g^2_{YM}} \int d^4x \; Tr(F_{\mu\nu} F^{\mu\nu})$$

or with the g_{YM} inside the field strength by the above rescaling.

They are equivalent except when we take the $g \to 0$ limit: the first form gives a free quantum theory with quadratic action, the second form gives a *classical* but non linear theory with a (cubic, quartic) action.

Finally coupling to matter fields - scalars and fermions - is easy:

$$\int d^4x[(D_\mu\phi)^\dagger(D_\mu\phi) + \bar{\psi}\not{D}\psi]$$

That completes our discussion of the classical Yang-Mills construction.