An Introduction to Exact RG

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Contents

1 Introduction

1. These are some notes for a collection of four lectures given at Matscience in September 2018. It is meant to be an *introduction* to some of the ideas in the Exact Renormalization Group (ERG). It assumes a background in Quantum Field Theory. While the material can be understood even by those who are not familiar with renormalization group, to appreciate it requires some background in the concepts of RG.

2. The basic reference for ERG is the original article in Physics Reports by Kogut and Wilson. There are other more recent reviews by Bagnuls and Bervillier and a very comprehensive one by O. Rosten. There is also a good review by Igarashi, Itoh and Sonoda. This last one is very systematic, compact but gives the mathematical details in a nice way. I have relied heavily on this one.

3. Some of the derivations and equations I give below are not given in any review but can easily be derived once the concepts are understood.

4. The notes go into some more detail than the actual lectures in some areas.

5. These notes only deal with scalar field theories and leaves out the vast subject of gauge theories and symmetry.

2 Wilson's RG equation

This is a prototype of Wilson's ERG equation. It is also called a diffusion equation.

$$\frac{\partial\psi'}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + x)\psi' \tag{1}$$

2.1 Solution and Interpretation as Coarse Graining

We use the following mapping to solve Wilson's RG:

$$y = xe^{t/2}, \quad \tau = e^t, \quad \psi' = e^{t/2}\psi \tag{2}$$

$$\frac{\partial}{\partial y} = \frac{\partial x}{\partial y}\frac{\partial}{\partial x} + \frac{\partial t}{\partial y}\frac{\partial}{\partial t} = e^{-t/2}\frac{\partial}{\partial x}\ ; \quad \frac{\partial^2}{\partial y^2} = e^{-t}\frac{\partial^2}{\partial x^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} = \frac{1}{2}y\frac{\partial}{\partial y} + e^t\frac{\partial}{\partial\tau}$$

So

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\psi}{\partial\tau}\frac{\partial\tau}{\partial t} = \underbrace{\frac{1}{2}y\frac{\partial\psi}{\partial y}}_{(i)} + \underbrace{e^t\frac{\partial\psi}{\partial\tau}}_{(ii)}$$

(i)

$$\frac{1}{2}y\frac{\partial\psi}{\partial y}=\frac{1}{2}x\frac{\partial\psi}{\partial x}=\frac{1}{2}e^{-t/2}x\frac{\partial\psi'}{\partial x}$$

(ii)

Consider the equation

$$\frac{\partial\psi}{\partial\tau}=\frac{1}{2}\frac{\partial^2\psi}{\partial y^2}$$

$$=\frac{1}{2}e^{-3t/2}\frac{\partial^2\psi'}{\partial x^2}$$

so

$$e^t\frac{\partial\psi}{\partial\tau}=\frac{1}{2}e^{-t/2}\frac{\partial^2\psi'}{\partial x^2}$$

So

$$\frac{\partial\psi}{\partial t}=(i)+(ii)=\frac{1}{2}e^{-t/2}\frac{\partial^2\psi'}{\partial x^2}+\frac{1}{2}e^{-t/2}x\frac{\partial\psi'}{\partial x}$$

Finally

$$\frac{\partial\psi'}{\partial t}=\frac{1}{2}\psi'+e^{t/2}\frac{\partial\psi}{\partial t}=\frac{1}{2}\psi'+\frac{1}{2}\frac{\partial^2\psi'}{\partial x^2}+\frac{1}{2}x\frac{\partial\psi'}{\partial x}=\frac{1}{2}\frac{\partial}{\partial x}(\frac{\partial}{\partial x}+x)\psi'$$

Thus the coordinate transformation (2) maps "Schroedinger" equation

$$\frac{\partial\psi}{\partial\tau}=\frac{1}{2}\frac{\partial^2\psi}{\partial y^2} \tag{3}$$

to Wilson's RG equation:

$$\frac{\partial\psi'}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x}(\frac{\partial}{\partial x}+x)\psi' \tag{4}$$

The solution to (3) is

$$\psi(y_f,\tau_f)\approx\frac{1}{\sqrt{2\pi(\tau_f-\tau_i)}}\int dy_i e^{-\frac{1}{2}\frac{(y_f-y_i)^2}{\tau_f-\tau_i}}\psi(y_i,\tau_i) \tag{5}$$

We are not concerned about overall normalization factors.

Thus the solution to (4) is

$$\psi(x_f,t_f)=e^{\frac{t_f}{2}}\frac{1}{\sqrt{2\pi(e^{t_f}-e^{t_i})}}\int dx_i e^{-\frac{1}{2}\frac{(x_f e^{\frac{t_f}{2}}-x_i e^{\frac{t_i}{2}})^2}{e^{t_f}-e^{t_i}}}\psi(x_i,t_i) \tag{6}$$

The solution can also be written as

$$\boxed{\psi(x_f,t_f)=\frac{1}{\sqrt{2\pi(1-e^{t_i-t_f})}}\int dx_i e^{-\frac{1}{2}\frac{(x_f-e^{\frac{t_i-t_f}{2}}x_i)^2}{1-e^{t_i-t_f}}}\psi(x_i,t_i)} \tag{7}$$

As $t_f \to \infty$ we can see that $\psi(x_f, t_f)$ becomes $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_f^2}$. As $t_f \to t_i$ the kernel becomes a delta function and $\psi(x_f, t_f) \to \psi(x_i, t_i)$.

So the final state is identical to the initial state in the beginning. As time progresses it has less and less information about the initial state. At $t_f \to \infty$, the initial state is completely integrated over and the final state is a Gaussian: all information is lost.

2.2 Generalizing to Field Theory

In field theory, x is replaced by $x(p)$, i.e. it becomes a mode of a field, and we have to integrate over all p. So one would like the rate of integrating out to depend on p: Higher p's should get integrated out faster. Thus t should be replaced by some function of p - call it $g(p,t)$. $g(p,t)$ is a time for each mode - and it should increase with p and also with t. Also $\Lambda = \Lambda_0 e^{-t}$ is the connection between t and scale - as t increases we move to the IR.

One can replace (4) by a functional DE:

$$\frac{\delta}{\delta g(p,t)}\psi[x(p),t] = \frac{\delta}{\delta x(p)}(\frac{\delta}{\delta x(-p)} + x(p))\psi[x(p),t] \tag{8}$$

The solution is obviously

$$\boxed{\psi[x_f(p),t] = \int \mathcal{D}x_i(p) e^{-\frac{1}{2}\int_p \frac{(x_f(p)-e^{-(g(p,t)-g(p,0))}x_i(p))(x_f(-p)-e^{-(g(p,t)-g(p,0))}x_i(-p))}{(1-e^{-2(g(p,t)-g(p,0))})}} \psi[x_i(p),0]} \tag{9}$$

$$\int_p = \int \frac{d^D p}{(2\pi)^D}$$

Overall normalization factors are not kept. In field theory ψ is typically some generating functional (or partition function), whose normalization is not important.

(8) can also be written as

$$\frac{\partial}{\partial t}\psi[x(p),t] = \int_p \dot{g}(p,t)\frac{\delta}{\delta x(p)}(\frac{\delta}{\delta x(-p)} + x(p))\psi[x(p),t] \tag{10}$$

Wilson's choice:

$$\dot{g} = c + 2p^2 e^{2t} = c + \frac{2p^2}{\Lambda^2}, \quad g(t) = ct + p^2 e^{2t} \tag{11}$$

Λ is the moving scale. g distinguishes high p from low. Also what is considered "high" changes with t because p is measured relative to the moving scale.

We identify the Bare action S_B and the coarse grained "Wilson action" S_Λ by:

$$\psi[x_i(p),t_i] = e^{-S_B[x_i(p),t_i]}, \quad \psi[x_f(p),t_f] = e^{-S_\Lambda[x_f(p),t_f]}$$

We will often write this equation suppressing the momentum label as:

$$\boxed{\frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2}\dot{g}(t)\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + x)\psi(x,t)} \tag{12}$$

this makes the analogy with ordinary q.m. very clear.
For completeness we write the equation for S suppressing momentum labels:

$$\frac{\partial S(x,t)}{\partial t} = \dot{g}(t)[\frac{\partial^2 S}{\partial x^2} - (\frac{\partial S}{\partial x})^2 + x\frac{\partial S}{\partial x}] \tag{13}$$

2.3 Relating Correlations

Consider the generating functional:

$$Z[J] = \int dx_f(p)\, e^{-S[x_f(p)] + \int_p J(p)\, x_f(-p)}.$$

$$\left.\frac{\delta^2 Z[J]}{\delta J(p_1)\,\delta J(p_2)}\right|_{J=0} = \langle x_f(p_1)\, x_f(p_2)\rangle_{S_f}$$

$$\left.\frac{\delta^3 Z[J]}{\delta J(p_1)\,\delta J(p_2)\,\delta J(p_3)}\right|_{J=0} = \langle x_f(p_1)\, x_f(p_2)\, x_f(p_3)\rangle_{S_f}.$$

Write

$$e^{-S[x_f(p)]} = \int dx_i(p)\, e^{-\frac{1}{2}\int A^2(p)\,(x_f(p) - Z(p)x_i(p))(x_f(-p) - Z(\ldots}\; e^{-S[x_i(p)]}$$

Then

$$Z[J] = \int dx_f(p)\, e^{-S[x_f(p)] + \int_p J(p)\, x_f(-p)}.$$

$$= \int dx_f(p) \int dx_i(p)\, e^{-\frac{1}{2}\int_p A^2\,(x_f(p) - Z x_i(p))^2 - S[x_i(p)] + \int_p J(p)\, x_f(-p)}$$

$$= \int dx_i(p)\, e^{-S[x_i(p)]} \int dx_f(p)\, e^{-\frac{1}{2}\int A^2 (\underbrace{x_f - Z x_i}_{x'})^2 + \int J x_f}.$$

$$x_f = x' + Z x_i$$

$$= \int dx_i(p)\, e^{-S[x_i(p)]} \int dx'\, e^{-\frac{1}{2}A^2 x'^2 + \int J x' + \int J Z x_i}.$$

$$= \int dx_i(p)\, e^{-S[x_i(p)]}\; e^{\frac{1}{2}\int_p \frac{J(p)\,J(-p)}{A^2(p)} + \int J(p)\, Z(p)\, x_i(-p)}.$$

$$\left.\frac{\delta^2 Z}{\delta J(p_1)\,\delta J(p_2)}\right|_{J=0} = Z(p_1)Z(p_2)\,\langle x_i(p_1)\, x_i(p_2)\rangle_{S_i} + \frac{\delta(p_1+p_2)}{A^2(p)}$$

$$\langle x_f(p_1)\, x_f(p_2)\rangle_{S_f} = Z(p_1)Z(p_2)\,\langle x_i(p_1)\, x_i(p_2)\rangle_{S_i} + \frac{\delta(p_1+p_2)}{A^2(p_1)}.$$

$$\langle x_f(p_1)\, x_f(p_2)\, x_f(p_3)\rangle_{S_f} = Z(p_1)Z(p_2)Z(p_3)\,\langle x_i(p_1)\, x_i(p_2)\, x_i(p_3)\rangle_{S_i}$$

$$\vdots \qquad\qquad \vdots$$

So as long as the $Z(p)$ are analytic fn. one can invert & relate correlations of S_i with those of S_f & vice versa.

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3 Wilson Action, Generating Functional

We have defined a coarse graining above which gave a Wilson action at low energies. We define another way of doing this - more suited for usual Feynman Diagram perturbation theory.

3.1 Wilson Action

Let us start by defining:

$$\boxed{Z[J] \equiv e^{W_B[J]} = \int \mathcal{D}\phi e^{-S_B[\phi]+\int_p J(-p)\phi(p)}} \tag{14}$$

Here S_B is the bare action. We separate out the kinetic and interaction terms as follows:

$$S_B[\phi] = \frac{1}{2}\phi\Delta^{-1}\phi + S_{B,I}[\phi]$$

$$\Delta = \frac{e^{-\frac{p^2}{\Lambda_0^2}}}{p^2} \equiv \frac{K_0(\frac{p^2}{\Lambda_0^2})}{p^2}$$

The proapagator is regulated. Λ_0 is a cutoff for the bare theory, which we may eventually take to ∞. The full propagator will now be written as a sum of two pieces, a high energy propagator and a low energy propagator: $\Delta = \Delta_h + \Delta_l$.

$$\Delta_h = \frac{K_0(p,\Lambda_0) - K(p,\Lambda)}{p^2} = \frac{e^{-\frac{p^2}{\Lambda_0^2}} - e^{-\frac{p^2}{\Lambda^2}}}{p^2} \tag{15}$$

propagates fields with momenta $\Lambda < p < \Lambda_0$. Λ_0 is the bare cutoff which can be taken to ∞ in the continuum limit. Note that as $p \to 0$, Δ_h is non singular.

Then we can show that $Z[J]$ can also be written as

$$Z[J] = \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l\frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h\frac{1}{\Delta_h}\phi_h + J\phi_h - S_{B,I}[\phi_l+\phi_h]}$$

$$= e^{-S_{B,I}[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l\frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h\frac{1}{\Delta_h}\phi_h + J\phi_h}$$

(See below for simple proof)

Another version of the Wilson action can then be defined by the following equations:

$$Z[0] = \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l\Delta_l^{-1}\phi_l} \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h\Delta_h^{-1}\phi_h - S_{B,I}[\phi_l+\phi_h]} \tag{16}$$

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l\Delta_l^{-1}\phi_l - S_{I,\Lambda}[\phi_l]} = \int \mathcal{D}\phi_l e^{-S_\Lambda[\phi_l]} \tag{17}$$

$S_{I,\Lambda}[\phi]$ is the interacting part of the Wilson action defined by integrating out the high momentum modes. We are left with an effective theory that is a QFT description of low energy phenomena:

$$\boxed{e^{-S_{I,\Lambda}[\phi_l]} \equiv \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h - S_{B,I}[\phi_l+\phi_h]}} \tag{18}$$

For this theory Λ acts as a UV cutoff as in all effective field theories. From the point of view of the original field theory S_B since we are only integrating modes with momentum above Λ, Λ is an IR cutoff. In fact because we have not done any low momentum integration, S_Λ is analytic at $p=0$.

Simple proof: Consider

$$Z[J] = e^{-S_{B,I}[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h + J\phi_h}$$

Do the integrals to get (ignoring field independent determinants)

$$Z[J] = e^{-S_{B,I}[\frac{\delta}{\delta J}]} e^{\frac{1}{2}J(\Delta_h+\Delta_l)J}$$

$$\begin{aligned} Z[J] &= e^{-S_{B,I}[\frac{\delta}{\delta J}]} \int \mathcal{D}\phi e^{-\frac{1}{2}\phi\Delta^{-1}\phi + J\phi} \\ &= \int \mathcal{D}\phi e^{-\frac{1}{2}\phi\Delta^{-1}\phi + J\phi - S_{B,I}[\phi]} \\ &= \int \mathcal{D}\phi e^{-S_B[\phi]+J\phi} \end{aligned}$$

All though S_Λ is a low energy effective field theory, because of the analytic nature of the cutoff, *in fact it has all the high energy information in it.* Thus in the vertices of the action, the coefficient functions are actually analytic functions of the momenta and one can extract high energy behaviour also from it. This will be clear below in (24) and (25).

3.2 $W_\Lambda[J]$

We start with some definitions:

$$\boxed{Z[J] \equiv e^{W_B[J]} = \int \mathcal{D}\phi e^{-S_B[\phi]+\int_p J(-p)\phi(p)}} \tag{19}$$

Now we have seen that $Z[J]$ can be written as

$$Z[J] = \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} \underbrace{\int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h + J\phi_h - S_{B,I}[\phi_l+\phi_h]}}_{I[J,\phi_l]} \tag{20}$$

$$I[J,\phi_l] \equiv \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h + J\phi_h - S_{B,I}[\phi_l+\phi_h]} \tag{21}$$

$$Z[J,\phi_l] \equiv \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h + J\phi_h - S_{B,I}[\phi_l+\phi_h] + J\phi_l} = I[J,\phi_l]e^{J\phi_l} \equiv e^{W_\Lambda[J,\phi_l]} \tag{22}$$

$I[J,\phi_l]$ can be rewritten as

$$\begin{aligned} I[J,\phi_l] &= \int \mathcal{D}\phi_h e^{-\frac{1}{2}\underbrace{(\phi_h - \Delta_h J)}_{\phi'}\frac{1}{\Delta_h}(\phi_h - \Delta_h J) + \frac{1}{2}J\Delta_h J - S_{B,I}[\phi_l+\phi_h]} \\ &= \int \mathcal{D}\phi' e^{-\frac{1}{2}\phi'\frac{1}{\Delta_h}\phi' + S_{B,I}[\phi'+\phi_l+\Delta_h J] + \frac{1}{2}J\Delta_h J} \\ &= e^{-S_{\Lambda,I}[\phi_l + \Delta_h J] + \frac{1}{2}J\Delta_h J} \end{aligned} \tag{23}$$

Also

$$I[J,\phi_l]e^{J\phi_l} = e^{W_\Lambda[J,\phi_l]}$$

Thus I is the generating functional when ϕ_h has been integrated out with a source. $\ln I[J,0] = W_\Lambda[J,0] = W_\Lambda[J]$ is the quantity for which we need an ERG. Thus we have

$$\boxed{W_\Lambda[J] = -S_{\Lambda,I}[\Delta_h J] + \frac{1}{2}J\Delta_h J} \tag{24}$$

The limit $\Lambda \to 0$ gives us $W_B[J]$

$$\lim_{\Lambda\to 0} W_\Lambda[J] = W_B[J] \tag{25}$$

Here we see explicitly, that S_Λ gives the correct behaviour at $p >> \Lambda$.

3.3 $\bar{W}_\Lambda[J]$

Associated with the Wilson action one can define a generating functional:

$$\bar{Z}_\Lambda[J] = e^{\bar{W}_\Lambda[J]} = \int \mathcal{D}\phi_l e^{-S_\Lambda[\phi_l] + J\phi_l} \tag{26}$$

One can use this to calculate correlations using the low energy Wilson action. As will be shown below, one can reconstruct the original correlations from this. So we conclude that the Wilson action has all the information about the original theory. This was also demonstrated earlier in the Wilsonian coarse graining defined in the beginning.

3.4 W_B

$$e^{W_B[J]} = \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h - S_{I,\Lambda_0}[\phi_l+\phi_h]+J\phi_h} \tag{27}$$

Completing the square and changing variables of integration gives:

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}(\phi_h - \Delta_h J) e^{-\frac{1}{2}(\phi_h-\Delta_h J)\frac{1}{\Delta_h}(\phi_h-\Delta_h J) - S_{I,\Lambda_0}[\phi_l+\Delta_h J+(\phi_h-\Delta_h J)]+\frac{1}{2}J\Delta_h J}$$

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} e^{-S_{I,\Lambda}[\phi_l+\Delta_h J]+\frac{1}{2}J\Delta_h J}$$

The last step follows from the definition of $S_{I,\Lambda}$. Let $\phi_l + \Delta_h J = \phi'$.

$$e^{W_B[J]} = \int \mathcal{D}\phi' e^{-\frac{1}{2}(\phi'-\Delta_h J)\frac{1}{\Delta_l}(\phi'-\Delta_h J)+J(\phi'-\Delta_h J)-S_{I,\Lambda}[\phi']+\frac{1}{2}J\Delta_h J}$$

$$= \int \mathcal{D}\phi' e^{-\frac{1}{2}\phi'\frac{1}{\Delta_l}\phi' + J\frac{\Delta_h}{\Delta_l}\phi' - \frac{1}{2}J\frac{\Delta_h^2}{\Delta_l}J - J\Delta_h J + J\phi' - S_{I,\Lambda}[\phi'] + \frac{1}{2}J\Delta_h J}$$

$$= \int \mathcal{D}\phi' e^{-\frac{1}{2}\phi'\frac{1}{\Delta_l}\phi' + J\frac{\Delta}{\Delta_l}\phi' - \frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J - S_{I,\Lambda}[\phi']}$$

$$= \int \mathcal{D}\phi' e^{J\frac{\Delta}{\Delta_l}\phi' - S_\Lambda[\phi'] - \frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J}$$

$$= e^{W_B[J]} = e^{\bar{W}_\Lambda[\frac{\Delta}{\Delta_l}J] - \frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J} \tag{28}$$

Here $\bar{W}_\Lambda[J]$ is the (log) of the generating functional of just the Wilson action. $\frac{\Delta}{\Delta_l}J$ couples only to ϕ_l. If we take $\Delta = \frac{K_0}{p^2}$, $\Delta_l = \frac{K}{p^2}$ and $\Delta = \frac{K_0-K}{p^2}$, then

$$W_B[J] = \bar{W}_\Lambda[\frac{K_0}{K}J] - \frac{1}{2}J\frac{(K_0-K)K_0}{Kp^2}J$$

Differentiate wrt J:

$$\langle\phi(p_1)\phi(p_2)\rangle_{S_B} = \prod_{i=1}^{2}\frac{K_0(p_i)}{K(p_i)}\langle\phi(p_1)\phi(p_2)\rangle_{S_\Lambda} - \frac{(K_0-K)K_0}{Kp_1^2}\delta(p_1+p_2)$$

$$\langle\phi(p_1)...\phi(p_n)\rangle_{S_B} = \prod_{i=1}^{n}\frac{K_0(p_i)}{K(p_i)}\langle\phi(p_1)...\phi(p_n)\rangle_{S_\Lambda}$$

upto disconnected pieces involving terms of the form $\frac{(K_0-K)K_0}{Kp_i^2}\delta(p_i+p_j)$.

As $\Lambda \to 0$, $\bar{W}_\Lambda$ has less and less infmn about high momentum modes. To squeeze out that infmn the argument of $\bar{W}_\Lambda$ becomes singular: $K \to 0$! Because of the analytic nature of the cutoff, even for very small Λ all the infmn about high momentum modes is there.

Thus as long as $\frac{K_0}{K}$ is analytic no information is lost. Note also that the RHS

$$\prod_{i=1}^{n} \frac{K_0(p_i)}{K(p_i)} \langle \phi(p_1)...\phi(p_n) \rangle_{S_\Lambda}$$

is independent of Λ.

3.4.1 In terms of R

The object R is often introduced in ERG literature. It is defined by the following equation:

$$\boxed{\Delta_h = \frac{1}{R + p^2}} \tag{29}$$

One can think of it as a generalized mass term - so that even at $p = 0$ fluctuations of the field are damped.

$$R + p^2 = \frac{p^2}{1-K} \implies R = \frac{p^2 K}{1-K}$$

For simplicity we have taken $K_0 = 1$ corresponding to Λ_0 being infinity. Also

$$\frac{R}{K} = \frac{p^2}{1-K} = \frac{1}{\Delta_h}$$

We already have:

$$W_\Lambda[J] = -S_{\Lambda,I}[\Delta_h J] + \frac{1}{2} J \Delta_h J$$

In terms of S_Λ:

$$W_\Lambda[J] = \underbrace{-S_{\Lambda,I}[\Delta_h J] - \frac{1}{2}\Delta_h J \frac{1}{\Delta_l} \Delta_h J}_{-S_\Lambda} + \frac{1}{2} J \Delta_h J + \frac{1}{2} \Delta_h J \frac{1}{\Delta_l} \Delta_h J$$

$$\implies W_\Lambda[J] = -S_\Lambda[\Delta_h J] + \frac{1}{2} J \frac{\Delta_h(\Delta_l + \Delta_h)}{\Delta_l} J$$

So

$$\boxed{W_\Lambda[J] = -S_\Lambda[\Delta_h J] + \frac{1}{2} J \frac{1}{R} J} \tag{30}$$

3.5 Polchinski's ERG

3.5.1 Interacting Wilson Action - $S_{I,\Lambda}$

Now the ERG for $S_\Lambda[\phi]$ is the Polchinski equation:

$$\boxed{\frac{\partial S_{I,\Lambda}[\phi]}{\partial t} = -\frac{1}{2}\dot{\Delta}_l[(\frac{\partial S_{I,\Lambda}}{\partial \phi})^2 - (\frac{\partial^2 S_{I,\Lambda}}{\partial \phi^2})]} \tag{31}$$

Derivation: Start with the definition of $S_{I,\Lambda}[\phi]$:

$$e^{-S_{I,\Lambda}[\phi]} = \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h - S_{B,I}[\phi_l+\phi_h]} \tag{32}$$

Note that the entire t dependence in the RHS is from Δ_h. So

$$\frac{d}{dt}e^{-S_{I,\Lambda}[\phi]} = \int \mathcal{D}\phi_h(-\frac{1}{2}\phi_h \frac{d}{dt}(\Delta_h^{-1}))\phi_h e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h - S_{B,I}[\phi_l+\phi_h]} \tag{33}$$

$$= \int \mathcal{D}\phi_h \frac{1}{2}\phi_h \frac{\dot{\Delta}_h}{\Delta_h^2}\phi_h e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h - S_{B,I}[\phi_l+\phi_h]}$$

Now consider

$$\frac{\partial^2}{\partial \phi_l^2}e^{-S_{I,\Lambda}[\phi_l]} = \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h}\frac{\partial^2}{\partial \phi_l^2}e^{-S_{B,I}[\phi_l+\phi_h]} \tag{34}$$

$$= \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h}\frac{\partial^2}{\partial \phi_h^2}e^{-S_{B,I}[\phi_l+\phi_h]}$$

$$= \int \mathcal{D}\phi_h \frac{\partial^2}{\partial \phi_h^2}e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h}e^{-S_{B,I}[\phi_l+\phi_h]}$$

$$= \int \mathcal{D}\phi_h[(\phi_h \Delta_h^{-1})^2 - \Delta_h^{-1}]e^{-\frac{1}{2}\phi_h \Delta_h^{-1}\phi_h}e^{-S_{B,I}[\phi_l+\phi_h]}$$

Comparing (33) with (34) we see that

$$\frac{d}{dt}e^{-S_{I,\Lambda}[\phi_l]} = \frac{1}{2}\dot{\Delta}_h \frac{\partial^2}{\partial \phi_l^2}e^{-S_{I,\Lambda}[\phi_l]} \tag{35}$$

which is the same as (31) upto a field independent term.

Note that $\dot{\Delta}_l = -\dot{\Delta}_h$.

We can also write this as

$$\frac{\partial \psi'[\phi,t]}{\partial t} = -\frac{1}{2}\dot{\Delta}_l \frac{\partial^2 \psi'[\phi,t]}{\partial \phi^2} = -\frac{1}{2}\dot{G}\frac{\partial^2 \psi'[\phi,t]}{\partial \phi^2} \tag{36}$$

with $\psi' = e^{-S_{I,\Lambda}[\phi_l]}$. We will use G interchangeably with Δ_l below. And also as

$$\frac{\partial \psi'(x,t)}{\partial t} = -\frac{1}{2}\dot{\Delta}_l \frac{\partial^2 \psi'(x,t)}{\partial x^2} = -\frac{1}{2}\dot{G}\frac{\partial^2 \psi'(x,t)}{\partial x^2} \tag{37}$$

where x is used in place of ϕ. This is the non relativistic Schroedinger equation - Euclideanized.

3.5.2 Full Wilson Action - S_Λ:

One can write an equation for the full action

$$S_\Lambda[\phi] = \frac{1}{2}\phi\Delta_l^{-1}\phi_l + S_{I,\Lambda}[\phi]$$

Let

$$\psi(x,t) = e^{-\frac{1}{2}G^{-1}x^2}\psi'(x,t) = e^{-S(x,t)}$$

where ψ' obeys (37) and S is now the full action. Then it is very easy to see by substitution that

$$\boxed{\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2}\dot{G}(t)\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + 2G^{-1}x)\psi(x,t)} \tag{38}$$

Note the similarity with (12) - Wilson's original equation.

In terms of S it reads:

$$\boxed{\frac{\partial S}{\partial t} = -\frac{1}{2}\dot{G}(t)[\frac{\partial^2 S}{\partial x^2} - (\frac{\partial S}{\partial x})^2 + 2G^{-1}x\frac{\partial S}{\partial x}]} \tag{39}$$

3.5.3 $W_\Lambda[J]$

We need

$$\frac{dW_\Lambda}{dt} = -\frac{dS_{\Lambda,I}[\Delta_h J]}{dt} + \frac{1}{2}J\dot{\Delta}_h J \tag{40}$$

We see that

$$\frac{dS_{\Lambda,I}[\Delta_h J]}{dt} = \frac{\partial S_{\Lambda,I}[\Delta_h J]}{\partial t} + \dot{\Delta}_h J \frac{\partial S_{\Lambda,I}[\phi]}{\partial \phi}|_{\phi=\Delta_h J} \tag{41}$$

We can substitute (31) in the RHS of (41) and the following (let $\Delta_h J = \Phi$) obtained from (24):

$$-\frac{\partial S_{\Lambda,I}[\Phi]}{\partial \Phi} + \frac{\Phi}{\Delta_h} = \frac{\partial W}{\partial J}\frac{1}{\Delta_h}$$

and

$$-\frac{\partial^2 S_{\Lambda,I}[\Phi]}{\partial \Phi^2} + \frac{1}{\Delta_h} = \frac{\partial^2 W}{\partial J^2}\frac{1}{\Delta_h^2}$$

to obtain

$$\begin{aligned}\frac{dW_\Lambda}{dt} &= \frac{1}{2}\frac{\dot{\Delta}_h}{\Delta_h^2}[(\frac{\partial W_\Lambda}{\partial J})^2 + \frac{\partial^2 W_\Lambda}{\partial J^2}] - \frac{1}{2}\frac{\dot{\Delta}_h}{\Delta_h} \\ &= -\frac{1}{2}\frac{d}{dt}(\frac{1}{\Delta_h})[(\frac{\partial W_\Lambda}{\partial J})^2 + \frac{\partial^2 W_\Lambda}{\partial J^2}] - \frac{1}{2}\frac{\dot{\Delta}_h}{\Delta_h}\end{aligned} \tag{42}$$

This is almost the same as the equation for $S_{I,\Lambda}$ except that Δ_h^{-1} replaces Δ_h.

3.5.4 In terma of R

Since $\frac{1}{\Delta_h} = R + p^2(\frac{\mu}{p})^\eta$, $\frac{d}{dt}(\frac{1}{\Delta_h}) = \dot{R}$. (This is a propagator with anomalous dimension.) So the equation for W is:

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{W_\Lambda[J]} = \int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)}e^{W_\Lambda[J]}\,. \tag{43}$$

3.5.5 Anomalous Dimension

$$W[J] = \frac{1}{2}\int_p \frac{J(p)J(-p)}{p^2(\frac{\mu}{p})^\eta + R_\Lambda(p)} \tag{44}$$

is a quadratic action with anomalous dimension.

Let us introduce explicit Λ dependence in J by defining

$$J' = (\frac{\Lambda}{\mu})^{\frac{\eta}{2}}J, \qquad \Lambda\frac{\partial}{\partial\Lambda}J' = \frac{\eta}{2}J'$$

$$W[J] = W'[J'] = \frac{1}{2}\int_p \frac{(\frac{\Lambda}{\mu})^{-\eta}J'(p)J'(-p)}{p^2(\frac{\mu}{p})^\eta + R_\Lambda(p)}$$

$$= \frac{1}{2}\int_p \frac{J'(p)J'(-p)}{p^2(\frac{\Lambda}{p})^\eta + (\frac{\Lambda}{\mu})^\eta R_\Lambda(p)} \equiv \frac{1}{2}\int_p \frac{J'(p)J'(-p)}{p^2(\frac{\Lambda}{p})^\eta + R'_\Lambda(p)}$$

This defines R'. Noting that

$$\Lambda\frac{\partial R_\Lambda}{\partial\Lambda} = (\frac{\Lambda}{\mu})^{-\eta}(-\eta R'_\Lambda + \Lambda\frac{\partial R'_\Lambda}{\partial\Lambda}), \qquad \frac{\delta}{\delta J} = (\frac{\Lambda}{\mu})^\eta\frac{\delta}{\delta J'}$$

and plugging into (43) we get an equation with anomalous dimension.

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{W'_\Lambda[J']} = \int_p \left[\frac{\eta}{2}J'(p)\frac{\delta}{\delta J'(p)} + \left(\Lambda\frac{\partial R'_\Lambda(p)}{\partial\Lambda} - \eta R'_\Lambda(p)\right)\frac{1}{2}\frac{\delta^2}{\delta J'(p)\delta J'(-p)}\right]e^{W'_\Lambda[J']}\,. \tag{45}$$

4 "Solution" or Integral Representation

4.1 Polchinski's Eqn for S_I

We would like to solve Polchinski's equation:

$$\frac{\partial \psi'(x,t)}{\partial t} = -\frac{1}{2}\dot{G}\frac{\partial^2 \psi'(x,t)}{\partial x^2} \tag{46}$$

This can be written as:

$$\frac{\partial \psi'(x,t)}{\partial G} = -\frac{1}{2}\frac{\partial^2 \psi'(x,t)}{\partial x^2} \tag{47}$$

The solution we know from Feynman:

$$\psi'(x_f,t_f) = \frac{1}{\sqrt{2\pi(G_f - G_i)}}\int dx_i e^{\frac{1}{2}\frac{(x_f-x_i)^2}{G_f-G_i}}\psi'(x_i,t_i) \tag{48}$$

4.2 Polchinski Eqn for S

We would like to now solve the equation for the full action:

$$\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2}\dot{G}(t)\frac{\partial}{\partial x}(\frac{\partial}{\partial x} + 2G^{-1}x)\psi(x,t) \tag{49}$$

One way is to use (48) and use $\psi(x,t) = e^{-\frac{1}{2}x^2G^{-1}}\psi'(x,t)$:

$$\psi(x_f,t_f) = \frac{1}{\sqrt{2\pi(G_f - G_i)}}\int dx_i\ e^{-\frac{1}{2}x_f^2G_f^{-1}+\frac{1}{2}\frac{(x_f-x_i)^2}{G_f-G_i}+\frac{1}{2}x_i^2G_i^{-1}}\psi(x_i,t_i)$$

This can be rewritten as in (55) below.

Another way is to change variables $(x,t)\to(y,\tau)$: Let $y = \frac{x}{\sqrt{G(t)}}$ and $\tau = t$.

$$\frac{\partial}{\partial t} = -\frac{1}{2}\frac{\dot{G}}{G}y\frac{\partial}{\partial y} + \frac{\partial}{\partial \tau}; \qquad \frac{\partial}{\partial x} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial y}$$

Let us write $\psi'(x,t) = \tilde{\psi}(y(x,t),\tau(t))$ - and drop the tildes! Thus LHS of (49) becomes

$$\frac{\partial \psi}{\partial t}(x,t) = -\frac{1}{2}\frac{\dot{G}}{G}y\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial \tau}$$

RHS becomes (dropping a constant term)

$$-\frac{1}{2}\frac{\dot{G}}{G}\frac{\partial^2 \psi}{\partial y^2} - \frac{\dot{G}}{G}y\frac{\partial \psi}{\partial y}$$

So

$$\frac{\partial\psi}{\partial\tau} = -\frac{1}{2}\frac{\dot{G}}{G}\frac{\partial^2\psi}{\partial y^2} - \frac{\dot{G}}{G}y\frac{\partial\psi}{\partial y} + \frac{1}{2}\frac{\dot{G}}{G}y\frac{\partial\psi}{\partial y}$$

Define $g = -\ln G$ and and use $dg = \dot{g}d\tau$ to get

$$\frac{\partial\psi}{\partial g} = +\frac{1}{2}[\frac{\partial^2\psi}{\partial y^2} + y\frac{\partial\psi}{\partial y}] \tag{50}$$

This becomes a version of Wilson's equation (4) whose solution was:

$$\psi(y_f, t_f) = \frac{1}{\sqrt{2\pi(1-e^{t_i-t_f})}}\int dy_i e^{-\frac{1}{2}\frac{(y_f - e^{\frac{t_i-t_f}{2}}y_i)^2}{1-e^{t_i-t_f}}}\psi(y_i, t_i) \tag{51}$$

We just replace $t \to g$ to get

$$\psi(y_f, g_f) = \frac{1}{\sqrt{2\pi(1-e^{g_i-g_f})}}\int dy_i e^{-\frac{1}{2}\frac{(y_f - e^{\frac{g_i-g_f}{2}}y_i)^2}{1-e^{g_i-g_f}}}\psi(y_i, g_i) \tag{52}$$

Now

$$e^{g_1-g_f} = \frac{G_f}{G_i}, \quad e^{\frac{g_i-g_f}{2}} = \sqrt{\frac{G_f}{G_i}}$$

so we get

$$\psi(y_f, g_f) = \frac{1}{\sqrt{2\pi(1-\frac{G_f}{G_i})}}\int dy_i e^{-\frac{1}{2}\frac{(y_f - \sqrt{\frac{G_f}{G_i}}y_i)^2}{1-\frac{G_f}{G_i}}}\psi(y_i, g_i) \tag{53}$$

In the last step we restore the variable $x = y\sqrt{G}$:

$$\psi(x_f, g_f) = \frac{1}{\sqrt{2\pi(1-\frac{G_f}{G_i})}}\int dx_i e^{-\frac{1}{2}\frac{(\frac{x_f}{\sqrt{G_f}} - \sqrt{\frac{G_f}{G_i}}\frac{x_i}{\sqrt{G_i}})^2}{1-\frac{G_f}{G_i}}}\psi(x_i, g_i) \tag{54}$$

Thus the solution to (49) can be written in a more symmetric way:

$$\boxed{\psi(x_f, t_f) = \frac{1}{\sqrt{2\pi(1-\frac{G_f}{G_i})}}\int dx_i e^{-\frac{1}{2}\frac{(\frac{x_f}{G_f} - \frac{x_i}{G_i})^2}{\frac{1}{G_f} - \frac{1}{G_i}}}\psi(x_i, t_i)} \tag{55}$$

A standard form of all such solutions is

$$\psi(x_f,t_f) = \int dx_i \; e^{-\frac{1}{2}A^2(x_f - Zx_i)^2}\psi(x_i,t_i) \tag{56}$$

$Z(p)$ in this case is $\frac{G_f}{G_i} \approx e^{-p^2/\Lambda^2}$ and damps the high frequency modes as we have already seen in Wilson's original equation.

5 Composite Operators

5.1 Field theoretic Treatment

1.
$$\langle\phi(p_1).......\phi(p_n)\rangle_{S_B} = \prod_{i=1}^{n}\frac{K_0(p_i)}{K(p_i)}\langle\phi(p_1).......\phi(p_n)\rangle_{S_\Lambda} \tag{57}$$

   This is because

$$Z_B[J] = \int \mathcal{D}\phi e^{-S_B[\phi]+J\phi} = \int \mathcal{D}\phi' e^{-\frac{1}{2}\phi'\frac{1}{\Delta_l}\phi' + J\frac{\Delta}{\Delta_l}\phi' - \frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J + S_{I,\Lambda}[\phi']}$$
$$= \int \mathcal{D}\phi' e^{-S_\Lambda[\phi'] + J\frac{\Delta}{\Delta_l}\phi' - \frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J}$$

   So

$$\frac{\delta}{\delta J(p_1)}...\frac{\delta}{\delta J(p_1)}Z_B[J] = \langle\phi(p_1).......\phi(p_n)\rangle_{S_B} = \prod_{i=1}^{n}\frac{K_0(p_i)}{K(p_i)}\langle\phi(p_1).......\phi(p_n)\rangle_{S_\Lambda}$$

   We have used $\frac{\Delta}{\Delta_l} = \frac{K_0}{K}$.

2. When
$$\int \mathcal{D}\phi_h \; O_B[\phi_l + \phi_h] e^{-\frac{1}{2}\phi_h\Delta_h^{-1}\phi_h - S_{I,B}[\phi_l+\phi_h]} = O_\Lambda[\phi_l]e^{-S_{I,\Lambda}[\phi_l]}$$

   Then we say that
$$[O_B[\phi]] = O_\Lambda[\phi]$$

   is the **composite operator** corresponding to O_B.

   One can think of it as follows. Perturb S_B by a term of order ϵ and get the change in S_Λ to order ϵ:

$$\int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h\Delta_h^{-1}\phi_h - S_{I,B}[\phi_l+\phi_h] + \epsilon O_B[\phi_l+\phi_h]} = e^{-S_{I,\Lambda}[\phi_l] + \epsilon O_\Lambda[\phi_l]}$$

   One can include sources in S_B to get

$$e^{W_B[J,\epsilon]} = \int \mathcal{D}\phi e^{-\frac{1}{2}\phi\frac{1}{\Delta}\phi+J\phi-S_{I,B}[\phi]+\epsilon O_B[\phi]} \tag{58}$$

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l\frac{1}{\Delta_l}\phi_l+J\phi_l} \int \mathcal{D}\phi_h e^{-\frac{1}{2}\phi_h\frac{1}{\Delta_h}\phi_h-S_{I,B}[\phi_l+\phi_h]+\epsilon O_B[\phi_l+\phi_h]+J\phi_h} \tag{59}$$

$$= \int \mathcal{D}\phi' e^{J\frac{\Delta}{\Delta_l}\phi'-S_\Lambda[\phi']+\epsilon O_\Lambda[\phi']-\frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J} \tag{60}$$

Differentiating wrt J we get

$$\langle O_B[p]\phi(p_1)\phi(p_2)....\phi(p_n)\rangle_{S_B} = \prod_{i=1,n}\frac{\Delta(p_i)}{\Delta_l(p_i)}\langle O_\Lambda[p]\phi(p_1)\phi(p_2)....\phi(p_n)\rangle_{S_\Lambda}$$

Note that if $p_i+p_j=0$ one can get extra disconnected pieces proportional to $\frac{\Delta_h\Delta}{\Delta_l}(p_i)$. The LHS is independent of Λ, therefore so is the RHS. This explains the raison d'etre for defining the composite operators.

3. Schematically

$$U(t_f,t_i)O(t_i)|\psi(t_i)\rangle = O(t_f)U(t_f,t_i)|\psi(t_i)\rangle$$

or

$$\boxed{O(t_f) = U(t_f,t_i)O(t_i)U^{-1}(t_f,t_i)}$$

4. Example of a composite operator:

$$\int \mathcal{D}\phi\frac{\delta S_B}{\delta\phi}e^{-S_B[\phi]+J\phi} = \int \mathcal{D}\phi J e^{-S_B[\phi]+J\phi} = \int \mathcal{D}\phi J e^{-S_\Lambda[\phi]+J\frac{K_0}{K}\phi-\frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J}$$

$$= \int \mathcal{D}\phi(\frac{K}{K_0}\frac{\delta}{\delta\phi}e^{+J\frac{K_0}{K}\phi})e^{-S_\Lambda[\phi]-\frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J} = -\frac{K}{K_0}\int \mathcal{D}\phi e^{+J\frac{K_0}{K}\phi-\frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J}(\frac{\delta}{\delta\phi}e^{-S_\Lambda[\phi]})$$

$$= \frac{K}{K_0}\int \mathcal{D}\phi\frac{\delta S_\Lambda[\phi]}{\delta\phi}e^{-S_\Lambda[\phi]}e^{+J\frac{K_0}{K}\phi-\frac{1}{2}J\frac{\Delta_h\Delta}{\Delta_l}J}$$

So

$$[K_0\frac{\delta S_B}{\delta\phi}] = K\frac{\delta S_\Lambda}{\delta\phi}$$

5. What is $[\phi]$?

$$\int \mathcal{D}\phi\phi e^{-S_B[\phi]+J\phi} = \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l\frac{1}{\Delta_l}\phi_l+J\phi_l}\int \mathcal{D}\phi_h(\phi_l+\phi_h)e^{-\frac{1}{2}\phi_h\frac{1}{\Delta_h}\phi_h-S_{I,B}[\phi_l+\phi_h]+J\phi_h}$$

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} \int \mathcal{D}\phi_h (\phi_l + \frac{\delta}{\delta J}) e^{-\frac{1}{2}\phi_h \frac{1}{\Delta_h}\phi_h - S_{I,B}[\phi_l+\phi_h] + J\phi_h}$$

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} (\phi_l + \frac{\delta}{\delta J}) e^{-S_{I,\Lambda}[\phi_l + \Delta_h J] + \frac{1}{2}J\Delta_h J}$$

$$= \int \mathcal{D}\phi_l e^{-\frac{1}{2}\phi_l \frac{1}{\Delta_l}\phi_l + J\phi_l} (\phi_l - \Delta_h \frac{\delta S_{I,\Lambda}}{\delta \phi_l}) e^{-S_{I,\Lambda}[\phi_l + \Delta_h J] + \frac{1}{2}J\Delta_h J}|_{J=0}$$

So

$$[\phi] = \phi_l - \Delta_h \frac{\delta S_{I,\Lambda}}{\delta \phi_l}$$

$$[\phi] = \phi_l - \frac{K_0 - K}{p^2} \frac{\delta S_{I,\Lambda}}{\delta \phi_l}$$

Use

$$\frac{\delta S_{I,\Lambda}}{\delta \phi_l} = \frac{\delta S_\Lambda}{\delta \phi_l} - \frac{1}{\Delta_l}\phi_l$$

to get

$$[\phi] = \frac{\Delta}{\Delta_l}\phi_l - \Delta_h \frac{\delta S_\Lambda}{\delta \phi_l}$$

$$= \frac{K_0}{K}\phi_l - \frac{K_0 - K}{p^2} \frac{\delta S_\Lambda}{\delta \phi_l}$$

Better derivation - without $J = 0$:

$$\frac{\delta}{\delta J} \int \mathcal{D}\phi e^{-S_B[\phi] + J\phi} = \frac{\delta}{\delta J} \int \mathcal{D}\phi e^{-\frac{1}{2}\phi \frac{1}{\Delta_l}\phi + J\frac{\Delta}{\Delta_l}\phi - \frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_{I,\Lambda}[\phi]}$$

$$= \int \mathcal{D}\phi (\frac{\Delta}{\Delta_l}\phi - \Delta_h \frac{\Delta}{\Delta_l} J) e^{-\frac{1}{2}\phi \frac{1}{\Delta_l}\phi + J\frac{K_0}{K}\phi - \frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_{I,\Lambda}[\phi]}$$

$$= \int \mathcal{D}\phi (\frac{K_0}{K}\phi - \frac{K_0 - K}{p^2}\frac{K_0}{K} J) e^{-\frac{1}{2}\phi \frac{1}{\Delta_l}\phi + J\frac{K_0}{K}\phi - \frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_{I,\Lambda}[\phi]}$$

The second term is

$$\int \mathcal{D}\phi \; \Delta_h (\frac{\delta}{\delta \phi} e^{J\frac{\Delta}{\Delta_l}\phi}) e^{-\frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_\Lambda[\phi]} = \int \mathcal{D}\phi \; \Delta_h \frac{\delta S_\Lambda[\phi]}{\delta \phi} e^{J\frac{\Delta}{\Delta_l}\phi - \frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_\Lambda[\phi]}$$

$$= \int \mathcal{D}\phi \; \frac{K_0 - K}{p^2} (\frac{\delta}{\delta \phi} e^{J\frac{K_0}{K}\phi}) e^{-\frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_\Lambda[\phi]} = \int \mathcal{D}\phi \; \frac{K_0 - K}{p^2} \frac{\delta S_\Lambda[\phi]}{\delta \phi} e^{J\frac{K_0}{K}\phi - \frac{1}{2}J\frac{\Delta_h \Delta}{\Delta_l}J - S_\Lambda[\phi]}$$

Thus

$$\boxed{[\phi] = \frac{\Delta}{\Delta_l}\phi_l - \Delta_h\frac{\delta S_\Lambda}{\delta\phi_l} = \frac{K_0}{K}\phi - \frac{K_0 - K}{p^2}\frac{\delta S_\Lambda[\phi]}{\delta\phi} \qquad (61)}$$

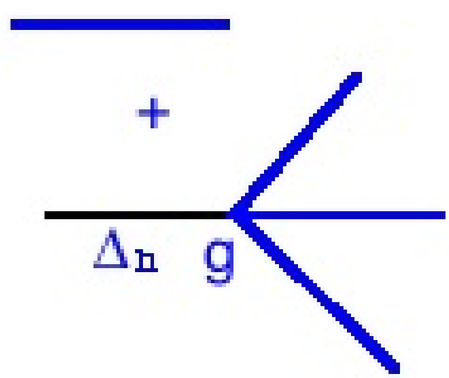
Proved without having to set $J = 0$.

What does the term $\Delta_h\frac{\delta S_\Lambda}{\delta\phi_l}$ mean intuitively?

Assume $g\phi^4$ theory: $\phi = \phi_l + \phi_h$ So

$$\phi^4 = (\phi_l + \phi_h)^4 = \phi_l^4 + 4\phi_h\phi_l^3 + ...$$

$$\phi \rightarrow\approx \phi_l + g\langle\phi_h\phi_h\rangle\phi_l^3 = \phi_l + g\Delta_h\phi_l^3$$



6. **Number Operator:** Acting on n field correlators it should give n. Thus

$$\int \mathcal{D}\phi(\int_p \phi(p)\frac{\delta}{\delta\phi(p)}e^{J\phi})e^{-S_B[\phi]} = -\int \mathcal{D}\phi\frac{d}{d\phi}(\phi e^{-S_B})e^{J\phi}$$

$$= \int \mathcal{D}\phi(\phi\frac{dS_B}{d\phi} - \frac{d\phi}{d\phi})e^{-S_B}e^{J\phi}$$

$$= J\int \mathcal{D}\phi\ \phi e^{J\phi - S_B[\phi]} = J\int \mathcal{D}\phi[\phi]e^{-S_\Lambda[\phi]+\frac{K_0}{K}J\phi-\frac{1}{2}\frac{K_0-K}{p^2}\frac{K_0}{K}JJ}$$

$$= \int \mathcal{D}\phi e^{-S_\Lambda[\phi]-\frac{1}{2}\frac{K_0-K}{p^2}\frac{K_0}{K}JJ}[\phi](\frac{K}{K_0}\frac{\delta}{\delta\phi}e^{\frac{K_0}{K}J\phi})$$

$$= -\int \mathcal{D}\phi\ (\frac{K}{K_0}\frac{\delta}{\delta\phi}[\phi]e^{-S_\Lambda[\phi]})e^{-\frac{1}{2}\frac{K_0-K}{p^2}\frac{K_0}{K}JJ+\frac{K_0}{K}J\phi}$$

So the number operator is

$$-\frac{K}{K_0}(\frac{\delta}{\delta\phi}[\phi] - [\phi]\frac{\delta S_\Lambda}{\delta\phi})$$

Inserting (61) we get

$$\boxed{[\phi\frac{dS_B}{d\phi}] = \phi\frac{\delta S_\Lambda[\phi]}{\delta\phi} + \frac{K}{K_0}\frac{K_0-K}{p^2}[\frac{\delta^2 S_\Lambda}{\delta\phi^2} - (\frac{\delta S_\Lambda}{\delta\phi})^2] = \mathcal{N} + \infty}$$

$$\boxed{= \phi\frac{\delta S_\Lambda[\phi]}{\delta\phi} + \frac{\Delta_l}{\Delta}\Delta_h[\frac{\delta^2 S_\Lambda}{\delta\phi^2} - (\frac{\delta S_\Lambda}{\delta\phi})^2]} \tag{62}$$

and up to a field independent constant $\int_p \frac{K_0}{K}\frac{K\delta\phi(p)}{K_0\delta\phi(p)} = \infty$ this is the number operator $\mathcal{N}$. Like a normal ordering constant.

5.2 A Conceptually Clearer Treatment: Number Operator

5.2.1

Consider the solution to Wilson's RG:

$$\psi(x,t) = U(t,t_i)\psi(x,t_i) = e^{\int_{t_i}^t dt'\mathcal{G}_{RG}(t')}\psi(x_i,t_i)$$

U is an"evolution " operator corresponding to the RG equation

$$\frac{d\psi}{dt} = \mathcal{G}_{RG}(t)\psi$$

It can be represented by an integration kernel:

$$\psi(x,t) = \int dx_i K(x,t;x_i,t_i)\psi(x_i,t_i)$$

$$= \int dx_i e^{\frac{1}{2}\frac{(x-x_i)^2}{G-G_i}-\frac{1}{2}\frac{x^2}{G}+\frac{1}{2}\frac{x_i^2}{G_i}}\psi(x_i,t_i) \tag{63}$$

One can ask: What operator acting on $\psi(x,t)$ has the same effect as $x_i \frac{\partial}{\partial x_i}$ on $\psi(x_i,t_i)$? Call it $\mathcal{N}$. Let

$$\mathcal{N} = Ax\frac{\partial}{\partial x} + B\frac{\partial^2}{\partial x^2}$$

$$\mathcal{N}\psi(x,t) = U(t,t_i)x\frac{\partial}{\partial x}\psi(x,t_i) \implies \mathcal{N}U(t,t_i)\psi(x,t_i) = U(t,t_i)x\frac{\partial}{\partial x}\psi(x,t_i)$$

$$\boxed{\mathcal{N} = U(t,t_i)x\frac{\partial}{\partial x}U(t,t_i)^{-1}} \tag{64}$$

In the integral kernel representation:

$$(Ax\frac{\partial}{\partial x} + B\frac{\partial^2}{\partial x^2})\int dx_i K(x,t;x_i,t_i)\psi(x_i,t_i) = \int dx_i K(x,t;x_i,t_i)x_i\frac{\partial}{\partial x_i}\psi(x_i,t_i) \tag{65}$$

RHS

$$= -\int dx_i \frac{\partial}{\partial x_i}(K(x,t;x_i,t_i)x_i)\psi(x_i,t_i) = -\int dx_i[(1+x_i\frac{\partial}{\partial x_i})K(x,t;x_i,t_i)]\psi(x_i,t_i)$$

$$= -\int dx_i[(1+x_i(-\frac{(x-x_i)}{G-G_i}+\frac{x_i}{G_i})K(x,t;x_i,t_i)]\psi(x_i,t_i)$$

$$= -\int dx_i[(1+(\frac{x_i^2}{G_i(G-G_i)}-\frac{xx_i}{G-G_i})]K(x,t;x_i,t_i)]\psi(x_i,t_i)$$

Now LHS:

$$Ax\frac{\partial}{\partial x}K(x,t;x_i,t_i) = A[x^2(\frac{1}{G-G_i}-\frac{1}{G})-\frac{xx_i}{G-G_i}]$$

$$B\frac{\partial^2}{\partial x^2}K(x,t;x_i,t_i) = B[x^2(\frac{1}{(G-G_i)^2}+\frac{1}{G^2}-\frac{2}{G(G-G_i)})+\frac{2xx_i}{G(G-G_i)}-\frac{2xx_i}{(G-G_i)^2}+\frac{x_i^2}{(G-G_i)^2}]$$

Comparing the coefficients of x_i^2 we get

$$B = -\frac{G(G-G_i)}{G_i}$$

Comparing the coefficients of xx_i we get (on substituting for B):

$$A = 1$$

One can check that the coefficient of x^2 is zero. Thus

$$\boxed{\mathcal{N} = U(t,t_i)x\frac{\partial}{\partial x}U(t,t_i)^{-1} = -1 + x\frac{\partial}{\partial x} + \frac{G(G_i-G)}{G_i}\frac{\partial^2}{\partial x^2}} \tag{66}$$

-1 is some kind of normal ordering constant - keeps cropping up.

5.2.2 Another way

$$e^{a\mathcal{N}}U(t,t_i) = U(t,t_i)e^{ax\frac{\partial}{\partial x}}$$

defines $\mathcal{N}$.

$$U = e^{\frac{A}{2}\frac{\partial^2}{\partial x^2}+sx\frac{\partial}{\partial x}}$$

$$\therefore Ue^{ax\frac{\partial}{\partial x}} = e^{\frac{(e^{ns}-1)}{ns}\text{“}x^n\text{”}}e^{ax\frac{\partial}{\partial x}}e^{sx\frac{\partial}{\partial x}} \qquad where \qquad \text{“}x^n\text{”} = \frac{A}{2}\frac{\partial^2}{\partial x^2}\ ; \quad n = -2$$

$$= e^{ax\frac{\partial}{\partial x}}e^{\frac{(e^{ns}-1)}{ns}e^{2a}\frac{A}{2}\frac{\partial^2}{\partial x^2}}e^{sx\frac{\partial}{\partial x}} = e^{ax\frac{\partial}{\partial x}}e^{\frac{(e^{ns}-1)}{ns}(e^{2a}-1)\frac{A}{2}\frac{\partial^2}{\partial x^2}}U$$

$$= e^{\frac{(e^{ns}-1)}{ns}2a\frac{A}{2}\frac{\partial^2}{\partial x^2}+ax\frac{\partial}{\partial x}}U = e^{a\mathcal{N}}U$$

$$\implies \mathcal{N} = \frac{(1-e^{-2s})}{2s}2\frac{A}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x}$$

What are A, s?

$$U = e^{\int_{g_i}^{g}dg\ \frac{1}{2}[\frac{\partial^2}{\partial y^2}+y\frac{\partial}{\partial y}]} = e^{(g-g_i)\frac{1}{2}\frac{\partial^2}{\partial y^2}+(g-g_i)\frac{1}{2}y\frac{\partial}{\partial y}}$$

$$A = (g-g_i); \quad s = \frac{1}{2}(g-g_i); \quad ns = -2s = g_i - g; \quad e^{ns} = e^{-2s} = e^{g_i-g} = \frac{G}{G_i}$$

$$e^{a\mathcal{N}} = e^{a[\frac{G_i-G}{G_i}\frac{\partial^2}{\partial y^2}+y\frac{\partial}{\partial y}]}$$

When acting on $\psi(y_f,t)$ ($x_f = \sqrt{G}_f y_f$). Writing x for x_f, $\frac{\partial^2}{\partial y^2} = G\frac{\partial^2}{\partial x^2}$. So acting on $\psi(x,t)$:

$$\boxed{e^{a\mathcal{N}} = e^{a[\frac{G(G_i-G)}{G_i}\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}]}} \tag{67}$$

5.3 Wilson's scaling operator

Wilson defines a uniform scaling by

$$f(x) = \int dy e^{\frac{1}{2}\frac{(x-e^{-a}y)^2}{1-e^{-2a}}}g(y)$$

This is obtained by

$$f(y) = e^{a\frac{\partial}{\partial y}(\frac{\partial}{\partial y}+y)}g(y)$$

This is a smooth transformation, in contrast to

$$f(x) = \int dy\delta(y-e^a x)g(y)$$

which is obtained by

$$f(y) = e^{ay\frac{\partial}{\partial y}}g(y)$$

Also Wilson's scaling operator $\frac{\partial}{\partial y}(\frac{\partial}{\partial y} + y)$, commutes with the ERG evolution, so it is marginal. Also the constant "c" which is the p independent coefficient of the ERG equation (12), multiplies precisely this operator and gives the dimension - including anomalous.

6 Fixed Points and Anomalous Dimension

6.1 What is a Fixed Point?

If we keep evolving RG for a long time one can expect that if a coupling constant is restricted to be finite, then either it has to go to a fixed value after some time or it has to enter some limit cycle. Fixed points are more common. (Though there are some examples of limit cycles recently - due to an angular variable that keeps increasing with time - since Physics is periodic in 2π we have an example of a limit cycle.)

At a fixed point physics is scale invariant - by definition. How can it be scale invariant if there is a scale -the RG scale Λ? Answer: If there is no *other* scale in the problem, then this scale has no physical meaning - everything is relative to this scale. So to see this behaviour one has to work in dimensionless variables. Everything is in units of Λ. Thus

$$p = \bar{p}\Lambda, \phi = \bar{\phi}\Lambda^{d_\phi}$$

where d_ϕ is the dimension of the field. In scalar field theory the dimension of $\phi(x)$ is $\frac{D}{2} - 1$.

$$[\phi(x)] = \frac{D}{2} - 1, \quad [\phi(p)] = -\frac{D}{2} - 1$$

Note that Λ keeps getting smaller as we evolve, so the dimensionless variable keeps changing w.r.t the dimensionful variable - which is the original physical object.

So at the fixed point one finds that the action, written in terms of $\bar{\phi}$ and $\bar{p}$ has dimensionless coupling constants. *At a fixed point, as you do RG transformation these dimensionless coupling constants remain the same. Thus they are independent of* Λ. As a simple example take a mass parameter m. We define dimensionless $\bar{m}$ by

$$m^2 = \bar{m}^2\Lambda^2$$

Normally, ie. when we do perturbation about the trivial fixed point or free theory, m^2 is fixed, so $\bar{m}^2$ being equal to m^2/Λ^2 will increase as Λ is decreased. This is the notorious naturalness problem for light scalars. However at a non trivial fixed point this is not the case. $\bar{m}^2$ is fixed. That means $m^2 = \bar{m}^2\Lambda^2$ has to "run" with Λ (due to interactions) in such a way that $\bar{m}^2$ is fixed.

6.1.1 Rescaling and Dimensionless Variables

First we define dimensionless variables:

$$\phi(p) = \Lambda^{d^c_\phi}\bar{\phi}(\bar{p}) \quad ; \quad p = \Lambda\bar{p}$$

The Wilson action is:

$$S_\Lambda[\phi] = \sum_n \int_{p_i,..,p_{n-1}} V_n(p_1, p_2, ..., p_n, \Lambda)\phi(p_1)\phi(p_2)....\phi(p_n)$$
$$= \sum_{n,m} \int_{p_i,..,p_{n-1}} V_{n,2m}(\Lambda)(p_i.p_j)^{2m}\phi(p_1)\phi(p_2)....\phi(p_n) \tag{68}$$

$V_{n,2m}(\Lambda)$ are the coupling constants - they can be dimensionful. The Λ dependence is due to RG evolution.

We write the RG equation (39) as

$$\frac{\partial S}{\partial t} = \mathcal{G}_{RG}S \tag{69}$$

In terms of dimensionless variables:

$$S_\Lambda[\bar{\phi}] = \sum_n \int_{\bar{p}_i,..,\bar{p}_{n-1}} \Lambda^{(n-1)D}V_n(p_1, p_2, ..., p_n, \Lambda)\Lambda^{nd^c_\phi}\bar{\phi}(\bar{p}_1)\bar{\phi}(\bar{p}_2)....\bar{\phi}(\bar{p}_n)$$
$$= \sum_{n,m} \int_{\bar{p}_i,..,\bar{p}_{n-1}} \Lambda^{(n-1)D}V_{n,2m}(\Lambda)\Lambda^{2m}(\bar{p}_i.\bar{p}_j)^{2m}\Lambda^{nd^c_\phi}\bar{\phi}(\bar{p}_1)\bar{\phi}(\bar{p}_2)....\bar{\phi}(\bar{p}_n)$$
$$= \sum_{n,m} \int_{\bar{p}_i,..,\bar{p}_{n-1}} \Lambda^{(n-1)D+2m+nd^c_\phi}V_{n,2m}(\Lambda)(\bar{p}_i.\bar{p}_j)^{2m}\bar{\phi}(\bar{p}_1)\bar{\phi}(\bar{p}_2)....\bar{\phi}(\bar{p}_n)$$
$$\equiv \sum_{n,m} \int_{\bar{p}_i,..,\bar{p}_{n-1}} \bar{V}_{n,2m}(t)(\bar{p}_i.\bar{p}_j)^{2m}\bar{\phi}(\bar{p}_1)\bar{\phi}(\bar{p}_2)....\bar{\phi}(\bar{p}_n) \tag{70}$$

Thus we have defined dimensionless coupling constants

$$\bar{V}_{n,2m}(t) = \Lambda^{(n-1)D+2m+nd^c_\phi}V_{n,2m}(\Lambda) \tag{71}$$

A fixed point is characterised by $\bar{V}_{n,2m}(t)$ being t-independent - they should not run. Thus if there is a mass parameter for instance: $m = \bar{m}\Lambda$, then $\bar{m}$ should not change by RG evolution. Normally when m is the mass parameter, it is m that is fixed and $\bar{m}$ goes as e^t. But at a fixed point, due to effects of interactions, $\bar{m}$ should be independent of t.

$$\frac{dV_{n,2m}(\Lambda)}{dt} = [\frac{\partial\bar{V}_{n,2m}(t)}{\partial t} + [(n-1)D + 2m + nd^c_\phi]\bar{V}_{n,m}(t)]\Lambda^{(n-1)D+2m+nd^c_\phi}$$

The t dependence of the LHS is due to RG evolution. The RHS has two contributions: One is the running of the dimensionless coupling due to tRG and the second is the effect of change of units as we evolve: all the factors of Λ which are used to make the coupling dimensionless - Λ is the unit and it is changing.

Thus the LHS is given by Wilson's (Polchinski's) eqn and the fixed point condition is for $\frac{\partial \bar{V}_{n,2m}(t)}{\partial t}$ to vanish. Let us concentrate on a particular term. The RG equation is:

$$\frac{dS}{dt} = \int_{p_i,..,p_{n-1}} \frac{dV_{n,2m}(\Lambda)}{dt}(p_i.p_j)^{2m}\phi(p_1)\phi(p_2)....\phi(p_n) = \mathcal{G}_{RG}S$$

$$LHS = \int_{p_i,..,p_{n-1}}[\frac{\partial \bar{V}_{n,2m}(t)}{\partial t}+[(n-1)D+2m+nd^c_\phi]\bar{V}_{n,m}(t)]\Lambda^{(n-1)D+2m+nd^c_\phi}](p_i.p_j)^{2m}\phi(p_1)\phi(p_2)....\phi(p_n)$$

$$= \int_{\bar{p}_i,..,\bar{p}_{n-1}}[\frac{\partial \bar{V}_{n,2m}(t)}{\partial t}+[(n-1)D+2m+nd^c_\phi]\bar{V}_{n,m}(t)]](\bar{p}_i.\bar{p}_j)^{2m}\bar{\phi}(\bar{p}_1)\bar{\phi}(\bar{p}_2)....\bar{\phi}(\bar{p}_n)$$

$$\equiv \frac{\partial S[\bar{\phi}]}{\partial t} - \mathcal{G}^c_{dil}S[\bar{\phi}]$$

So

$$\mathcal{G}^c_{dil}S = -[(n-1)D+2m+nd^c_\phi] = -[N_\phi(D+d^c_\phi)+N_p-D] \tag{72}$$

So now the RG eqn becomes

$$\frac{\partial S[\bar{\phi}]}{\partial t} - \mathcal{G}^c_{dil}S[\bar{\phi}] = \mathcal{G}_{RG}S[\bar{\phi}] \tag{73}$$

$\frac{\partial S[\bar{\phi}]}{\partial t} = 0$ is the fixed point condition, modulo a subtlety: We must ensure that the kinetic term has standard normalization before imposing this. Thus if we have two actions related by a fixed rescaling of the field, these two are equivalent, even though the coupling constants are all different.

6.1.2 A more formal way of doing the same thing:

Define dimensionless variables as before:

$$p = \Lambda\bar{p} \;\;;\;\; \phi(p) = \Lambda^{d_\phi}\bar{\phi}(\bar{p}) = \Lambda^{d_\phi}\bar{\phi}(\frac{p}{\Lambda}) \tag{74}$$

From this one can undertand how a scaling changes the fields. Thus think of a transformation law:

$$p' = \lambda p \;\;;\;\; \phi'(p') = \lambda^{d^p_\phi}\phi(p) = \lambda^{d^p_\phi}\phi(p'/\lambda) \tag{75}$$

(Note that scalars under coordinate transf obey $\phi'(p') = \phi(p)$. In this case because it has a scaling dimension it is not a scalar under these transformations - it transforms multiplicatively by λ^{d_ϕ}.)

Thus

$$\phi'(p') = (1+\epsilon)^{d^p_\phi}\phi(p(1-\epsilon))$$

$$= \phi(p) + \epsilon(d^p_\phi - p\frac{d}{dp})\phi(p)$$

So

$$\delta\phi(p) = \epsilon(d^p_\phi - p\frac{d}{dp})\phi(p) \tag{76}$$

Here d^p_ϕ is the dimension of the scalar field in *momentum* space. Thus

$$[\phi(x)] = \frac{1}{2}(D-2) = d^x_\phi \quad ; \quad [\phi(p)] = -\frac{1}{2}(D+2) = d^p_\phi$$

So when we have an action $S[\phi(p_i)]$ we implement this by - note the sign convention chosen to match (72):

$$\delta S[\phi(p_i)] = \int_q (q\frac{d}{dq} - d^p_\phi)\phi(q)\frac{\delta}{\delta\phi(q)}S[\phi(p_i)] \tag{77}$$

Let us apply this to

$$S[\phi(p_i)] = \int_{p_1,p_2,...p_n} u(p_1,p_2,...,p_n)\phi(p_1)\phi(p_2)...\phi(p_n)\delta^D(p_1+p_2+...+p_n)$$

$$\delta S = \int_{p_1,p_2,...p_n} u(p_1,p_2,...,p_n)\delta^D(p_1+p_2+...+p_n)\sum_{i=1}^n(p_i\frac{d}{dp_i} - d^p_\phi)[\phi(p_1)\phi(p_2)...\phi(p_n)]$$

Integrate by parts on p_i to get

$$= \int_{p_1,p_2,...p_n}(\sum_{i=1}^n(-p_i\frac{d}{dp_i} - d^p_\phi - D)[u(p_1,p_2,...,p_n)\delta^D(p_1+p_2+...+p_n)]\phi(p_1)\phi(p_2)...\phi(p_n)$$

$$= \int_{p_1,p_2,...p_n}\{(\sum_{i=1}^n(-p_i\frac{d}{dp_i} - d^p_\phi - D)[u(p_1,p_2,...,p_n)]\delta^D(p_1+p_2+...+p_n)$$

$$+u(p_1,p_2,...,p_n)\sum_{i=1}^n(-p_i\frac{d}{dp_i}[\delta^D(p_1+p_2+...+p_n)]\}\phi(p_1)\phi(p_2)...\phi(p_n)$$

The scaling dimension of the delta function is $-D$. So we get

$$= \int_{p_1,p_2,...p_n}\{(\sum_{i=1}^n(-p_i\frac{d}{dp_i})[u(p_1,p_2,...,p_n)]\delta^D(p_1+p_2+...+p_n)$$

$$-nd^p_\phi - (n-1)D[u(p_1,p_2,...,p_n)]\delta^D(p_1+p_2+...+p_n)\}\phi(p_1)\phi(p_2)...\phi(p_n)$$

$$= [-N_\phi(D+d^p_\phi) + D - N_p]S \tag{78}$$

where we have set $n = N_\phi$ and $\sum_{i=1}^n p_i\frac{d}{dp_i} = N_p$ which is (72).

So to conclude: a compact way is to write

$$\mathcal{G}_{dil}S = \int_q (q\frac{d}{dq} - d^p_\phi)\phi(q)\frac{\delta}{\delta\phi(q)}S \tag{79}$$

6.2 Rescaling or Wave Function Renormalization

Consider an illustrative example. Suppose the fixed point action has a form

$$S_\Lambda = \int \frac{d^D\bar{p}}{(2\pi)^D}\ [\frac{1}{2}\bar{\phi}(\bar{p})\bar{p}^2\bar{\phi}(-\bar{p}) + \frac{1}{2}\bar{m}^2\bar{\phi}^2 + \frac{\bar{\lambda}}{4!}\bar{\phi}^4 + ...]$$

with $\phi = \Lambda^{d_\phi}\bar{\phi}$ And suppose after an evolution from $\Lambda \to \frac{1}{2}\Lambda$ it becomes

$$S_{\Lambda/2} = \int \frac{d^D\bar{p}}{(2\pi)^D}\ [\frac{1}{2}(\sqrt{2}\bar{\phi}(p))\bar{p}^2(\sqrt{2}\bar{\phi}(-\bar{p})) + \frac{1}{2}\bar{m}^2(\sqrt{2}\bar{\phi})^2 + \frac{\bar{\lambda}}{4!}(\sqrt{2}\bar{\phi})^4 + ...]$$

where now $\phi(p) = (\frac{\Lambda}{2})^{d_\phi}\bar{\phi}(\bar{p})$. Note that since ϕ is fixed, $\bar{\phi}$ is larger now by a factor of 2^{d_ϕ}. This is just due to the engineering dimension of ϕ. The new action $S_{\Lambda/2}$ is clearly physically the same as the old one. One can see this by rescaling:

$$\sqrt{2}\bar{\phi} = \tilde{\phi}$$

In terms of $\tilde{\phi}$ the actions are identical. This for every factor of 2 (decrease) in Λ there is a factor of $\sqrt{2}$ in $\tilde{\phi}$. This is then over and above the factor of 2^{d_ϕ} that is already there for engineering dimensions. Net scaling factor is $2^{d_\phi+\frac{1}{2}}$. Thus we can say that the *scaling* dimension of ϕ has changed from $d_\phi \to d_\phi + \frac{1}{2}$. And we can write:

$$\phi(p) = \Lambda^{d_\phi}(\frac{\Lambda}{\Lambda_0})^{\frac{1}{2}}\tilde{\phi}(\bar{p})$$

In terms of $\tilde{\phi}$ the action is really fixed. Note that the engineering dimension of ϕ has not changed - only the *scaling* dimension has changed - the change (in our example, $\frac{1}{2}$) is called the **anomalous dimension**. We have a dimensionless ratio of scales for the anomalous dimension - so that the engineering dimension is unchanged.

Thus it makes sense to require time independent couplings only after we perform a rescaling over and above that required to make the field dimensionless. We can then impose a given normalization condition - say for the kinetic term. This in general will require a continuous time dependent field rescaling as we evolve, because the kinetic term will keep getting corrections. This additional time dependent rescaling of the field variable effectively changes the **scaling dimension** of the field - this change is denoted by $\frac{\eta}{2}$. [1]

To summarize: When the evolution is continuous and parametrized by t one obtains a factor of the form $e^{\eta t}$ multiplying the kinetic term. Thus *if one wants a fixed point solution* one has to rescale the field by this factor before comparing coupling constants and deciding whether they are fixed or not. Thus we define a new field by

$$\bar{\phi} = e^{-\frac{\eta}{2}t}\tilde{\phi} = (\frac{\Lambda}{\Lambda_0})^{\frac{\eta}{2}}\tilde{\phi} \tag{80}$$

[1] Of course it is worth repeating that the engineering dimension cannot change.

or

$$\phi = \Lambda^{d_\phi}(\frac{\Lambda}{\Lambda_0})^{\frac{\eta}{2}}\tilde{\phi} \tag{81}$$

The engineering dimension of $\tilde{\phi}$ continues to be zero. The *scaling dimension* of ϕ is thus said to be modified. Expressed in terms of $\tilde{\phi}, \bar{p}$ the fixed point action is really fixed and does not change under RG evolution.

The parameter $\frac{\eta}{2}$ is the **anomalous dimension** of ϕ. Note that it is introduced in order to be able to write down a fixed point action. So it is tied to the physics of a particular fixed point - whatever scaling is necessary to keep the action invariant has to be performed and the amount is decided by the interactions.

This is the same factor that is called "wave function renormalization" in renormalization theory : $\phi_B = Z^{\frac{1}{2}}\phi_R$. and $\frac{d\ln Z^{\frac{1}{2}}}{d\ln\Lambda} = \gamma = \frac{\eta}{2}$ is the usual terminology.

6.3 Effect of η on two point function

Suppose we calculate the two point function using the fixed point action:

$$\langle\tilde{\phi}(\bar{p})\tilde{\phi}(\bar{q})\rangle$$

This is a dimensionless quantity and must be of the form $\bar{p}^a\delta^D(\bar{p}+\bar{q})$ for some a. Using (81) we see that

$$\langle\phi(p)\phi(q)\rangle = \Lambda^{2d_\phi}(\frac{\Lambda}{\Lambda_0})^\eta(\frac{p}{\Lambda})^a\Lambda^D\delta^D(p+q) = \frac{\Lambda^{-2+\eta-a}}{\Lambda_0^\eta}p^a$$

But LHS must be independent of Λ. Thus $a = -2+\eta$. This gives

$$\langle\phi(p)\phi(-p)\rangle = \frac{1}{p^{2-\eta}\Lambda_0^\eta}$$

6.4 Fixed Point equation

Use the above concepts to derive the fixed point equation. The fixed point equation can be understood as an equation for an action where the kinetic term is kept always normalized by a field redefinition.

We go back to the notation where x is the field and $\psi(x,t) = e^{-S[x,t]}$.

Imagine continuously redefinition of the starting bare field so that the kinetic term at time t is always normalized. So $\psi(x_i, t_i)$ we write as

$$e^{ax_i\frac{\partial}{\partial x_i}}\psi_n(x_i, t_i)$$

and choose $a = \frac{\eta}{2}t$ so that at time t, $\psi(x,t)$ has a normalized kinetic term which fact we indicate by writing $\psi_n(x,t)$.[2]

[2]In terms of $\phi(p)$ the operator is $e^{a\int_p\phi(p)\frac{\delta}{\delta\phi(p)}}$.

6.4.1 A property of number operator

Start with:

$$U(t,t_i)e^{ax\frac{\partial}{\partial x}}U(t,t_i)^{-1} = e^{a\mathcal{N}(t)}$$

$$\therefore U(t+\Delta d,t_i)e^{ax\frac{\partial}{\partial x}}U(t+\Delta t,t_i)^{-1} = U(t+\Delta t,t)e^{a\mathcal{N}(t)}U(t+\Delta t,t)^{-1} = e^{a\mathcal{N}(t+\Delta t)}$$

Thus

$$(1+\Delta t\mathcal{G}_{RG}(t))e^{a\mathcal{N}(t)}(1-\Delta t\mathcal{G}_{RG}(t)) = e^{a\mathcal{N}(t+\Delta t)}$$

So

$$\boxed{\Delta t[\mathcal{G}_{RG}(t),e^{a\mathcal{N}(t)}] = e^{a\mathcal{N}(t+\Delta t)} - e^{a\mathcal{N}(t)} = e^{a\mathcal{N}(t)+a\Delta t\mathcal{N}'(t))} - e^{a\mathcal{N}(t)}} \tag{82}$$

Since $\mathcal{N}'(t)$ and $\mathcal{N}(t)$ do not commute we have to retain them as exponentials.

Useful Result:

Lemma:

$$e^{[A(1+\epsilon_1),\epsilon_2 B]} - e^{[A,\epsilon_2 B]} = O(\epsilon_1\epsilon_2) = e^{[A(1+\epsilon_1),[A(1+\epsilon_1),\epsilon_2 B]]} - e^{[A,[A,\epsilon_2 B]]} = ... \tag{83}$$

Because if either ϵ_1,ϵ_2 are zero the answer is zero.

Now the Baker Campbell Hausdorf formula gives

$$e^{A+B} = e^A e^B e^{c_1[A,B]+c_2[A,[A,B]+...} \tag{84}$$

where the dots denote various commutators involving A and B. Using these two results we can say that to linear order in ϵ_1 or ϵ_2:

$$e^{A(1+\epsilon_1)+\epsilon_2 B} - e^{A+\epsilon_2 B} = e^{A(1+\epsilon_1)}e^{\epsilon_2 B}e^{c_1[A(1+\epsilon_1),\epsilon_2 B]+...} - e^A e^{\epsilon_2 B}e^{c_1[A,\epsilon_2 B]+...}$$

$$= e^A(1+\epsilon_1 A)e^{\epsilon_2 B}e^{c_1[A(1+\epsilon_1),\epsilon_2 B]+...} - e^A e^{\epsilon_2 B}e^{c_1[A,\epsilon_2 B]+...}$$

$$= e^A e^{\epsilon_2 B}\underbrace{[e^{c_1[A(1+\epsilon_1),\epsilon_2 B]+...} - e^{c_1[A,\epsilon_2 B]+...}]}_{O(\epsilon_1\epsilon_2)} + e^A\epsilon_1 A e^{\epsilon_2 B}$$

Thus

$$e^{A(1+\epsilon_1)+\epsilon_2 B} - e^{A+\epsilon_2 B} = \epsilon_1 A e^A e^{\epsilon_2 B} + O(\epsilon_1\epsilon_2) \tag{85}$$

6.4.2 Fixed Point ERG equation

$$\psi(t) = U(t,t_i)\psi(t_i)$$

Write

$$\psi(t) = e^{\frac{\eta}{2}t\mathcal{N}(t)}\psi_n(t)$$

where $\psi_n(t)$ has a normalized kinetic term. This fixes η.

LHS:

$$\psi(t+\Delta t)-\psi(t)=e^{\frac{\eta}{2}t\mathcal{N}(t)+\frac{\eta}{2}\Delta t\mathcal{N}(t)+\frac{\eta}{2}t\Delta t\mathcal{N}'(t)}\psi_n(t+\Delta t)-e^{\frac{\eta}{2}t\mathcal{N}(t)}\psi_n(t)$$
$$=e^{\frac{\eta}{2}t\mathcal{N}(t)+\frac{\eta}{2}\Delta t\mathcal{N}(t)+\frac{\eta}{2}t\Delta t\mathcal{N}'(t)}\psi_n(t)-e^{\frac{\eta}{2}t\mathcal{N}(t)}\psi_n(t)+e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\psi_n'(t)$$

RHS:

$$=\Delta t\mathcal{G}_{RG}(t)\psi(t)=\Delta t\mathcal{G}_{RG}(t)e^{\frac{\eta}{2}t\mathcal{N}(t)}\psi_n(t)$$
$$=e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\mathcal{G}_{RG}(t)\psi_n(t)+\Delta t[\mathcal{G}_{RG}(t),e^{\frac{\eta}{2}t\mathcal{N}(t)}]\psi_n(t)$$
$$=e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\mathcal{G}_{RG}(t)\psi_n(t)+[e^{\frac{\eta}{2}t\mathcal{N}(t)+\frac{\eta}{2}t\Delta t\mathcal{N}'(t))}-e^{\frac{\eta}{2}t\mathcal{N}(t)}]\psi_n(t)$$

Equating LHS and RHS we get

$$e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\dot{\psi}_n(t)=e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\mathcal{G}_{RG}(t)\psi_n(t)+[e^{\frac{\eta}{2}t\mathcal{N}(t)+\frac{\eta}{2}t\Delta t\mathcal{N}'(t))}-e^{\frac{\eta}{2}t\mathcal{N}(t)+\frac{\eta}{2}\Delta t\mathcal{N}(t)+\frac{\eta}{2}t\Delta t\mathcal{N}'(t)}]\psi_n(t)]$$

Set

$$A=\frac{\eta}{2}t\mathcal{N}(t),\quad \epsilon_1=\frac{\Delta t}{t},\qquad B=\frac{\eta}{2}t\mathcal{N}'(t),\quad \epsilon_2=\Delta t$$

Using (85) we see that to order Δt the equation becomes

$$e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\dot{\psi}_n(t)=e^{\frac{\eta}{2}t\mathcal{N}(t)}\Delta t\mathcal{G}_{RG}(t)\psi_n(t)-\frac{\eta}{2}\Delta t\mathcal{N}(t)e^{\frac{\eta}{2}t\mathcal{N}(t)}\psi_n(t)$$

or finally

$$\boxed{\dot{\psi}_n(t)=\mathcal{G}_{RG}(t)\psi_n(t)-\frac{\eta}{2}\mathcal{N}(t)\psi_n(t)}\tag{86}$$

with η fixed by the requirement that $\psi_n(t)$ have a normalized kinetic term.

6.5 Wilson's Equation

In Wilson's equation

$$\frac{\partial}{\partial t}\psi[x(p),t]=\int_p\dot{g}(p,t)\frac{\delta}{\delta x(p)}(\frac{\delta}{\delta x(-p)}+x(p))\psi[x(p),t]\tag{87}$$

there is a function

$$\dot{g}=c+2p^2e^{2t}=c+\frac{2p^2}{\Lambda^2},\quad g(t)=ct+p^2e^{2t}\tag{88}$$

Wilson chose:

$$c=1-\frac{\eta}{2}$$

Thus η multiplies the operator [3]

$$\frac{1}{2}\frac{\delta}{\delta x(p)}(\frac{\delta}{\delta x(-p)}+x(p))$$

We have already seen that this implements a momentum independent scaling of x_i

$$\psi(x_f,t_f)=\frac{1}{\sqrt{2\pi(1-e^{t_i-t_f})}}\int dx_i e^{-\frac{1}{2}\frac{(x_f-e^{\frac{t_i-t_f}{2}}x_i)^2}{1-e^{t_i-t_f}}}\psi(x_i,t_i) \tag{89}$$

So we see that Wilson's ERG equation already has incorporated in it the rescaling necessary to find a fixed point.

6.6 Integral Representation for Wilson-Polchinski with Anomalous Dimension

The equation is really the fixed point equation (86) reproduced here:

$$\frac{\partial\psi_n(t)}{\partial t}=\mathcal{G}_{RG}(t)\psi_n(t)-\frac{\eta}{2}\mathcal{N}(t)\psi_n(t)=[\frac{1}{2}(-\dot{G}-\eta\frac{G(G_0-G)}{G_0})\frac{\partial^2}{\partial x^2}-(\frac{\dot{G}}{G}+\frac{\eta}{2})x\frac{\partial}{\partial x}]\psi_n \tag{90}$$

Change variables: $(x,t)\rightarrow(y,\tau)$:

$$y=\frac{x}{G}e^{-\frac{\eta}{2}t};\quad \tau=t$$

$$\frac{\partial}{\partial x}=\frac{\partial y}{\partial x}\frac{\partial}{\partial y}+\frac{\partial\tau}{\partial x}\frac{\partial}{\partial\tau}=\frac{e^{-\frac{\eta}{2}t}}{G}\frac{\partial}{\partial y};\quad \frac{\partial^2}{\partial x^2}=\frac{e^{-\eta t}}{G^2}\frac{\partial^2}{\partial y^2}$$

$$\frac{\partial}{\partial t}=\frac{\partial y}{\partial t}\frac{\partial}{\partial y}+\frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau}=-(\frac{\dot{G}}{G}+\frac{\eta}{2})y\frac{\partial}{\partial y}+\frac{\partial}{\partial\tau}$$

LHS becomes

$$\frac{\partial\psi_n}{\partial\tau}-(\frac{\dot{G}}{G}+\frac{\eta}{2})y\frac{\partial}{\partial y}\psi_n$$

RHS becomes

$$\frac{1}{2}(-\dot{G}-\eta\frac{G(G_0-G)}{G_0})\frac{e^{-\eta t}}{G^2}\frac{\partial^2}{\partial y^2}\psi_n-(\frac{\dot{G}}{G}+\frac{\eta}{2})y\frac{\partial}{\partial y}\psi_n$$

Thus we get (replacing t by τ)

$$\frac{\partial\psi_n}{\partial\tau}=\frac{1}{2}\frac{d}{d\tau}\underbrace{[(\frac{1}{G}-\frac{1}{G_0})e^{-\eta\tau}]}_{H^{-1}=h}\frac{\partial^2}{\partial y^2}\psi_n$$

[3] The constant 1 denotes a specific choice of engineering dimension for the field and can be changed by rescaling the field with powers of Λ.

Thus the equation can be written as

$$\frac{\partial \psi_n}{\partial h} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\psi_n$$

Thus the solution is easily written down:

$$\psi_n(y,\tau) = \int dy_i e^{-\frac{1}{2}\frac{(y-y_i)^2}{h-h_i}}\psi_n(y_i,\tau_i)$$

Putting back the original variables:

$$\boxed{\psi_n(x_f,t_f) = \int dx_i e^{-\frac{1}{2}\frac{(\frac{x_f}{G_f}e^{-\frac{\eta}{2}t_f}-\frac{x_i}{G_i}e^{-\frac{\eta}{2}t_i})^2}{H_f^{-1}-H_i^{-1}}}\psi_n(x_i,t_i)} \tag{91}$$

with

$$\boxed{H^{-1} = (\frac{1}{G}-\frac{1}{G_0})e^{-\eta t}} \tag{92}$$