Bell inequalities under one sided relaxation of physical constraints

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Bell inequalities are violated by quantum mechanics. So, one of the properties, like, determinism, no signaling and measurement independence which are required to frame Bell inequality, is incompatible with quantum correlations. Thus, any model simulating quantum correlations must either individually or jointly give up these physical constraints. Recently, M. J. W. Hall (Phys Review A, **84**, 022102 (2011)) derived different forms of Bell inequalities under the assumption of individual or joint relaxation of those physical properties on both sides. One sided relaxation can also be a useful resource for simulating singlet correlations. For that we have derived a Bell-type inequality under the assumption of joint relaxation of determinism and no signaling on one side. It is shown that for the previous case, if we increase measurement settings per party, then the Bell-type inequality takes a different form. We also relaxed no signaling, determinism and measurement independence all at the same time on one side and framed corresponding Bell-type inequality. In each case, we have obtained the minimum degree of relaxation of these physical properties for any model which violates a standard Bell inequality.

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In 1964, John S. Bell introduced an *inequality* [1] thereby showing *no realistic physical theory which is also local in a specified sense can agree with all of the statistical implications of Quantum Mechanics* (under conditions which were relaxed in later work by Bell [2–4] himself and also by others [5–8]). Different versions of the theorem, inspired by the paper [1], are considered as a family which is uniquely termed as "*Bell's Thoerem*" and the corresponding inequalities are termed as "*Bell-type inequalities*". Each of these inequalities is statistical in nature. Violation of any "*Bell-type inequality*" thus may be considered as one of the remarkable features of quantum theory. Till date violation of Bell inequality serves various objectives in the context of better understanding of the behaviour of composite quantum systems. For example, it serves as a criteria necessary for categorizing correlations, helps us in quantifying nonlocality, etc.

Now, various plausible physical postulates are at the background of framing Bell inequalities. Some of the physical constraints in this regard are: no signaling, measurement independence, locality, determinism. Therefore, violation of these inequalities by any physical theory thereby give rise to some queries: *are the predictions made by the theory incorrect?* or, *whether at least one of these applied postulates incompatible with the description of the natural phenomena?* As Quantum mechanical predictions tally with the experimental data so only the second query is relevant in this regard. Till now various literatures have dealt with the relaxation of physical constraints for framing Bell inequalities [9–14]. In [11], M.J.W.Hall considered relaxation of measurement independence. He argued that for violation of Bell-CHSH inequality by any singlet correlations at least 86% percentage of measurement independence must be relaxed. In [12], he introduced a Bell inequality considering joint relaxation of no signaling and determinism thereby focusing on the complementary relationship shared by signaling and indeterminism by any physical model. He showed that at least 60% of signaling and 41% of indeterminism must be introduced in the Bell-CHSH model to justify the violation shown by singlet correlations. In [13], a relaxed Bell-type inequality was introduced under the assumption of joint relaxation of no signaling, determinism and measurement independence. The main objective of all these papers [11–13] was to simulate singlet correlations assuming both side relaxation of these physical constraints. The question that naturally arises in this context is, whether relaxation of these physical properties on one side can simulate singlet correlations? In [14], Banik et. al., dealt with one sided relaxation of measurement independence. It was shown that 59% of measurement independence by one of the two parties was optimal to generate singlet correlations.

In this work, we have investigated whether relaxation of one sided no signaling is more useful as a resource for simulation of singlet correlations than that of both sided relaxation of the same. We have succeeded in showing that the minimum degree of relaxation has decreased from 60% to 17%. The *complementary relation* [12] between determinism and signaling remains invariant in this case. We also investigate one sided joint relaxation of no sig-

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naling, determinism and measurement independence. Interestingly, for one sided relaxation, maximal violation of Bell inequality cannot be achieved unlike that in the case of both sided relaxation [12, 13]. Moreover, depending on the *complementary relation*, we obtain two subintervals (2,3) and (2,4) of the total violation (2,4]. We now briefly describe our work.

Model Behind Any Experiment: Consider a joint experiment between two parties, Alice and Bob. Suppose each party has dichotomic measurement settings with inputs x and y for Alice and Bob respectively. Let $a, b \in \{-1, 1\}$ label the possible outcomes of Alice and Bob respectively. The results of this experiment can be expressed in terms of the statistical correlations $p(a, b|x, y)$. Now, for the correlations that depend on underlying hidden variable λ, we have by Bayes' theorem: $p(a, b|x, y) = \int d\lambda p(a, b|x, y, \lambda)p(\lambda|x, y)$. In the standard Hilbert space model of quantum mechanics, one can consider the underlying variable λ as the density operator ρ and the joint measurement setting by a POVM M_{ab}^{xy} with $p(a, b|x, y, \rho) = tr[\rho M_{ab}^{xy}], \quad p(\rho|x, y) = \delta(\rho - \rho_0)$. For a pure state, $p(a, b|x, y, \psi) = \langle\psi|M_{ab}^{xy}|\psi\rangle, p(\psi|x, y) = p_0(\psi)$ where the hidden variable λ being confined to the set of unit vectors $\{|\psi\rangle\}$ of the Hilbert space and $\rho_0 \equiv \int d\psi p_0(\psi)|\psi\rangle\langle\psi|$. Now, given a model behind an experiment, it may or may not satisfy various physically acceptable properties, such as locality, no signaling, determinism, measurement independence, etc. For instance, one may consider a model of singlet correlations. Such a model violates Bell inequalities implying that at least one of the aforementioned physical constraints must be relaxed to frame a model of these type of quantum correlations. The necessary degrees of relaxation of the above properties to model quantum correlations were derived in [11–13]. In this work, we also deal with the task of relaxing one or more physical constraints in a slightly different approach.

No signaling: In a system of distant parties if we assume that no communication among the parties can take place when the measurements are performed, then the obtained correlations must obey the principle of no signaling: *the choices of observable by one party cannot influence the statistics observed by the remaining parties.* In other words, if $p(a|x, y, \lambda) = p(a|x, y', \lambda)$, $p(b|x, y, \lambda) = p(b|x', y, \lambda)$ hold for all pairs (x, y), (x, y') and (x', y) of the model.

The degree of signaling is defined by the maximum shift possible in an underlying marginal probability for one observer, due to the alteration of measurement setting of the other. One may formulate it as follows [12, 13]: $S_{1\to2} := \sup_{x,x',y,b,\lambda} |p^{(2)}(b|x, y, \lambda) - p^{(2)}(b|x', y, \lambda)|$, $S_{2\to1} := \sup_{x,y,y',a,\lambda} |p^{(1)}(a|x, y, \lambda) - p^{(1)}(a|x, y', \lambda)|$ where a, b, x, x', y and y' have their usual meanings. Thus, $S_{1\to2}$ is the maximum possible deviation in an underlying marginal probability distribution for the second observer, induced via the change of a measurement settings of the first observer. The overall degree of signaling, for a given underlying model is defined by, $S := \max\{S_{1\to2}, S_{2\to1}\}$.

Determinism: A model is said to be deterministic if the observed statistical correlations are generated by averaging over a set of all possible values of the underlying variable (λ) such that for any fixed value of the variable(λ) all measurement outcomes are fully determined [12]. In a deterministic correlation model all the outcomes being predictable with certainty for any given knowledge of λ, correlation terms are either 0 or 1, i.e., $p(a, b|x, y, \lambda) \in \{0, 1\}$. The underlying marginal probabilities are also deterministic, i.e., $p(a|x, y, \lambda), p(b|x, y, \lambda) \in \{0, 1\}$.

The degree of indeterminism may be defined as the measure of deviation of the marginal probabilities from the deterministic values of 0 and 1. The local degree of indeterminism I_j may be defined [12, 13] as the smallest positive number, such that the corresponding marginal probabilities lie in $[0, I_j] \bigcup [1 - I_j, 1]$, i.e., $I_j := \sup_{\{x,y,\lambda\}} \min_z\{p^j(z|x, y, \lambda), 1-p^j(z|x, y, \lambda)\}$. Thus, $I_j = 0$ if and only if the corresponding marginal is deterministic. The overall degrees of indeterminism for the model may be defined as, $I := \max\{I_1, I_2\}$. Hence $0 \le I \le 1/2$, with $I = 0$ if the model is fully deterministic.

If a model of two parties be such that the determinism is relaxed for the second party while the measurement outcomes of the first party are deterministic, then such a model is said to be the *one sided indeterministic* model. In such a model $I_1 = 0, \quad 0 < I_2 \le 1/2$. Hence, $I = I_2$.

A Complementary Relation Between Signaling And Determinism: The degrees of indeterminism and signalling are dependent on each other to some extent. This is due to the fact that any deviation in a marginal probability value p, due to signaling, must either keep the value in the same subinterval $[0, I]$ (or, $[1 - I, 1]$) $(S \le I)$, or shift the value across the gap between the subintervals $(S \ge 1 - 2I)$ which leads to $I \ge \min\{S, (1 - S)/2\}$.

Measurement Independence: Measurement independence of a model is the property that the distribution of the underlying variable is independent of the measurement settings chosen by the experimenters. More specifically, $p(\lambda|x, y) = p(\lambda|x', y')$ for every joint settings (x, y), (x', y'). This condition is satisfied by quantum system. By Bayes' theorem, alternately one can obtain, $p(x, y|\lambda) = p(x, y), \quad p(x, y, \lambda) = p(x, y)p(\lambda)$ assuming the existence of a well defined distribution $p(x, y)$ of joint measurement settings [13]. Thus, *Measurement dependence(M)* may be interpreted as a measure to quantify the degree of violation of measurement independence by the underlying model. It is defined as [11]: $M := \sup_{x,x',y,y'} \int d\lambda|p(\lambda|x, y) - p(\lambda|x', y')|$. Therefore, for measurement independence $M = 0$. The maximum possible value of M is given by $M_{max} = 2$ implying complete measurement dependence. The fraction of measurement independence corresponding to a given model is defined by [11], $F := 1 - M/2$. Thus $0 \le F \le 1$, with $F = 0$ for the case where $M = 2$. Geometrically, F represents the minimum degree of overlap between any two underlying distributions $p(\lambda|x, y)$ and $p(\lambda|x', y')$. Also, local degrees of measurement dependence are defined analogously [11], $M_1 := \sup_{x,x',y} \int d\lambda |p(\lambda|x,y) - p(\lambda|x',y)|$; $M_2 := \sup_{x,y,y'} \int d\lambda |p(\lambda|x,y) - p(\lambda|x,y')|$. So, local degrees of measurement independence are defined by, $F_1 := 1 - M_1/2$; $F_2 := 1 - M_2/2$.

Bell Inequality Under Relaxation of Determinism and No signaling on One Side: In order to generate models violating Bell inequality, the properties of no signaling and that of determinism are relaxed to some extent, thereby introducing signaling and indeterminism in the models. The extent of relaxation of these two properties can be quantified with the help of the corresponding relaxed Bell-type inequality [12]. Below we describe a model where one sided signaling and indeterminism are introduced.

In a system of two parties (Alice and Bob), it is assumed that determinism and no signaling are preserved by the correlations shown by Alice's measurement. A signal is sent to Bob by Alice and it is also assumed that the correlations in Bob's part are indeterministic. So for this model $S = S_{1\to 2}$ and $I = I_2$. The extent of minimum possible relaxation in this context is given by the following theorem.

Theorem 1: Let x, x' and y, y' denote possible measurement settings for Alice and Bob, respectively, and label each measurement outcome by 1 or -1. Suppose, $\langle XY\rangle$ being the average product of the measurement outcomes for joint measurement settings. Then, for any underlying model having values of indeterminism and signalling of at most I and S, respectively, Bell inequality takes the form: $\langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle - \langle X'Y'\rangle \leq B(I,S)$ with upper bound $B(I,S) = 2+2I$ for $S < 1-2I$(tight upper bound), and < 4 for $S \geq 1-2I$.

Immediate implications: Original form of Bell inequality is derivable with $I = S = 0$, i.e., $B(0,0) = 2$. *Degree of relaxation for V, the amount of violation:* Suppose, the Bell-CHSH inequality be violated by an amount V. The left hand side of above equation gives $2+V$. Hence the corresponding model must satisfy the relation $B(I,S) \geq 2+V$. Also, $I \geq I_V := V/2$ and/or $S \geq S_V := 1-V$. This theorem, thus exerts bounds on the minimum possible degrees of indeterminism and signalling that must exist in any corresponding model. For singlet state correlations, $V = 2\sqrt{2} - 2$ [15]. Thus, any singlet state model must assign at least 82% of indeterminism, and/or communicating at least 17% of signaling. *Can Maximal Violation Be Reached?* For $I = 0$ and/or $S = 1$, $B(I,S) = 4$. But in that case Bob's outcome correlations become deterministic. Hence, unlike the case of both sided relaxation on no signaling and determinism [12], for one sided signaling and indeterminism maximal violation cannot be reached. *Division of the range of violation:* Depending on the complementary relation between indeterminism and signaling, we get two subintervals $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of the interval $\mathfrak{R} = (2,4]$ of violation. For $S < 1-2I$, $\mathfrak{R}_1 = (2,3)$. For $S \geq 1-2I$, we get the other subinterval $\mathfrak{R}_2 = (2,4)$.

Bell Inequalities Under Relaxation of Determinism, No signaling and measurement independence on One Side: A relaxed Bell-type inequality is framed to investigate the extent to which Bell inequalities are violated when jointly no signaling, determinism and measurement independence are relaxed from one side. This quantifies the amount of individual and/or joint degrees of relaxation required to model a given violation of a standard Bell inequality. Without loss of generality, we assume that joint relaxation is done only on Bob's side, i.e., $I_2 > 0$, $S_{1\to 2} > 0$ and $M_2 > 0$, whereas at the same time Alice maintains $I_1 = 0, S_{2\to 1} = 0$ and $M_1 = 0$. Thus, $I > 0$ and $S > 0$.

Theorem 2: Suppose, x, x' and y, y' be the measurement settings for Alice and Bob respectively and the measurement outcomes for each party be 1 or -1. If $\langle XY\rangle$ denote the average of the product of the measurement outcomes for joint measurement settings X and Y, then $\langle XY\rangle + \langle XY'\rangle + \langle X'Y\rangle + \langle X'Y'\rangle \leq B(I,S,M)$ where I, S and M are the values of indeterminism, signaling and measurement dependence respectively, for any underlying models. $B(I,S,M) = 4 - (1-I)(2-M)$, for $S < 1-2I$ and $M < 2$ (tight upper bound) and < 4, otherwise.

As immediate implications we will discuss here two cases:*Degree of relaxation for generation of singlet state correlations:* Here $V = 2\sqrt{2} - 2$ [15]. Thus any singlet state model must either assign at least 82% of uncertainty and/or predict a change of at least 17% and/or relax measurement independence by 59% for one party in response to a measurement performed on the other party. *Local deterministic model:* In this case, both $I = 0$ and $S = 0$ and $B(0,0,M) = min\{2+M, 4\}$. Hence, a local deterministic model exists for simulating a singlet state correlation if and only if $M \geq V = 2\sqrt{2} - 2 \approx 0.82$. So 59% measurement independence is optimal for simulating singlet correlation when measurement dependency is allowed only on one side. These results have been obtained in a recent paper [14]. So, our relaxed Bell inequality gives a general result and [14] can be obtained as a particular case.

In our work, we then considered relaxed Bell inequalities for more than two settings. Thus, we have considered here only one sided relaxation of constraints such as, no signaling, determinism and measurement independence and have framed Bell-type inequalities under such assumptions. With the view of all discussed in our work, we can safely conclude that the results derived in one sided relaxation scenarios do not tally exactly with that of in the both sided cases. The existing complementary relation between both sided signaling and indeterminism also holds in the one sided case. But the minimal degree of relaxation of the constraints in the one sided cases differ from that of both sided cases. The form of *relaxed Bell-type inequalities* have also changed. There still remain many other topics of discussion in the context of one sided relaxation of physical constraints.

For details of our work we refer arXiv version arXiv:1304.7409

[1] J. S. Bell, Physics **1**, 195 (1964).
[2] J.S. Bell, "Introduction to the hidden-variable question," in Foundations of Quantum Mechanics (Proceedings of the International School of Physics 'Enrico Fermi', course IL), B. d'Espagnat (ed.), New York: Academic, 171–181 (1971).
[3] J.S. Bell, "The theory of local beables," Dialectica, **39**: 97–102 (1985).
[4] J.S. Bell, Speakable and Unspeakable in Quantum Mechanics, Cambridge: Cambridge University Press, revised edition, 2004. (Contains reprints of the preceding and related papers (1987).
[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23** 880-884 (1969).
[6] J. F. Clauser, M. A. Horne, Phys. Rev. D, **10**, 526 (1974).
[7] N.D. Mermin, "Generalizations of Bell's Theorem to higher spins and higher correlations," in Fundamental Questions in Quantum Mechanics, L.M. Roth and A. Inomato (eds.), New York: Gordon and Breach, 7–20 (1986).
[8] A.Aspect, Trois tests expérimentaux des inégalités de Bell par mesure de corrélation de polarization de photons, Orsay: Thèse d'Etat (1983).
[9] C. Branciard et al., Nature Physics **4**, 681 (2008).
[10] Jonathan Barrett and Nicolas Gisin Phys. Rev. Lett. **106**, 100406 (2011)..
[11] M.J.W. Hall, Phys. Rev. Lett.**105**, 250404 (2010).
[12] M.J.W. Hall, Phys. Rev. A **82**, 062117 (2010).
[13] M.J.W. Hall, Phys. Rev. A **84**, 022102 (2011).
[14] Manik Banik, MD Rajjak Gazi, Subhadipa Das, Ashutosh Rai, Samir Kunkri, J. Phys. A: Math Theor. **45**, 475302 (2012).
[15] B.S. Csirel'son, Lett. Math. Phys. **4**, 93 (1980).