Local Quantum Uncertainty and Bounds on Quantumness for $\mathcal{O}\otimes\mathcal{O}$ invariant class of states

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We derive closed form of local quantum uncertainty and bounds for post-entanglement correlation measures - geometric discord and measurement-induced nonlocality for highly symmetric orthogonal invariant sates. This class of states includes both the Werner and Isotropic class. We provide analytical formula for local quantum uncertainty for $\mathcal{O}\otimes\mathcal{O}$ invariant class of states in $n\otimes n$ systems.

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Quantum mechanics shows several counter-intuitive results when we are dealing with composite systems [1–3]. There exist peculiar type of correlations between different parts of a composite system commonly known as non-classical correlations. Entanglement is one of the most powerful non-classical correlation that establishes its importance in different information processing tasks. However, several post entanglement correlation measures have generated a lot of interests in recent years. Discord, quantum deficit, measurement-induced nonlocality(in short, MIN) [4–7] are a few of them. Even there are different non equivalent versions of discord [8]. Recently Girolami *et. al.*[9] introduced the concept of local quantum uncertainty which quantifies the uncertainty in a quantum state due to measurement of a local observable. Nevertheless, such quantifier has strong reasons to be considered as a faithful measure of quantumness in quantum states. But due to inherent optimization, finding closed formula is a difficult problem for most of the correlations measures. The value of quantum discord is not even known for general bipartite qubit system. In higher dimensional bipartite systems, the results are known for only some special classes of states [10, 11]. Geometric discord has explicit formula for qubit-qudit system and its lower bound is calculated in [12, 13] for higher dimensions. MIN has a closed formula for qubit-qudit systems and it has tight upper bound in higher dimensions [6]. It is possible to derive closed formula for MIN, geometric discord and also for quantum discord for Werner and Isotropic classes of states due to their highly inherent symmetry in the structures. However, local quantum uncertainty (LQU) has analytical form only for any qubit-qudit system.

Here, we will consider orthogonal invariant class of states which is a larger class of symmetric states and it contains both Werner and Isotropic classes. We will derive closed form of LQU for this class of states in two qudit system. We will also evaluate bounds of geometric discord and MIN for this symmetric class of states.

Classically, it is possible to measure any two observable with arbitrary accuracy. However, such measurement is not always possible in quantum systems. Uncertainty relation gives the statistical nature of errors in these kind of measurement. Measurement of single observable can also help to detect uncertainty of a quantum observable. For a quantum state ρ, an observable is called *quantum certain* if the error in measurement of the observable is due to only the ignorance about the classical mixing in ρ. A good quantifier of this uncertainty of an observable is the skew information, defined by Wigner and Yanase [14] as

$$I(\rho, K) := -\frac{1}{2}\mathrm{tr}\{[\sqrt{\rho}, K^A]^2\} \tag{1}$$

Wigner and Yanase introduced this quantity as a measure of information content of the ensemble ρ_{AB} skew to a fixed conserved quantity K^A. Since it quantifies non-commutativity between a quantum state and an observable so it serves as a measure of uncertainty of the observable K^A in the state ρ_{AB}. This type of measure helps to quantify the quantum part of error in measuring an observable. $I = 0$ indicates quantum certain nature of the observable K^A. It is also convex and non-increasing under classical mixing. For a bipartite quantum state ρ_{AB}, Girolami *et.al.* [9] introduced the concept of local quantum uncertainty(LQU) and it is defined as

$$\mathcal{U}_A^\Lambda := \min_{K^\Lambda} I(\rho_{AB}, K^A) \tag{2}$$

The minimization is performed over all local *maximally informative observable* (or non-degenerate spectrum Λ) $K^\Lambda = K_A^\Lambda \otimes \mathbb{I}$. This quantity quantifies the minimum amount of uncertainty in a quantum state. Non-zero value of this quantity indicates the non existence of any quantum certain observable for the state ρ_{AB}. This quantity also possess many interesting properties, such as:

- it vanishes for all zero discord state w.r.t. measurement on party A.

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- it is invariant under local unitary.
- it reduces to entanglement monotone for pure state. In fact, for pure bipartite states it reduces to linear entropy of reduced subsystems. So, LQU can be taken as a measure of bipartite quantumness.

LQU is believed to be the reason behind quantum advantage in DQC1 model and it also works as a lower bound of quantum Fisher Information in parameter estimation. It has geometrical significance in terms of Hellinger distance. LQU is inherently an asymmetric quantity and explicit its closed form is available only for some simple system. For a quantum state ρ of $2 \otimes n$ system, LQU reduces to $1 - \lambda_{max}(\mathcal{W})$ where λ_{max} is the maximum eigenvalue of the matrix $\mathcal{W} = (w_{ij})_{3\times 3}$, $w_{ij} = \mathrm{tr}\{\sqrt{\rho}(\lambda_i \otimes \mathbb{I})\sqrt{\rho}(\lambda_j \otimes \mathbb{I})\}$ and λ_i's are standard Pauli matrices in this case.

Now any state of a $n \otimes n$ quantum system can be written in general, as of the form:

$$\rho = \frac{1}{n^2}[\mathbb{I}_n \otimes \mathbb{I}_n + \mathbf{x^t}\lambda \otimes \mathbb{I}_n + \mathbb{I}_n \otimes \mathbf{y^t}\lambda + \sum t_{ij}\lambda_i \otimes \lambda_j] \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n^2-1})^t$ and λ_i's are the generators of SU(n). For $n = 2$, Pauli matrices can be used as the generators of SU(2). While for $n = 3$, generallly, Gell-Mann matrices are taken as the generators of SU(3). In this way we can construct traceless, orthogonal generators (*generalized Gell-Mann matrices*) for SU(n), containing $n^2 - 1$ elements. The generators satisfy some commutation and anti-commutation relations.

Any $\mathcal{O}\otimes\mathcal{O}$ invariant state from a $n \times n$ system can be taken as

$$\rho = a\,\mathbb{I}_{n^2} + b\,\mathbb{F} + c\,\hat{\mathbb{F}} \tag{4}$$

with $n(na + b + c) = 1$ (trace condition) and proper positivity constraints. This is an important class of states of bipartite systems. This class can have both PPT(positive partial transpose) and NPT(negative partial transpose) states depending on the extra constraints on the parameters. When $b = c$ the positivity conditions of ρ implies the corresponding positivity of partial transpositions ρ^{T_A} or ρ^{T_B}. In case of $b \neq c$ we can find NPT states. We can choose any A-observable K_A=$\mathbf{s}.\lambda$ with $\mathbf{s} = (s_1, s_2, ..., s_{n^2-1})$, $|\mathbf{s}| = 1$ and $\lambda = (\lambda_1,\ \lambda_2, ..., \lambda_{n^2-1})^t$. From the definition of local quantum uncertainty(LQU), we can derive the value of Λ dependent $\mathcal{U}_A$ in terms of maximum eigenvalue λ_{max} of $\mathcal{W}$ as

$$\mathcal{U}_A = \frac{2}{n} - \lambda_{max}(\mathcal{W}) \tag{5}$$

The above result (5) also holds for the large class of states with $\mathrm{tr}(\rho\lambda_i \otimes \mathbb{I}_n) = 0,\ i = 1, 2, ..., n^2 - 1$. Hence, closed form of LQU is possible for a large class of bipartite states, depending on the previous condition. Here, we will deal with the orthogonal invariant class of states for our purpose. However, for qubit-qudit system(with observable on the qubit system) our result recovers the result of [9].

For a general $\mathcal{O}\otimes\mathcal{O}$ invariant state (4) we have, $\mathbf{x} = \mathbf{0}$ and the correlation matrix elements,

$$t_{kk} = \frac{n^2}{2}\begin{cases}(b+c) \text{ for } k = 1, 2, ..., \frac{n^2+n-2}{2}\\ (b-c) \text{ for } k = \frac{n^2+n}{2}, ..., n^2-1\end{cases} \tag{6}$$

Now, we can evaluate the bounds for the geometric discord and measurement-induced nonlocality. Since, $\mathbf{x} = \mathbf{0}$, the extra constraints in the definition of MIN is automatically satisfied. Hence, discord and MIN becomes minimum and maximum value of the same optimization problem respectively. So, $D(\rho) \leq N(\rho)$. It follows, if $bc \leq 0$

$$\begin{aligned}0 \leq (n^2-n)(b^2+c^2) + 4(n-1)bc \leq D(\rho)\\ \leq N(\rho) \leq (n^2-n)(b^2+c^2)\end{aligned} \tag{7}$$

and if $bc \geq 0$

$$\begin{aligned}0 \leq (n^2-n)(b^2+c^2) \leq D(\rho) \leq N(\rho)\\ \leq (n^2-n)(b^2+c^2) + 4(n-1)bc\end{aligned} \tag{8}$$

Thus, we obtain bounds for both geometric discord and MIN for $\mathcal{O}\otimes\mathcal{O}$ invariant class of states. The bounds saturate when at least one of b and c is zero. It is also interesting to note that whenever $b \neq 0$ or $c \neq 0$ the lower bounds are strictly positive. Hence, all $\mathcal{O}\otimes\mathcal{O}$ invariant class of states possess quantum correlation.

For full technical details we refer our arXiv version entitled, "Local Quantum Uncertainty and Bounds on Quantumness for $\mathcal{O}\otimes\mathcal{O}$ invariant class of states", arXiv:1304.7019

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