Entropic uncertainty relation for successive measurements

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Abstract

In quantum mechanics, the uncertainty is fundamental concept, so its underlying meaning has been argued intensively due to its weird properties. On this account, there has been various formulations expressing the uncertainty principle in different ways. In this paper, the new entropic uncertainty relations will be suggested showing different aspects of uncertainty as we consider successive measurements. Already, Deutsch and Uffink suggested entropic uncertainty relations, so our work will be to modify it as focusing on another aspect (error-disurbance) of uncertainty.

• The uncertainty relations

Heisenberg is the first man who suggested the uncertainty principle. In 1927 [1], he proposed the uncertainty relation,

$$\epsilon(Q)\eta(P) \geq \frac{\hbar}{2}$$  \hspace{1cm} (1)

where $\epsilon(Q)$ is the mean error occurring when an observer measures the position of a particle, and $\eta(Q)$ is the disturbance caused by the position measurement $Q$ and $\hbar$ is Planck's constant divided by $2\pi$. The equation (1) shows that we cannot measure position exactly without disturbing momentum.

The Heisenberg’s relation (1) was later generalized to arbitrary pair of observables by Robertson [2]. Instead of position and momentum, he considered the generalized observables $A$ and $B$. Additionally, the lower bound was also generalized to commutator of the observables, as follows.

$$\sigma(A)\sigma(B) \geq \frac{1}{2} |\langle \psi | [A,B] | \psi \rangle|$$  \hspace{1cm} (2)

where $\sigma(Q)$ is standard deviation defined by

$$\sigma(Q)^2 = |\langle \psi | (Q - \langle Q \rangle)^2 | \psi \rangle|$$

and $[A,B] = AB - BA$. This above relation (2) claims that in an arbitrary state $|\psi\rangle$ a pair of noncommuting observables can not be well localized simultaneously. In other words, it can be said that we can not prepare well localized states simultaneously for noncommuting observables.

Afterwards Heisenberg’s relation (1) was revised and improved in 2004 by M. Ozawa [7], which is called a universally valid error-disturbance uncertainty relation,

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2} |\langle \psi | [A,B] | \psi \rangle|.$$  \hspace{1cm} (3)

In 2012 [3], it is proved that the old relation (1) is violated in spin measurements, but the improved one (3) is valid. Consequently, nowadays we can summarize the interpretation of the uncertainty principle in three statements, [4]

(i) It is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized.

(ii) It is impossible to measure simultaneously position and momentum.

(iii) It is impossible to measure position without disturbing momentum.

In these statements, position and momentum represent conjugate variables.

According to the statements, we can classify the above relations. Firstly, the Robertson’s (2) relation is equivalent with the statement (i), since from these relations we can conclude that there is a limitation of preparation of states in which noncommuting observables are well localized. Secondly, the error-disturbance relations (1)(3) are equivalent with the statement (iii), since they describe a situation that a measurement for observable $A$ can not avoid the disturbance about $B$ caused by the measurement $A$, when we consider two noncommuting observables $A$ and $B$.

• Entropic uncertainty relations

In quantum information theory, in 1983 the entropic uncertainty relation was firstly introduced by Deustch [5] and improved by Uffink [6] in 1988. When probability distribution is defined as $X = \{p_1, p_2, ..., p_n\}$ and $Y = \{q_1, q_2, ..., q_m\}$ and $H(X)$ is Shannon entropy described as $H(X) = -\sum p_i \log p_i$, the entropic uncertainty relation is

$$H(X) + H(Y) \geq -2 \log c, \text{ where } c = \max_{i,j} |x_i y_j|.$$  \hspace{1cm} (4)
where \(|\{x_i\}\rangle\) and \(|\{y_i\}\rangle\) are the corresponding complete sets of normalized eigenvectors with respect to operators \(X\) and \(Y\), and then \(p_{xi}, p_{yj}\) is defined as \(|\langle x_i|\psi\rangle|^2\) and \(|\langle y_j|\psi\rangle|^2\). In the relation (4), the lower bound \(-2 \log c\) is independent of the initial state \(|\psi\rangle\), but contrastively the relations (2)~(3) in the previous section is dependent. That means the lower bound, \(|\langle \psi|[A,B]|\psi\rangle|\), is varied with an initial state, so in a specific state it is diminishing.

- Entropic uncertainty relation for successive measurements

In order to deal with unavoidable disturbance, we have to consider successive measurements, like as the relation (3). In successive measurements, we obtain results about \(X\) and \(Y\), in regular sequence. That means we observe the result of measurement \(X\) and the outcome will be measured for observable \(Y\), so we get the results of \(X\) and \(Y\) finally. This situation is described in Fig 1.

![Figure 1: Probability distribution in successive measurements \(X\) and \(Y\).](image)

Eigenvalue set of \(X,Y\) is \(|\{x_i\}, \{y_i\}\rangle\) and each eigenvector that corresponds to the eigenvalues \(x_i, y_i\) is \(|x_i\rangle\) and \(|y_i\rangle\). In this assumption, \(P_{xi}\) is defined as \(|\langle \psi|x_i\rangle|^2\) and \(P_{(x_i,y_j)}\) means \(|\langle x_i|y_j\rangle|^2\). The probability distribution of whole possible outcomes is depicted in Fig 1.

Thus, the entropy of the probability distribution for the successive measurements is

\[
H(X,Y) = - \sum_{i,j} p_{xi} p_{(x_i,y_j)} \log p_{xi} p_{(x_i,y_j)} \quad \text{(5)}
\]

\[
H(X) = - \sum_{i,j} p_{xi} p_{(x_i,y_j)} \log p_{(x_i,y_j)} \quad \text{(6)}
\]

\(H(X,Y)\) quantifies naturally occurring uncertainty when a state is measured for observables \(X\) and \(Y\) in succession. In the situation, from the statement (iii) and the noise-disturbance relations, it can be easily conjectured that it is also limited by a lower bound, likewise with the entropic uncertainty relation. Hence, by simple calculation, the relation is

\[
H(X) - \sum_{i,j} p_{xi} p_{(x_i,y_j)} \log p_{(x_i,y_j)} \geq -2 \log c, \quad \text{(7)}
\]

where \(c = \max_{i,j} |\langle x_i|y_j\rangle|\).

References


