Precision-guaranteed quantum tomography for finite-dimensional systems

Takanori Sugiyama[1 *] Peter S. Turner[1] Mio Murao[1 2]

[1] *Department of Physics, Graduate School of Science, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo, Japan.*

[2] *Institute for Nano Quantum Information Electronics, The University of Tokyo, 4-6-1 Komaba, Meguro-ku,Tokyo , Japan.*

Abstract. Quantum state tomography is the standard tool in current experiments for verifying that a state prepared in the lab is close to an ideal target state, but up to now there were no rigorous methods for evaluating the precision of the state preparation in tomographic experiments. We propose a new estimator for quantum state tomography, and prove that the (always physical) estimates will be close to the true prepared state with high probability. We derive an explicit formula for evaluating how high the probability is for an arbitrary finite-dimensional system and explicitly give the one- and two-qubit cases as examples. Using the formula, we can evaluate not only the difference between the estimated and prepared states, but also the difference between the prepared and target states. This is the first result directly applicable to the problem of evaluating the precision of estimation and preparation in quantum tomographic experiments.

Keywords: Quantum tomography, Finite data analysis, Confidence level

1 Introduction

Quantum tomography is a method for estimating an unknown quantum state and process and the standard tool in current quantum information experiments for verifying a successful realization of states and operations [1]. Let us consider the case of state preparation, where ρ_* denotes a target state that we are trying to prepare in the lab. In real experiments, the true prepared state ρ does not coincide with ρ_* because of imperfections. We wish to evaluate the precision of this preparation, that is, the difference between ρ_* and ρ - however we do not know ρ. Instead, we perform quantum state tomography; let ρ_N^{est} denote an estimate of the state made from N sets of data obtained in a tomographic experiment. To date the best we have been able to do is to evaluate the difference between ρ_* and ρ_N^{est} (see Fig. 1), but even if the difference is small, it does not guarantee that the prepared state ρ is close to the target state ρ_*, because ρ_N^{est} is given probabilistically and can deviate from ρ when N is finite. In this context, we refer to the difference between ρ_N^{est} and ρ the *precision* of the estimation.

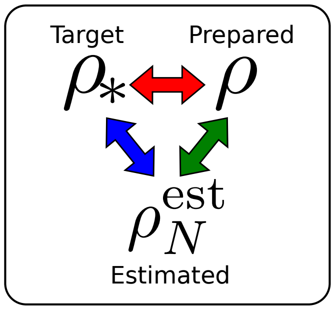

Figure 1: Three-fold relation between target, prepared, and estimated states: ρ_* is a target state that an experimentalist tries to prepare, ρ is the true prepared state. ρ_N^{est} is an estimate made from N tomographic data sets.

There are many proposals for evaluating the precision of estimation [2, 3, 4] and preparation [5, 6]. However the settings considered in [2, 3, 4, 5, 6] are different from the standard setting in quantum tomography, and their results are not directly applicable for evaluating

*sugiyama@eve.phys.s.u-tokyo.ac.jp; Current email address:sugiyama@itp.phys.ethz.ch; Current affiliation:Institute for Theoretical Physics, ETH Zurich, Switzerland.

the precision of estimation or preparation in current tomographic experiments. This is a crucial problem in the current theory of quantum tomography.

In this presentation, we give a solution to this problem. We propose a new estimator for quantum state tomography in finite-dimensional systems, and prove that the estimated states are within a distance threshold δ from the prepared state with high probability. We derive an explicit formula for evaluating how high the probability is in arbitrary finite-dimensional systems. This formula applies for any informationally complete set of measurements, for an arbitrary finite amount of data, and for general loss functions including the infidelity, the Hilbert-Schmidt, and the trace distances. Importantly, for a given experimental setup we can calculate the value of the formula without knowing the true prepared state, and so the formula can be used to evaluate the precision of state preparation. To our knowledge this is the first result directly applicable to evaluating the precisions of both estimation and preparation in quantum tomography. We demonstrate the technique for the example of one- and two-qubit state tomography.

2 Setting

We consider a finite $d(<\infty)$ dimensional quantum system, with Hilbert space $\mathcal{H}$. A state of the system is described by a density matrix, which is a positive-semidefinite and trace-one matrix, the space of which we denote by $\mathcal{S}(\mathcal{H})$. Let ρ denote the density matrix describing the true prepared state on $\mathcal{H}$. It is unknown, and we make no further assumptions on ρ. Suppose that N identical copies of the unknown true state, $\rho^{\otimes N}$, are available, and we can perform a measurement on each copy. Our aim is to estimate the unknown true state ρ from measurement results.

The statistics of a quantum measurement are described by a positive operator-valued measure (POVM), which is a set of positive-semidefinite matrices that sum to the identity. In standard setting of quantum tomography we choose a combination of measurements. A set of POVMs is called informationally complete (IC) if it spans the vector space of Hermitian matrices on $\mathcal{H}$ [7]. Such a set allows for the reconstruction of an arbitrary quantum state, and we will assume that the set of POVMs describing our measurements is always IC.

A map from a data set to the space of interest – in this case the space of quantum states – is called an estimator, and an estimation result is called an estimate. A loss function is a measure for evaluating the difference between two states. We analyze the following three loss functions:

$$\Delta^{\mathrm{HS}}(\rho',\rho) := \frac{1}{\sqrt{2}}\,\mathrm{Tr}\left[(\rho'-\rho)^2\right]^{1/2}, \quad (1)$$

$$\Delta^{\mathrm{T}}(\rho',\rho) := \frac{1}{2}\,\mathrm{Tr}\left[|\rho'-\rho|\right], \quad (2)$$

$$\Delta^{\mathrm{IF}}(\rho',\rho) := 1-\mathrm{Tr}\left[\sqrt{\sqrt{\rho'}\rho\sqrt{\rho'}}\right]^2. \quad (3)$$

called the Hilbert-Schmidt distance, the trace distance, and the infidelity, respectively. In current quantum tomography experiments the trace distance and the infidelity are most often used.

3 Main results

We propose a new nonlinear estimator ρ^{ENM} whose estimates are always physical. We call ρ^{ENM} an extended norm-minimization (ENM) estimator.

We prove that the ENM estimates will be close to the true prepared state with high probability.

Theorem 1 *For arbitrary true density matrix $\rho\in\mathcal{S}(\mathcal{H})$, any IC set of POVMs, positive integer N, and positive number δ,*

$$\Delta(\rho_N^{\mathrm{ENM}},\rho)\le\delta \quad (4)$$

holds with probability at least

$$CL := 1-2\sum_{\alpha=1}^{d^2-1}\exp\left[-\frac{b}{c_\alpha}\delta^2 N\right], \quad (5)$$

where b is determined by our choice of the loss function as

$$b := \begin{cases} 8/(d^2-1) & \text{if } \Delta=\Delta^{\mathrm{HS}} \\ 16/d(d^2-1) & \text{if } \Delta=\Delta^{\mathrm{T}} \\ 4/d(d^2-1) & \text{if } \Delta=\Delta^{\mathrm{IF}} \end{cases}, \quad (6)$$

and c_α are determined by our choice of the measurement setting.

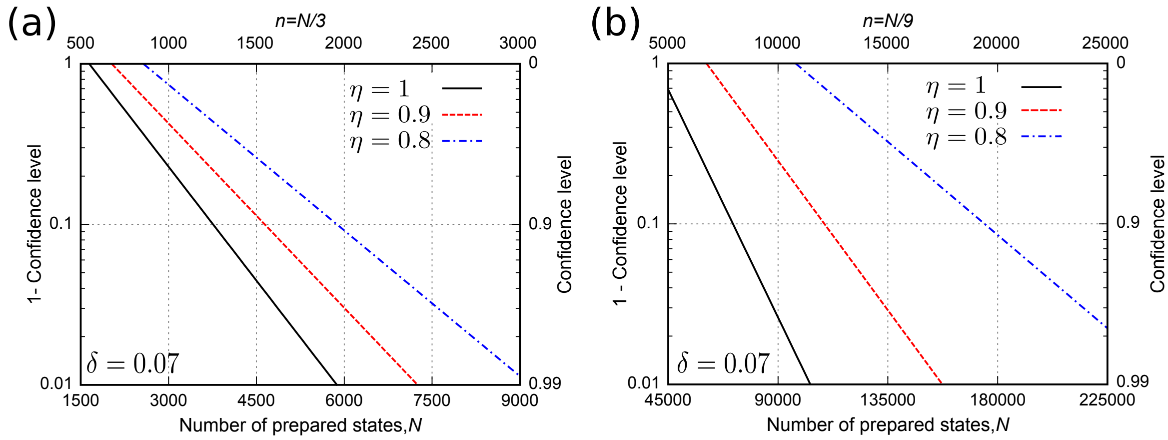

Figure 2: Confidence level of ρ^{ENM} for error threshold $\delta = 0.07$ in quantum state tomography: panel (a) is the one-qubit case ($k = 1$) and panel (b) is the two-qubit case ($k = 2$). In both panels, the left and right vertical axes are $1 - CL(k)$ and $CL(k)$ in Eq. (9), respectively. The lower and upper horizontal axes are the number of prepared states N and the number of observations for each tensor product of Pauli matrices $n = N/3^k$, respectively. In both panels, the line styles are fixed as follows: solid (black) line for detection efficiency $\eta = 1$, dashed (red) line for $\eta = 0.9$, chain (blue) line for $\eta = 0.8$.

We call CL in Eq. (5) the confidence level of ρ^{ENM} at the (user-specified) error threshold δ.

4 Analysis

The most important point in Theorem 1 is that Eq. (5) is independent of the true prepared state ρ. Therefore we can use it to evaluate $\Delta(\rho_*, \rho)$ without knowing ρ. Suppose that we choose a loss function Δ that satisfies properties of a mathematical distance. Then from the triangle inequality and Theorem 1, we have

$$\begin{aligned} \Delta(\rho_*, \rho) &\leq \Delta(\rho_*, \rho_N^{\mathrm{ENM}}) + \Delta(\rho_N^{\mathrm{ENM}}, \rho) \quad (7) \\ &\leq \Delta(\rho_*, \rho_N^{\mathrm{ENM}}) + \delta, \quad (8) \end{aligned}$$

where Eq. (8) holds at the confidence level in Eq. (5). We can calculate the value of the R.H.S. of Eq. (8) without knowing the true prepared state ρ and use it to evaluate the size of $\Delta(\rho_*, \rho)$.

Let us consider quantum state tomography of a k-qubit system and suppose that we make the three Pauli measurements with detection efficiency η on each qubit. There are 3^k different tensor products of Pauli matrices ($J = 3^k$), and suppose that we observe each equally $n := N/3^k$ times. When we choose $\Delta = \Delta^{\mathrm{T}}$, we obtain the explicit form of CL for k-qubit state tomography as

$$CL(k) = 1 - 2\sum_{l=0}^{k-1} 3^{k-l} \binom{k}{l} \exp\left[-\frac{2}{2^{2k}-1} \frac{\eta^{2(k-l)}}{3^{k-l}} \delta^2 N \right].$$

Figure 2 shows plots of Eq. (9) for the one-qubit ($k = 1$) and two-qubit ($k = 2$) cases in panels (a) and (b), respectively. The error threshold is $\delta = 0.07$ and detection efficiency is $\eta = 1,\ 0.9,\ 0.8$. Both panels indicate that smaller detection efficiency requires a larger number of prepared states. The plots tell us what value of N sufficient for guaranteeing a fixed confidence level. For example, if we want to guarantee 99% confidence level for $\delta = 0.07$ in one-qubit state tomography with $\eta = 0.9$, panel (a) indicates that $N = 7500$ is sufficient for that.

Acknowledgments

TS thanks Steven T. Flammia, David Gross, and Fuyuhiko Tanaka for helpful comments on least squares estimators. This work was supported by the JSPS Research Fellowships for Young Scientists (22-7564), the JSPS Postdoctoral Fellowships for Research Abroad (H25-32), and the Project for Developing Innovation Systems of the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan.

References

[1] M. Paris and J. Řeháček, eds., *Quantum State Estimation.* Lecture Notes in Physics (Springer, Berlin, 2004).

[2] R. Blume-Kohout, "Robust error bars for quantum tomography.", arXiv:1202.5270 [quant-ph] (2012).

[3] M. Christandl and R. Renner, "Reliable quantum state tomography," *Phys. Rev. Lett.* **109**, 120403 (2012).

[4] S. T. Flammia, D. Gross, Y.-K. Liu, and J. Eisert, "Quantum tomography via compressed sensing: error bounds, sample complexity and efficient estimators," *New J. Phys.* **14**, p. 095022 (2012).

[5] S. T. Flammia and Y.-K. Liu, "Direct fidelity estimation from few Pauli measurements," *Phys. Rev. Lett.* **106**, 230501 (2011).

[6] M. P. da Silva, O. Landon-Cardinal, and D. Poulin, "Practical characterization of quantum devices without tomography," *Phys. Rev. Lett.* **107**, 210404 (2011).

[7] E. Prugovečki, "Information-theoretical aspects of quantum measurement," *Int. J. Theor. Phys.* **16**, 321 (1977).