Nonclassicality-breaking = Entanglement-breaking for bosonic Gaussian channels

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Two notions that have been particularly well explored in the context of quantum information of continuous variable states are *nonclassicality* [1] and *entanglement* [2]. The 'older' notion of entanglement has become one of renewed interest in recent decades for its central role and applications in (potential as well as demonstrated) quantum information processes [3], while the concept of nonclassicality which emerges directly from the *diagonal representation* [1] had already been well explored in the quantum optical context [4], long before the emergence of the present quantum information era. While nonclassicality can be defined even for states of a single mode of radiation, the very notion of entanglement requires two or more parties. Nevertheless, it turns out that the two notions are not entirely independent of one another; they are rather intimately related [5–7]. In fact, nonclassicality is a prerequisite for entanglement [7]. Since a nonclassical bipartite state whose nonclassicality can be removed by local unitaries could not be entangled, one can assert, at least in some intuitive sense, that '*entanglement is nonlocal nonclassicality*'.

An important aspect in the study of nonclassicality and entanglement is in regard of their evolution under the action of a channel. A noisy channel acting on a state can degrade its nonclassical features [8]. Simlarly, bipartite entanglement can be degraded by channels acting locally on the constituent parties or modes [9–11]. In fact, there are channels that render every bipartite state separable by acting on just one of the parties [10–12]. Such channels are said to be *entanglement-breaking*.

A class of channels that has been of particular interest in the continuous variable quantum information processing context is the family of Gaussian channels. These are physical processes that map Gaussian states into Gaussian states. A (centered) Gaussian state is completely specified by its variance matrix V, and under the action of Gaussian channel Γ specified by the pair of matrices $(X,\,Y)$, $V\,\rightarrow\,V'\,=\,X^T\,V\,X\,+\,Y$. Let $\mathcal{S}$ denote an element of the symplectic group $Sp(2n,\,R)$ of linear canonical transformation and $\mathcal{U}(\mathcal{S})$ the corresponding unitary (metaplectic) operator [17]. One often encounters situations wherein the aspects one is looking for are invariant under local unitary operations, entanglement being an example. In such cases a Gaussian channel Γ is 'equivalent' to $\mathcal{U}(\mathcal{S}')\,\Gamma\,\mathcal{U}(\mathcal{S})$, for arbitrary symplectic group elements $\mathcal{S},\ \mathcal{S}'\,\in\,Sp(2n,\,R)$. The orbits or double cosets of equivalent channels in this sense are the ones classified and enumerated by Holevo and collaborators [14, 15]. The canonical forms so determined are useful, for instance, in the study of entanglement-breaking Gaussian channels [11].

In this work we address the following issue: *which Gaussian channels have the property that they rid every input state of its nonclassicality*? We recall that the density operator $\hat{\rho}$ representing any state of radiation field is '*diagonal*' in the coherent state '*basis*' [1], and this happens because of the over-completeness property of the coherent state basis. An important notion that arises from the diagonal representation is the *classicality-nonclassicality divide*. Since coherent states are the most elementary of all quantum mechanical states exhibiting classical behaviour, any state that can be written as a convex sum of these elementary classical states is deemed classical. Any state which cannot be so written as a convex sum of coherent states is deemed nonclassical. This classicality-nonclassicality divide leads to the following natural definition, inspired by the notion of entanglement breaking channels:

Definition: A channel Γ is said to be *nonclassicality-breaking* if and only if the output state $\hat{\rho}_{\rm out}\,=\,\Gamma(\hat{\rho}_{\rm in})$ is classical *for every* input state $\hat{\rho}_{\rm in}$, i.e., if and only if the diagonal 'weight' function of every output state is a genuine probability distribution.

Now, the close connection between nonclassicality and entanglement alluded to earlier raises a related and important second issue : *what is the connection, if any, between entanglement-breaking channels and nonclassicality-breaking channels*? To appreciate the nontriviality of this second issue, it suffices to simply note that the very definition of entanglement-breaking refers to bipartite states whereas the notion of nonclassicality-breaking makes no such reference. In this paper we show that both these issues can be completely answered in the case of bosonic Gaussian channels.

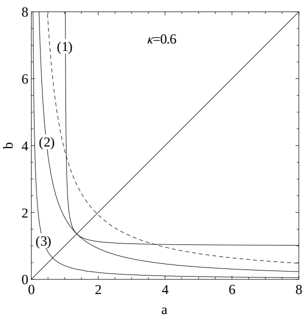
To this end we first derive the *nonclassicality-based* canonical forms for Gaussian channels. The classification of Holevo and collaborators is *entanglement-based*, and so it is not suitable for our purpose, since the notion of nonclassicality-breaking has a more restricted local invariance. A nonclassicality-breaking Gaussian channel Γ preceded by any Gaussian unitary $\mathcal{U}(\mathcal{S})$ is nonclassicality-breaking if and only if Γ itself is nonclassicality breaking. In contradistinction, the nonclassicality-breaking aspect of Γ and that of $\mathcal{U}(\mathcal{S})\,\Gamma$ [Γ followed the Gaussian unitary $\mathcal{U}(\mathcal{S})$] are not equivalent in general. They are equivalent if and only if $\mathcal{S}$ is in the intersection $Sp(2n,\, R) \cap SO(2n,\, R)$ of symplectic phase space rotations, or passive elements in the quantum optical sense [17]. In the single-mode case this intersection is just the rotation group $SO(2) \subset Sp(2,\, R)$. We thus need to classify single-mode Gaussian channels Γ into orbits or double cosets $\mathcal{U}(\mathcal{R})\,\Gamma\,\mathcal{U}(\mathcal{S})$, $\mathcal{S} \in Sp(2,\, R)$, $\mathcal{R} \in SO(2) \subset Sp(2, R)$. Equivalently, we classify (X, Y) into orbits $(\mathcal{S}\, X\, \mathcal{R},\, \mathcal{R}^T\, Y\, \mathcal{R})$. It turns out there are three distinct canonical forms for $(X,\, Y)$. These are then used to derive the necessary and sufficient conditions on a single-mode Gaussian channel to be nonclassicality-breaking. The canonical forms and the corresponding necessary and sufficient conditions for nonclassicality-breaking are listed in Table I. The conditions for entanglement-breaking and complete-positivity are also listed for comparison. The content of Table I is made transparent in the plots listed in Table II (see caption of Table. II).

For all three canonical forms we show that a nonclassicality-breaking channel is necessarily entanglement-breaking. There are channel parameter ranges where in the channel is entanglement-breaking but not nonclassicality-breaking, but the nonclassicality of the output state is of a 'weak' kind in the following sense : For every entanglement-breaking channel, there exists a particular value of squeeze-parameter r_0, depending only on the channel parameters and not on the input state, so that the entanglement-breaking channel followed by unitary squeezing of extent r_0 always results in a nonclassicality-breaking channel. It is in this precise sense that nonclassicality-breaking channels and entanglement-breaking channels are essentially one and the same.

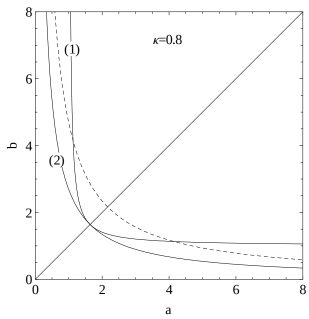
Squeezing is not the only form of nonclassicality. Our result not only says that the output of an entanglement-breaking channel could at the most have a squeezing-type nonclassicality, it further says that the nonclassicality of **all** output states can be removed by a **fixed** unitary squeezing transformation.

Canonical form	Nonclassicality-breaking condition	Entanglement-breaking condition	Complete-positivity condition
$(\kappa\, 1\!\!1,\, \mathrm{diag}(a,b))$	$(a-1)(b-1) \geq \kappa^4$	$ab \geq (1+\kappa^2)^2$	$ab \geq (1-\kappa^2)^2$
$(\kappa\,\sigma_3,\, \mathrm{diag}(a,b))$	$(a-1)(b-1) \geq \kappa^4$	$ab \geq (1+\kappa^2)^2$	$ab \geq (1+\kappa^2)^2$
$(\mathrm{diag}(1,0),\, Y)$,	$a,\, b \geq 1$, $a,\, b$ being eigenvalues of Y	$ab \geq 1$	$ab \geq 1$
$(\mathrm{diag}(0,0),\, \mathrm{diag}(a,b))$	$a,\, b \geq 1$	$ab \geq 1$	$ab \geq 1$

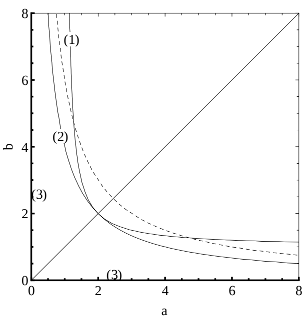
TABLE I: Here $1\!\!1$ is the 2×2 identity matrix, and σ_3 is the diagonal Pauli matrix.



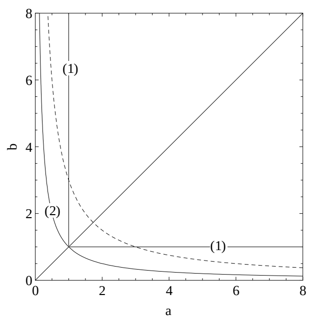
(a) Canonical form I: $(\kappa 1\!\!1,\, \mathrm{diag}(a,b))$



(b) Canonical form II: $(\kappa\,\sigma_3,\, \mathrm{diag}(a,b))$



(c) Limiting case of I: $(1\!\!1,\, \mathrm{diag}(a,b))$



(d) Canonical form III: Singular X

TABLE II: In all the four frames, the region to the right of (above) curve (1) corresponds to nonclassicality-breaking channels; the region to the right of (above) curve (2) corresponds to entanglement-breaking channels; curve (3) depicts the CP condition, so the region to the right of (above) it alone corresponds to physical channels. In frames (b) and (d), curves (2) and (3) coincide. The dotted curve indicates the orbit of a typical Gaussian channel under a unitary squeezing after the channel action. Note that the orbit of every entanglement-breaking channel passes through the nonclassicality-breaking region, showing that the nonclassicality in the output of an entanglement-breaking channel can be removed by a fixed unitary squeezing.

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