A gambling interpretation of some quantum information-theoretic quantities

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Abstract. It is known that repeated gambling over the outcomes of independent and identically distributed (i.i.d.) random variables gives rise to alternate operational meaning of entropies in the classical case in terms of the doubling rates. We give a quantum extension of this approach for gambling over the measurement outcomes of tensor product states. Under certain parameters of the gambling setup, one can give operational meaning of von Neumann entropies. We discuss two variants of gambling when a helper is available and it is shown that the difference in their doubling rates is the quantum discord. Lastly, a quantum extension of Kelly's gambling setup in the classical case gives a doubling rate that is upper bounded by the Holevo information.

Keywords: Quantum entropies, gambling, log-optimal strategies.

Quantum information theory [1, 2, 3] deals with the information content in the quantum systems and is a generalisation of classical information theory (see Ref. [4] for example) for quantum systems.

The measurement outcome of a quantum system is a random variable and the measurement alters the quantum state in general. We confine ourselves to finite dimensional Hilbert spaces that describe the quantum states and a probability mass function would describe the measurement outcome random variable that can be computed using the postulates of quantum mechanics.

If after a measurement, a quantum system is prepared again in the same state as before the measurement and the same measurement process is repeated, the sequence of the measurement outcomes is a sequence of i.i.d. random variables.

As an example, consider a quantum system prepared each time before the measurement in the quantum state $\rho = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$, where $0 \leq p \leq 1$. The measurement operators are $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. The measurement outcomes form a sequence of i.i.d. binary random variables each of which take values 0, 1 with probabilities p, $1-p$ respectively.

A classical gambling device such as a roulette consists of a revolving wheel onto which a ball is dropped and the ball settles down to one of the numbered slots or compartments on the wheel. Alice, the roulette player, bets on a number or a subset of numbers on which the ball comes to rest. There is a probability associated with winning on each gamble.

If bets are placed on the measurement outcomes of a quantum system, then the apparatus becomes a gambling device or a quantum roulette. Quantum gambling has been studied before in different contexts. Goldenberg *et al* invented a zero-sum game where a player can place bets at a casino located in a remote site [5]. Hwang *et al* considered its extensions using non-orthogonal and more than 2 states [6, 7]. Betting on the outcomes of measurements of a quantum state was considered by Pitowsky [8].

We note that none of the above references study the log-optimal gambling strategies, which, on the other hand, have been well-studied for the classical case (see for example [9, 4, 10] and references therein).

Quantum systems exhibit certain characteristics that are not possible classically. Bell inequalities [11, 12] give classical limits to the figure of performance for certain setups and these inequalities could be violated by quantum systems. We show that the quantum gambling devices too exhibit certain characteristics that are impossible to replicate classically.

At an information-theoretic level, the von Neumann entropy of the composite quantum systems A and B can be smaller than the von Neumann entropy of the subsystem B alone giving rise to negative conditional entropies. This is, as is well known, impossible for classical Shannon entropies.

Kelly defined a log-optimal gambling strategy by applying the law of large numbers to the factor (a random variable) by which Alice's wealth grows in a gamble. Thus, one can loosely claim that Alice's wealth is an exponential function of the number of gambles [9]. (We define this more precisely in the expanded version.) This approach has been developed further with the side information (or a helper) in Ref. [4].

The exponent (or the doubling rate if the base of the logarithm is 2) is a function of payoffs that the casino owner, Charlie, offers for each outcome, outcome probability distribution, and Alice's strategy. When we optimise the strategy under certain conditions, then the entropy (Shannon or von Neumann) appears in the exponent.

We note that these entropies (and certain information measures) have deep operational interpretations in classical and quantum information theory (see Ref. [1] and references therein).

In the classical case, Alice chooses how the wealth with which she is gambling is going to be distributed across the various outcomes. As an example, for two outcomes, Alice could bet half of her money on each of the outcomes.

For the quantum case, Alice can additionally make

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a choice of the measurement operators. Any classical roulette would be a special case of a quantum roulette.

We also consider a case when a helper named Bob is available for the gambler to make more money. Bob has access to a quantum system that is correlated with Alice's quantum system. Bob is broke and has no money to gamble on his system and offers Alice help in two ways.

In the first variant, Bob reports the measurement outcome to Alice who now knows the collapsed state of her quantum system and uses this information to further optimise her exponent (or the doubling rate). Alice may or may not have control over the measurement operators applied by Bob.

In the second variant, Bob leases out his quantum system to Alice who then gambles on the composite quantum system consisting of her and Bob's systems. In return, Bob demands a share in Alice's accrued wealth and wants Alice to retain the fraction of wealth that Alice would have accrued by gambling only on her system and had completely ignored the correlations between the two systems. Bob's argument is that Alice can win more by taking the correlations into account and it thus a win-win situation for both him, since he earns money after being broke, and Alice since she earns more money.

Under certain conditions, for the classical gambling, these two variants give rise to the same doubling rates whereas, for quantum gambling, they give rise to different doubling rates whose difference is equal to the quantum discord, a quantity that has been studied in a completely different context [13, 14].

Quantum discord is interpreted as purely the quantum part of the total correlation between the two quantum systems. That these two variants are the same classically in terms of the doubling rate lends support to the above interpretation.

Kelly gave another interesting interpretation of the mutual information [9]. Suppose Alice, knowing the output of a communication channel, bets on the inputs to the channel, then under certain conditions, the optimised doubling rate is equal to the mutual information. We extend Kelly's result to the quantum case and get a doubling rate in terms of a certain mutual information that is a function of the measurement operators which, using the Holevo bound, is upper bounded by the Holevo information.

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