Quantum discord for two-qubit X-states: An approach inspired by classical polarization optics

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Abstract. We present a comprehensive approach to quantum discord of two-qubit X-states. Our approach is geometric in nature, and employs methods that have been in use in classical polarization optics for several decades. We believe that our treatment is exhaustive: all known results can be reproduced, often more simply and economically; several new insights emerge, including clarification on some not so accurate claims in the literature.

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The study of correlations in bipartite systems has been invigorated over the last couple of decades or so. Various measures and approaches to segregate the classical and quantum contents of correlations have been explored. Entanglement has been the most popular of these correlations, owing to its inherent advantages in performing quantum computation and communication tasks. More recently, however, there has been a rapidly growing interest in the study of correlations from a more direct measurement perspective; in particular quantum discord and classical correlation have been attracting much attention.

In this work, we undertake a comprehensive analysis of the problem of computation of correlations in two-qubit system X-states which have come to be accorded a distinguished status in this regard. The problem of X-states has been considered by many authors. Here we present an approach that exploits the very geometric nature of the problem, and it may be noted that the geometric methods used here have been the basic tools of (classical) polarization optics for a very long time, and involve constructs like Stokes vectors, Poincaré sphere, and Mueller matrix [1].

The present study is largely caused by the work of M. Ali, A. R. P. Rau, and G. Alber [2] who *simply asserted* that the optimal measurement for X-states is *always* the von-Neumann measurement along either 'x' or 'z' direction. This assertion itself is based on an unusual symmetry argument which goes like thus: *since the problem has a symmetry, the optimal von-Neumann measurement must be invariant under this symmetry.* We begin by exhibiting an X-state for which, despite the symmetry, the x and z-projections return the **worst** value among **all** von-Neumann measurements. Several authors have more recently presented isolated examples wherein the Ali *et al.* assertion either fails or is doubtful, *numerically*; but the need for our comprehensive study originates in our position that the Ali *et al.* argument is non-maintainable even in cases where their assertion returns numerically correct values.

Now, classical correlation in a bipartite state is given by the expression

$$C(\hat{\rho}_{AB}) = \max_{\Pi} S(\hat{\rho}_A) - \sum_j p_j S(\hat{\rho}_j^A), \qquad (1)$$

where the probabilities $\{p_j\}$ and the state of system A after the measurement Π_j are given by

$$p_j = \mathrm{Tr}[(\mathbb{1}_A \otimes \Pi_j^B)\hat{\rho}_{AB}],\ \ \hat{\rho}_j^A = \frac{\mathrm{Tr}_B[\Pi_j^B \hat{\rho}_{AB}]}{p_j}. \qquad (2)$$

The set $\Pi = \{\Pi_j\}$ meets the defining conditions $\sum_j \Pi_j = \mathbb{1}$ and $\Pi_j \geq 0$ for all j. That is, the set $\{\Pi_j\}$ forms a POVM. The second term in the expression (1) for classical correlation is the (minimum) conditional entropy post measurement, and we may denote it by

$$S^A_{\min} = \min_{\Pi} \sum_j p_j S(\hat{\rho}_j^A), \qquad (3)$$

so that the expression for quantum discord reads as

$$\mathcal{D}(\hat{\rho}_{AB}) = S(\hat{\rho}_B) - S(\hat{\rho}_{AB}) + S^A_{\min}. \qquad (4)$$

We note that the first two terms of this expression for quantum discord are known as soon as the bipartite state $\hat{\rho}_{AB}$ is specified. Therefore the only quantity of computational interest is the conditional entropy $S^A_{\min}$ of system A post measurement (on B): this alone involves an optimization.

In classical polarization optics the state of a light beam is represented by a 2×2 complex positive matrix Φ called the *polarization matrix.* The intensity of the beam is identified with $\mathrm{Tr}\,\Phi$, and so the normalized matrix $(\mathrm{Tr}\Phi)^{-1}\Phi$ represents the actual state of polarization. The polarization matrix Φ is thus analogous to the density matrix of a qubit, the only distinction being that the trace of the latter needs to assume unit value. Even this one little difference is gone when one deals with conditional quantum states post measurement: the probability of obtaining a conditional state becomes analogous to intensity = $\mathrm{Tr}\Phi$ of the classical context.

The Mueller-Stokes formalism itself arises from the following simple fact: any (complex) 2×2 matrix Φ can be invertibly associated with a (generally complex) four-vector S, called the Stokes vector, through

$$\Phi = \frac{1}{2}\sum_{k=0}^{3} S_k\sigma_k, \quad S_k = \mathrm{Tr}(\sigma_k\Phi). \qquad (5)$$

This representation is an immediate consequence of the fact that the Pauli triplet σ_1, σ_2, σ_3 together with $\sigma_0 = \mathbb{1}$, the unit matrix, form a complete orthonormal set of (hermitian) matrices. Clearly, hermiticity of the polarization matrix Φ is equivalent to reality of the associated four-vector S and $\mathrm{Tr}\Phi = S_0$. Positivity of Φ reads $S_0 > 0$, $S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0$ corresponding, respectively, to the pair $\mathrm{Tr}\,\Phi > 0$, $\det\Phi \geq 0$. Thus positive 2×2 matrices (or their Stokes vectors) are in one-to-one correspondence with points of the *positive branch of the solid light cone.* Unit trace (intensity) restriction corresponds to the section of this (four-dimensional) cone at unity along the 'time' axis, $S_0 = 1$. The resulting three-dimensional unit ball $\mathcal{B}_3 \in \mathcal{R}^3$ is the more familiar Bloch (Poincaré) ball, whose surface or boundary $\mathcal{S}^2$ representing pure states is often called the Bloch (Poincaré) sphere $\mathcal{P}$.

Optical systems which map Stokes vectors *linearly* into Stokes vectors have been of particular interest in polarization optics. Such a linear system is represented by a 4×4 real matrix M, the Mueller matrix $M : S^{\rm in} \to S^{\rm out} = MS^{\rm in}$.

To see the connection between Mueller matrices and two-qubit states unfold naturally, use a single index rather than a pair of indices to label the computational basis two-qubit states $\{|jk\rangle\}$ in the usual manner: $(00, 01, 10, 11) = (0, 1, 2, 3)$. Now, a two-qubit density operator $\hat{\rho}_{AB}$ can be expressed in *two distinct ways*:

$$\hat{\rho}_{AB} = \sum_{j,k=0}^{3} \rho_{jk} |j\rangle\langle k| = \frac{1}{4} \sum_{a,b=0}^{3} M_{ab}\, \sigma_a \otimes \sigma_b^*, \qquad (6)$$

the second expression simply arising from the fact that the sixteen hermitian matrices $\{\sigma_a \otimes \sigma_b^*\}$ form a complete orthonormal set of 4×4 matrices. Hermiticity of operator $\hat{\rho}_{AB}$ is equivalent to *reality* of the matrix $M = ((M_{ab}))$, but the same hermiticity is equivalent to $\rho = ((\rho_{jk}))$ being a *hermitian* matrix.

The Stokes vector of the conditional state of A resulting from measurement element corresponding to Stokes vector S on the B side turns out from Eq. (6) to be $S_a^{'} = \sum_{k=0}^{3} M_{ak} S_k$, which may be written in the suggestive form

$$S^{\rm out} = MS^{\rm in}. \qquad (7)$$

Comparison with the polarization scenario prompts us to call M the *Mueller matrix associated with two-qubit state* $\hat{\rho}_{AB}$. The state of subsystem A resulting from measurement of any POVM element on the B-side of $\hat{\rho}_{AB}$ is the Stokes vector resulting from the action of the associated Mueller matrix on the Stokes vector of the POVM element. In the rank-one case, the input Stokes vectors correspond to points on the (surface $\mathcal{S}^2 = \mathcal{P}$) of the Bloch ball.

Every two-qubit state has associated with it a unique *output ellipsoid of all possible conditional states.* Since the ellipsoid degenerates to a single point in the case of product states, we may call it the *correlation ellipsoid* associated with the given bipartite state. The X-states are distinguished by the fact that C (center of the ellipsoid), I (image when identity is the input), and the origin of $\mathcal{P}$ are collinear with one of the principal axes of the ellipsoid. In other words, C and I become one-dimensional rather than three-dimensional variables, rendering X-states a 11-parameter subfamily of the 15-parameter state space. This geometric rendering is manifestly invariant under local unitaries as against the characterization in terms of 'shape' X in the computation basis.

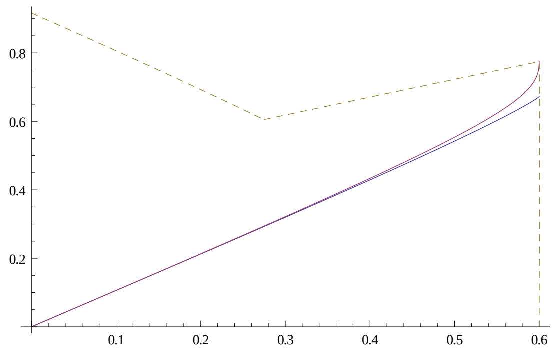

Figure 1: Depicting a two-section of the manifold of X-states. It is the wedge-like tiny region at the top right end that corresponds to X-states for which the optimal POVM is not a von-Neumann projection.

We develop a optimal scheme for computation of the quantum discord for any X-state of a two-qubit system. Our treatment itself is *both comprehensive and self-contained and, moreover, it is geometric in flavour.* We begin by exploiting symmetry to show, without loss of generality, that the problem itself is one of *optimization over just a single variable.* The analysis is entirely based on the output or correlation ellipsoid.

Not all parameters of a two-qubit X-state influence the correlation ellipsoid, and since our entire analysis is anchored on the correlation ellipsoid, the parameters that influence and those which do not influence play very different roles. The correlation ellipsoid has an invariance group which is much larger that the group of local unitary symmetries. An appreciation of this larger invariance turns out to be essential to the simplification of the present analysis.

A typical two-section of the manifold of X-states is depicted in Fig. 1 to underline the fact that the region where the assertion of Ali *et al.* is numerically misplaced is really tiny. But the X-states in this tiny region have *the same symmetry* as those outside, perhaps implying that if the symmetry argument of Ali *et al.* is misplaced it is likely to be so everywhere, and not just in this region.

References

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