How well can one jointly measure two incompatible observables on a given quantum state?

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Abstract. We consider the approximate joint measurement of two incompatible observables on a given quantum state, and present a tight relation characterizing the optimal trade-off between the error on one observable vs. the error on the other. As a particular case, our approach allows us to characterize the disturbance of an observable induced by the approximate measurement of another one; we introduce an even stronger error-disturbance relation for this scenario.

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1 Introduction

Uncertainty Relations, first suggested by Heisenberg in 1927 [1], are among the main pillars of quantum theory. One of the best known versions is for instance the one due to Robertson [2], which writes $\Delta A \Delta B \geq |C_{AB}|$, where $\Delta A$, resp. $\Delta B$, is the standard deviations of the measurement of an observable $A$, resp. $B$, on a quantum state $|\psi\rangle$, and $C_{AB} = \langle \psi | [A, B] |\psi\rangle / 2i$.

It is often argued that such uncertainty relations imply that incompatible observables cannot be jointly measured on quantum states such that $C_{AB} \neq 0$, or that the measurement of one necessarily implies a disturbance on the other (which is in fact the original idea presented by Heisenberg in [1]). This is however not what the Robertson relation above tells, as it only bounds the statistical deviations of the measurement results of $A$ and $B$, when either of each measurement is performed many times, on several independent copies of $|\psi\rangle$.

Instead of considering such statistical deviations on many measurements of either $A$ or $B$, we address here the following problem: if $A$ and $B$ are incompatible and cannot be perfectly jointly measured on $|\psi\rangle$, it may still be possible to jointly approximate the measurement of both observables, at the price of introducing errors; what is then the optimal trade-off between the errors $\epsilon_A$ and $\epsilon_B$ introduced in the measurement of $A$ and $B$, respectively?

2 Approximate joint measurements

In order to approximate the measurement of an observable $A$ on a quantum system in the state $|\psi\rangle$, a general strategy consists in measuring another, “approximate” observable $\mathcal{A}$, possibly on an extended Hilbert space—i.e., on the joint system composed of the state $|\psi\rangle$, and of an ancillary system in the state $|\xi\rangle$. In this picture, the impossible joint measurement of two incompatible observables $A$ and $B$ on $|\psi\rangle$ can thus be approximated by the perfect joint measurement of two compatible (i.e., commuting) observables $\mathcal{A}$ and $\mathcal{B}$ on $|\psi, \xi\rangle = |\psi\rangle \otimes |\xi\rangle$.

Following Ozawa (e.g. [3, 4]), we characterize the quality of the approximations $\mathcal{A}$ and $\mathcal{B}$ of $A$ and $B$, respectively, by defining the root-mean-square (rms) errors

\[
\epsilon_A = \langle \psi, \xi | (A - \mathcal{A} \otimes 1)^2 | \psi, \xi \rangle^{1/2},
\]

\[
\epsilon_B = \langle \psi, \xi | (B - \mathcal{B} \otimes 1)^2 | \psi, \xi \rangle^{1/2}.
\]

3 Error-trade-off relations for approximate joint measurements

The fact that quantum theory forbids perfect joint measurements of incompatible observables implies that the rms errors $(\epsilon_A, \epsilon_B)$ can in general not take arbitrary values.

A common misconception is that Robertson’s relation should still hold if the standard deviations $\Delta A$ and $\Delta B$ are simply replaced by the rms errors $\epsilon_A$ and $\epsilon_B$, so that $\epsilon_A \epsilon_B \geq |C_{AB}|$. While this relation, often attributed to Heisenberg himself or to Arthurs and Kelly [5], can indeed be proven under some restrictive assumptions, it is worth emphasizing that in general it does not hold [6].

Only recently did Ozawa show [4] how one could derive a universally valid “uncertainty relation” for joint measurements, namely

\[
\epsilon_A \epsilon_B + \Delta B \epsilon_A + \Delta A \epsilon_B \geq |C_{AB}|.
\]

The three terms in Ozawa’s relation come from three independent uses of Robertson’s relation to different pairs of observables. While this indeed leads to a valid relation and allows one to exclude a large set of impossible values $(\epsilon_A, \epsilon_B)$, this is not optimal, as the three Robertson’s relations (and therefore Ozawa’s relation) in general cannot be saturated simultaneously.

3.1 A new, tight error-trade-off relation for joint measurements

Using a general geometric inequality for vectors in a Euclidean space, we could show [7] how to improve upon the sub-optimality of Ozawa’s proof, and derive the following error-trade-off relation for approximate joint mea-
measurements:
\[ \Delta B^2 \epsilon_A^2 + \Delta B \epsilon_B^2 + 2 \sqrt{\Delta A^2 \Delta B^2} - C_{AB}^2 \epsilon_A \epsilon_B \geq C_{AB}^2. \]

It can easily be checked that Ozawa’s relation (3) can directly be derived from our new relation above. Furthermore, not only is our relation stronger than Ozawa’s, it is actually tight: for any \(A, B\) and \(|\psi\rangle\), any values \((\epsilon_A, \epsilon_B)\) saturating inequality (4) can be obtained [7]. Hence, contrary to previously derived relations, our new one does not only tell what cannot be done quantum mechanically, but also what can be done.

3.2 The error-disturbance scenario and the same-spectrum assumption

The error-disturbance scenario, as first discussed by Heisenberg [1], can be treated as a particular case of the general framework for approximate joint measurements.

In this context, one considers the disturbance \(\eta_B\) in the statistics of one observable, \(B\), due to the unsharp measurement of another observable, \(A\). The approximation of \(A\) can again be described by the measurement of an observable \(A\), while the subsequent measurement of \(B\) on the disturbed system can be written as the measurement of an observable \(B\) on the original state. Using the same formalism as in the joint measurement framework, the rms error \(\epsilon_B\) is now interpreted as the rms disturbance \(\eta_B\) of \(B\), with formally the same definition [3]: \(\eta_B = \epsilon_B\) as defined in (2).

Any error-trade-off relation derived in the more general framework of joint measurements thus remains valid in this error-disturbance scenario. In particular, when interpreting \(\epsilon_B\) as the rms disturbance \(\eta_B\), Ozawa’s relation (3) writes

\[ \epsilon_A \eta_B + \Delta B \epsilon_A + \Delta A \eta_B \geq |C_{AB}|. \] (5)

This error-disturbance relation was actually introduced by Ozawa before its previous version (3) for joint measurements [3]. In a similar manner, our new error-trade-off relation (4) also implies a new error-disturbance relation, by simply replacing \(\epsilon_B\) by \(\eta_B\).

The difference with the previous, more general scenario of joint measurements is however not merely in the interpretation of \(\epsilon_B\). A crucial point is that as the approximate measurement \(B\) corresponds to the actual measurement of \(B\) on the disturbed system, it necessarily has the same spectrum as \(B\)—which was not assumed previously. Because of this constraint, one may expect stronger restrictions on the possible values of \(\eta_B\) to hold, and that stronger “error-disturbance relations” can be derived.

To illustrate this, we could show [7] that for the case of a dichotomic observable \(B\) with eigenvalues \(\pm 1\) (such that \(B^2 = \mathbb{1}\)), and for a state \(|\psi\rangle\) for which \(|B\rangle = 0\) (which implies \(\Delta B = 1\)), if one imposes the same spectrum assumption to \(B\) (i.e. that \(B^2 = \mathbb{1}\) as well), then an analogous relation to (4) holds, where \(\epsilon_B\) is replaced by \(\eta_B \sqrt{1 - \eta_B^2} \frac{1}{4}\). If \(A\) is also such that \(A^2 = \mathbb{1}\) and \(\langle A \rangle = 0\), and if one also imposes the same-spectrum assumption to \(A\), then one can derive the error-disturbance [7]

\[ \epsilon_A^2 \left(1 - \frac{\epsilon_A^2}{4}\right) + \eta_B^2 \left(1 - \frac{\eta_B^2}{4}\right) + 2 \sqrt{1 - C_{AB}^2} \epsilon_A \eta_B \sqrt{1 - \eta_B^2} \geq C_{AB}^2. \] (6)

This new error-disturbance relation is strictly stronger than (4) (and than Ozawa’s relation (5)). Furthermore, we could show [7] that it is tight when \(|\langle \psi | AB | \psi \rangle| = 1\): for any \(A, B\) and \(|\psi\rangle\) satisfying the constraints above, one can reach any values \((\epsilon_A, \eta_B)\) that saturate the inequality, using approximate measurements such that \(A^2 = B^2 = \mathbb{1}\).

4 Conclusion

Our new error-trade-off relation (4) quantifies precisely the optimal trade-off between the rms errors \(\epsilon_A\) and \(\epsilon_B\) introduced in any approximate joint measurement of \(A\) and \(B\), thus answering the question posed in our title. In the case where one imposes that the approximations should have the same spectrum as \(A\) and \(B\), one can derive stronger constraints, as e.g. our error-disturbance relation (6) for the case of \(\pm 1\)-valued observables.

The tightness of these relations is a crucial feature: they do not only indicate what cannot be done quantum mechanically, but also what can be done. Two recent experiments [8, 9] verified the validity of Ozawa’s error-disturbance relation (5) (and the violation of the “Heisenberg-Arthurs-Kelly relation” \(\epsilon_A \eta_B \geq |C_{AB}|\), but were nowhere near saturating it—which is indeed in general not possible. However, an adequate setup (such as the one of [9]) should allow one to saturate our error-disturbance relation (6), as well as our error-trade-off relation (4) if the approximate measurements are not restricted to output eigenvalues of \(A\) and \(B\).

References