Using partial transpose and realignment to generate local unitary invariants.

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Abstract.
Motivated by link transformations of lattice gauge theory, a method for generating local unitary invariants, especially for a system of qubits, has been pointed out in an earlier work [M. S. Williamson *et. al.*, Phys. Rev. A 83, 062308 (2011)]. This paper first points the equivalence of the so constructed transformations to the combined operations of partial transpose and realignment. This allows construction of local unitary invariants of any system, with subsystems of arbitrary dimensions. Some properties of the resulting operators and consequences for pure tripartite higher dimensional states are briefly discussed.

Keywords: Quantum Entanglement, Quantum Information and Local Unitary Invariants

To detect and measure entanglement in a general state, represented by a density matrix, is a difficult problem. It has been solved for the case of two qubit states via concurrence, bipartite states of a qubit and a qutrit via the partial transpose, and all bipartite pure states. There are various measures of entanglement, for example the von Neumann entropy of any one subsystem in a bipartite pure state is such a measure, the concurrence introduced by Hill and Wootters measures entanglement between two qubits in a pure or mixed state, while negativity and log negativity is invoked for a general bipartite mixed state which uses the positive, but not completely positive, map of partial transposition (PT). The partial transpose was introduced by Peres to detect entanglement, and provides a sufficient but not necessary condition. One other such criterion uses the operation of "realignment". The operation of realignment is also associated with entanglement detection [1] and provide necessary conditions for separability. This operation is found to detect some bound entangled states, these being positive under PT and hence not being detected by the corresponding criterion.

The question of entanglement in the case of pure bipartite states is solved completely by using the Schmidt decomposition, where the Schmidt coefficients are invariant under local unitary (LU) operations [2]. The Schmidt coefficients or functions of them, for *e.g.* the von Neumann entropy, are used to characterize bipartite entanglement. As there is no such decomposition, in general, for multipartite states, another approach to study entanglement, is by studying the LU invariants of the system which consists of invariants under arbitrary unitary transforms restricted to the individual subsystems. In as much as entanglement quantifies non-local properties, entanglement measures remain invariant under LU operations and hence the importance of their study. The spectra of the density matrix itself and the various reduced density matrices got by tracing out subsystems are such LU invariants. However, it helps to have invariants that are polynomials in the entries of the density matrix [3], and those whose physical interpretation in terms of entanglement is available.

These invariants uniquely determine the orbit of the state under these local operations. For example three qubit pure states have five independent LU invariants [2, 4] excluding the trace. In a recent paper Williamson *et.al* [5], inspired by lattice gauge theory, have given a method of generating these invariants by associating them with a closed path joining some or all the qubits where two consecutive qubits on the path are connected by a "link transformation". In this paper, it is first shown that the link transformation in [5] is unitarily equivalent to the combined operations of partial transpose and realignment [1]. Thus it is interesting that two rather independent operations on which entanglement criteria are based come together in the construction of local invariants. This result immediately suggests a way to generalize to a system of an arbitrary set of qudits, each not necessarily of identical dimensions. Such a generalization is then shown to be LU invariant as well. One of the advantages of this method is that it does not need the generalization of Pauli matrices in higher dimensions to get the invariants as required for the link transformation approach of [5].

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