Entanglement certainty from Heisenberg's uncertainty

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Abstract. We present a new paradigm for capturing the complementarity of two observables (see original article, arXiv:1305.3442). It is based on the entanglement created by the interaction between the system observed and the two measurement devices used to measure the observables sequentially. Our main result is a lower bound on this entanglement and resembles well-known entropic uncertainty relations. Besides its fundamental interest, this result directly bounds the effectiveness of measurement operations for generating entanglement, and has further application for decoupling and coherent teleportation.

Keywords: entanglement, complementarity, uncertainty, Heisenberg, sequential, measurement

Heisenberg's original formulation of the uncertainty principle considered sequential measurements of complementary observables, like position and momentum, performed on the *same* physical system, and the principle was that the second observable is *unavoidably disturbed* by the measurement of the first [1]. An alternative scenario considers *unavoidable uncertainty* for independent measurement of the two observables, with the measurements performed on *distinct but identically prepared* quantum systems [2, 3].

The latter formulation of the uncertainty principle seems to receive more attention in modern times. For example, entropic uncertainty relations [4] typically capture this unavoidable uncertainty; consider a well-known example from Maassen and Uffink [5]. For any state ρ_S of a finite-dimensional quantum system S they find

$$H(X) + H(Z) \geqslant \log_2(1/c), \tag{1}$$

where $X = \{|X_j\rangle\}$ and $Z = \{|Z_k\rangle\}$ are any two orthonormal bases of $\mathcal{H}_S$, $H(X) := -\sum_j p(X_j)\log_2 p(X_j)$ is the Shannon entropy associated with the probability distribution $p(X_j) := \langle X_j|\rho_S|X_j\rangle$ (similarly for $H(Z)$), and $c := \max_{j,k} |\langle X_j|Z_k\rangle|^2$ quantifies the complementarity between the X and Z observables.

Ref. [6] showed that an entropic uncertainty relation like (1) has a correspondent *entanglement certainty relation.* They considered the generation of entanglement between measurement devices and independent, identically-prepared copies of some system, and proved that, when dealing with complementary observables, there is *unavoidable creation of entanglement* between at least one copy of the system and one measuring device.

Main result.—In this work, we offer a new point of view on complementarity. As Heisenberg did originally, we consider sequential measurements performed on the same physical system, rather than idependent copies of the system; on the other hand, following [6, 7, 8], we focus on the entanglement generated between the system and the measurement devices. In general, for any X and Z, we can lower-bound the entanglement $E(X, Z)$ between the system and the measurement devices created from sequentially measuring X and Z with

$$E(X, Z) \geqslant \log_2(1/c), \tag{2}$$

where the c factor appearing here is *precisely the same c appearing in Eq.* (1). Here, we take E to be the distillable entanglement, i.e., the optimal rate to distill Einstein-Podolsky-Rosen (EPR) pairs using local operations and classical communication (LOCC) in the asymptotic limit of infinitely many copies of the state.

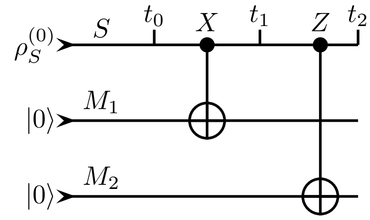

Figure 1: Circuit diagram for the sequential measurement of the X and Z observables on system S.

Our approach relates in a novel way two basic concepts of quantum mechanics: complementarity—in the sequential-measurement scenario—and entanglement. Besides this fundamental interest, our results have direct operational interpretations. On one hand, they provide bounds on the usefulness of sequential bipartite operations—corresponding to the measurement interactions—for entanglement generation. On the other hand, we discuss below the application of our results to decoupling [9] and coherent teleportation [10].

Figure 1 depicts the basic setup for our main result. At the initial time, t_0, the system is in an arbitrary state $\rho_S^{(0)}$. It firsts interacts with device M_1, which measures observable X, and later it interacts with device M_2, which measures observable Z. The measurements are depicted with the controlled-NOT symbols although more generally they represent controlled-shift unitaries. We are interested in the bipartite entanglement $E(X, Z)$ between S and the joint system M_1M_2 at the final time, t_2.

The case where X and Z are fully complementary, so-called mutually unbiased bases (MUBs), corresponds to

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$|\langle X_j|Z_k\rangle|^2 = 1/d$ for all j, k, and hence $c = 1/d$, where $d = \dim(\mathcal{H}_S)$. In the uncertainty relation (1), this leads to the maximum tradeoff in knowledge, with the r.h.s. becoming $\log_2 d$, and hence perfect knowledge of X implies complete ignorance of Z. Likewise, in our main result, the r.h.s. of (2) becomes $\log_2 d$. This implies that, *for any input state* $\rho_S^{(0)}$, sequentially measuring X and Z results in a maximally entangled state between S and M_1M_2. It may seem surprising that this is even true if we feed in a maximally-mixed state $\rho_S^{(0)} = I/d$.

Eq. (2) also allows us to say that if X and Z are *almost* MUBs, then for any input state, S is *almost* maximally entangled to M_1M_2 at time t_2. Furthermore, as long as $\log_2(1/c) > 0$, then there is guaranteed to be distillable entanglement at time t_2.

Ref. [11] provides two alternative proofs of (2). One is based on the uncertainty principle with quantum memory [12], applied at time t_1 in Fig. 1. The other invokes the monotonicity of entanglement under LOCC, which allows us to derive a slightly stronger version of (2) given in [11].

In what follows, we discuss the implications of our main result for decoupling and coherent teleportation.

Decoupling.—The correlations between two quantum systems can be destroyed, turning an arbitrary bipartite state $\rho_{SS'}$ into some tensor product $\sigma_S \otimes \sigma_{S'}$, with appropriate local operations. This idea, called decoupling [9] has specific applications in state merging [13] and quantum cryptography [14]. Our work identifies sequential complementary measurements as one such method to decouple. This is due to the monogamy principle: because S is highly entangled to M_1M_2 at time t_2, then S cannot be too correlated with any other system S' at t_2.

We make this precise by considering the relative entropy $D(\sigma\|\tau) := \mathrm{Tr}(\sigma \log_2 \sigma) - \mathrm{Tr}(\sigma \log_2 \tau)$. Letting $\rho_{SS'}^{(2)}$ denote the state of S and some other system S' at time t_2, we find that, for any initial state $\rho_{SS'}^{(0)}$,

$$D(\rho_{SS'}^{(2)} || I/d \otimes \rho_{S'}^{(2)}) \leqslant \log_2(d \cdot c), \tag{3}$$

which is a corollary of Eq. (2). Indeed this implies that the final state $\rho_{SS'}^{(2)}$ is *almost* completely decoupled if the X and Z observables are *almost* fully complementary.

Coherent teleportation.—We have shown that the ability to produce entanglement and to decouple using sequential measurements is a quantification of the complementarity of those measurements. It turns out there is a third perspective on complementarity. In the case when X and Z are MUBs, there exists a local unitary applied to M_1M_2 at time t_2 that recovers the input state $\rho_S^{(0)}$ on device M_1, i.e., we can "teleport" the input state of S to one of the measurement devices. This is commonly known as coherent teleportation [10]. In this case, the channel $\mathcal{E} : S(t_0) \to S(t_2)$ is completely noisy, while the channel $\mathcal{E}^c : S(t_0) \to M_1M_2(t_2)$ is perfect. As we reduce the complementarity of X and Z, the channel $\mathcal{E}^c$ becomes less perfect, so we can consider the quantum capacity Q of $\mathcal{E}^c$, i.e., the optimal rate at which $\mathcal{E}^c$ allows for the reliable transmission of quantum information [15], as a measure of the complementarity of X and Z. We make this idea quantitative with the following corollary of (2),

$$Q(\mathcal{E}^c) \geqslant \log_2(1/c). \tag{4}$$

Eq. (4) allows us to say that we can *approximately* teleport the state $\rho_S^{(0)}$ when X and Z are *almost* MUBs.

Conclusions.—We give an alternative take on complementarity. Instead of discussing a trade-off of knowledge, as in uncertainty relations, we propose that a signature and quantification of complementarity of two observables is given by the entanglement generated when the observables are sequentially measured on the same system via a coherent interaction with corresponding measurement devices. We also offer the perspectives of decoupling and coherent teleportation. We find it intriguing that the same complementary factor c appearing in uncertainty relations also appears in these operational contexts.

References

[1] W. Heisenberg, Zeitschrift für Physik **43**, 172 (1927).

[2] E. Kennard, Z. Phys **44**, 326 (1927).

[3] H. P. Robertson, Phys. Rev. **34**, 163 (1929).

[4] S. Wehner and A. Winter, New J. Phys. **12**, 025009 (2010).

[5] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988).

[6] P. J. Coles, Phys. Rev. A **86**, 062334 (2012).

[7] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. **106**, 160401 (2011).

[8] M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, Phys. Rev. Lett. **106**, 220403 (2011).

[9] B. Schumacher and M. D. Westmoreland, Quantum Information Processing **1**, 5 (2002), ISSN 1570-0755.

[10] G. Brassard, S. L. Braunstein, and R. Cleve, Physica D: Nonlinear Phenomena **120**, 43 (1998), ISSN 0167-2789, proceedings of the Fourth Workshop on Physics and Consumption.

[11] P. J. Coles and M. Piani, ArXiv e-prints (2013), 1305.3442.

[12] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nature Physics **6**, 659 (2010).

[13] M. Horodecki, J. Oppenheim, and A. Winter, Nature **436**, 673 (2005).

[14] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).

[15] M. M. Wilde, arXiv preprint arXiv:1106.1445 (2011).