Tight bounds on the distinguishability of quantum states under separable measurements

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One of the many interesting features of quantum nonlocality is that the states of a multipartite quantum system cannot always be distinguished as well by local measurements as they can when all quantum measurements are allowed. In this work we address a basic question, which is how much can be learned about a given quantum system using separable measurements – those which contain the class of local measurements but nevertheless are free of entanglement between the component systems. We consider two quantities: The separable fidelity – a truly quantum quantity – which measures how well we can "clone" the input state, and the classical probability of success, which simply gives the optimal probability in identifying the state correctly.

We obtain lower and upper bounds on the separable fidelity and give several examples in the bipartite and multipartite settings where these bounds are optimal. Moreover the optimal values in these cases can be attained by local measurements. We further show that for distinguishing orthogonal states under separable measurements, a strategy that maximizes the probability of success is also optimal for separable fidelity. We point out that the equality of fidelity and success probability does not depend on an using optimal strategy, only on the orthogonality of the states. To illustrate this, we present an example where two sets (one consisting of orthogonal states, and the other non-orthogonal states) are shown to have the same separable fidelity even though the success probabilities are different.

Suppose a composite quantum system is known to be in one of many states, not necessarily orthogonal, such that its parts are distributed among spatially separated observers. The goal is to learn about the state of the system using only local quantum operations and classical communication between the parties (LOCC). This problem, known as local state discrimination, is of considerable interest [1–4, 9, 12, 13, 15, 17, 20, 27], as in many instances the information obtainable by LOCC is strictly less than that achieved with global measurements [4, 6–8]. This gives rise to a new kind of nonlocality [4, 12, 27], conceptually different from that captured through the violation of Bell inequalities [21, 22]. Thus the problem of local state discrimination and the phenomenon of nonlocality serve to explore fundamental questions related to local access of global information [3, 10, 11], and the relationship between entanglement and local distinguishability [4, 12, 16**?**]. Moreover, it has found novel applications such as quantum/classical data hiding [29–31] and secret sharing [32].

There are many celebrated results identifying sets of states for which perfect local discrimination is possible and sets for which it is not. In particular: any two pure states can be optimally distinguished with LOCC [1, 36] but no more than d maximally entangled states on $\mathbb{C}^d \otimes \mathbb{C}^d$ can be [13, 14]; a complete basis of a composite space which can be distinguished with separable measurements must be a product basis but this condition is not sufficient in general [4, 6, 12]; and sometimes increasing the average entanglement in a set can enable state discrimination [12]. More recent studies include distinguishing states (pure or mixed) when many copies are provided [26–28, 34, 35].

The class of LOCC measurements does not have a simple mathematical characterization, and optimization is often analytically intractable. In this paper, we will focus on the class of separable measurements – those which are free of entanglement between the component systems. These comprise a strict superset of LOCC measurements and are much more amenable to analytic results (as in [18, 23]). It should be noted however that while every LOCC protocol can be realized by a rank one separable measurement, the converse is known not to be true [4, 5].

The focus of this work is in quantifying imperfect local discrimination, a question which has been settled in the case of a pair non-orthogonal pure states [36] but has generally not been explored as deeply. In [13] bounds on the error probability in distinguishing bipartite orthogonal states were obtained, and in [14] upper bounds on the maximum probability of perfect local discrimination were derived for special sets of maximally entangled states. In a different approach, a complementary relation between locally accessible information and final average entanglement was observed [10, 11] which provides upper bounds on the locally accessible information and are known to be optimal for some classes of states. Other approaches used measurements with positive partial transpose [14, 28]; the set of such measurements contains the separable ones as a strict subset.

We will use two measures of distinguishability, the average fidelity and the success probability. The notion of *average fidelity*, first considered by Fuchs and Sasaki in the theory of the so called "quantumness of Hilbert space" [24, 25] and later by Navascués in the problem of state estimation and separability [23] can be understood as follows: Suppose a state $|\psi_i\rangle$ is drawn with some probability p_i from a known collection of states $\{p_i, |\psi_i\rangle\}$, and the goal is perform a measurement to maximize our knowledge of the input state. The average fidelity is defined as the expected value of the overlap between the input state and the "best-guess" state that we prepare following the measurement outcome. In our restricted problem, the objective is to maximize the average fidelity over all separable measurements, yielding the *separable fidelity* [23]. We derive lower and upper bounds on the separable fidelity and provide examples in bipartite and multipartite settings where the bounds are shown to be optimal. This is shown by an explicit local strategy for each example. These general bounds are useful, as explicit expressions for fidelity and success probability are hard to find even in specific cases.

The second figure of merit that we consider is the probability of identifying the state which was prepared. Note that, while the fidelity is truly a quantum quantity, the probability of success is a classical measure of how well a quantum protocol encodes and decodes classical information. We show that, when the states are mutually orthogonal, the separable fidelity coincides with the maximum success probability, which relates our results to bounds obtained in [13]. We point out that this equality between separable fidelity and probability of success depends crucially on the orthogonality of the states. To illustrate this, we present an example where two sets (one consisting of orthogonal states, and the other non-orthogonal states) are shown to have the same separable fidelity even though the success probabilities are different.

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