Continuous Variable Entropic Uncertainty Relations in the Presence of Quantum Memory

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Abstract. We generalize entropic uncertainty relations in the presence of quantum memory [Nature Physics 6 (659), 2010], and [Physical Review Letters 106 (110506), 2011] in two aspects. First, we consider measurements with a continuum of outcomes, and, second, we allow for infinite-dimensional quantum memory. To achieve this, we introduce conditional differential entropies for classical-quantum states on von Neumann algebras, and show approximation properties for these entropies. As an illustration, we evaluate the uncertainty relations for position-momentum measurements, which has applications in continuous variable quantum cryptography and quantum information theory.

Keywords: Quantum Information, Uncertainty Relations, Continuous Variables

1 Introduction

Uncertainty relations play a fundamental role in quantum mechanics and quantum information theory. For the latter purpose entropies are commonly used to quantify the statistical uncertainties of the measurement outcomes. For finite-dimensional observables Maassen and Uffink [6] proved the sharp inequality

$$H(X) + H(Y) \geq \log \frac{1}{c} , \tag{1}$$

where $H(X)$ and $H(Y)$ are the Shannon entropies of the outcome distributions of non-degenerate measurements X and Y and $c = \max_{i,j} |\sup x_i y_j|^2$ with $|x_i\rangle$ and $|y_j\rangle$ the eigenvectors of X and Y, respectively. The inequality was further generalized to observables described by positive operator valued measures, and to different entropies.

It was recently realized that uncertainty should not be treated as absolute, but with respect to the knowledge of an observer [1]. In particular, if the observer has a quantum memory at hand, one obtains a subtle interplay between the observed uncertainty, and the entanglement between the measured system and the quantum memory. This can be quantified by an entropic uncertainty relation stated by conditional entropies [1]. Because of the monogamy property of entanglement, the tripartite scenario allows a particularly elegant formulation: it holds for any tripartite quantum state ρ_{ABC} and measurements as above that

$$H(X|B) + H(Y|C) \geq \log \frac{1}{c} , \tag{2}$$

where $H(X|B)$ ($H(Y|C)$) is the conditional von Neumann entropy of the measurement outcome X (Y) conditioned on B (C).[1] Note that the constant c is the same as in inequality (1) and the tripartite formulation can be seen as a further generalization of the one shown by Massen and Uffink.

The entropic uncertainty relations with quantum side information have various applications in quantum information theory. Most prominently, the tripartite version in (2) can be used as a straightforward tool to prove security against arbitrary (coherent) attacks of certain quantum key distribution protocol [1, 8]. For that purpose, the uncertainty relation has been extended to the smooth min- and max-entropies [8], which quantify the extractable key length in the one-shot scenario.

The above mentioned results including quantum memory are restricted to quantum systems with finitely many degrees of freedom. A first generalization of the tripartite uncertainty relation to infinite-dimensional quantum systems has been derived for the smooth min- and max-entropy by some of the authors in [2]. While the quantum side-information could be arbitrary, only measurements withe a finite-number of outcomes were considered. Based on this uncertainty relation, the first quantitative security analysis of a continuous variable quantum key distribution proto-

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[1] More precisely, $H(X|B)$ is the von Neumann entropy of the post-measurement state $\rho_{XB} = \sum_i (|x_i\rangle\langle x_i|_A \otimes \mathbb{1}_B)\, \rho_{AB}\, (|x_i\rangle\langle x_i|_A \otimes \mathbb{1}_B)$.

col against arbitrary attacks has been presented in [4].

The extension of (2) to infinite number or continuous outcomes was recently also addressed in [5], which also apply to continuous position-momentum measurements. Yet, they only consider a restricted definition of the conditional von Neumann entropy.

2 Results.

In this work, we present tripartite entropic uncertainty relations with quantum side-information for infinite-dimensional quantum systems without restrictions on the observables and the quantum side information. The uncertainty relations are derived for the von Neumann entropy as well as the min- and max-entropy. For that purpose, we introduce differential conditional von Neumann entropy $h(X|B)$ and differential conditional min- and max-entropy, $h_{\min}(X|B)$ and $h_{\max}(X|B)$, for classical systems X described by a measure space and quantum side-information B modeled by an observable algebra.

Intuitively, continuous classical systems may be thought of as being approximated by discrete systems in the limit of infinite precision. Hence, we expect that operational quantities have similar behavior. We make this precise by proving that the differential conditional entropies $h(X|B)$, $h_{\min}(X|B)$ and $h_{\max}(X|B$ can be approximated by their discretized counterparts. In particular, if X is a classical system with outcome range being the real line, and X_δ its restriction to a partition of $\mathbb{R}$ into intervals of length δ, then

$$h(X|B) = \lim_{\delta\to 0} \big(H(X_\delta|B) + \log\delta \big) \tag{3}$$

if the differential entropy $h(X|B)$ is finite. A similar result is derived for the differential min- and max-entropies $h_{\min}(X|B)$ and $h_{\max}(X|B)$.

The tripartite uncertainty relation for measurements with a continuous outcome range are then derived by means of these approximation results from the discrete outcome case. The relations hold in the general case where the measurements are general positive operator valued measures. The inequality reads exactly like in equation (2) except that the conditional entropies are exchanged by their differential versions and the overlap is computed by a limit along finer and finer partitions of the continuous outcome range.

We analyzed the uncertainty relations for position and momentum observables in the case of finite and infinite precision measurements. In the case of finite precision modeled by a binning of the outcome range into intervals of length δ, the overlap only depends on δ and can be explicitly determined [7]. The behavior for small δ is given by $c(\delta) \approx \delta^2\backslash(2\pi)$. We then show that the obtained uncertainty relation in terms of the min- and max-entropy is sharp even without quantum memory. In particular, a pure state which has only support on one interval of the measurement for which the max-entropy is evaluated achieves equality. The sharpness question is more subtle in the case of the von Neumann entropy. There we can only show that in the case of trivial side information equality cannot be achieved for small δ.

In the case of infinite precision measurements, and thus, continuous outcomes the constant can be obtained by taking the limit for $\delta \to 0$. The resulting inequality reads as

$$h(Q|B)_\omega + h(P|C)_\omega \geq \log 2\pi,$$

which generalizes previous results without quantum memory (see e.g. [3]). The uncertainty relation is again sharp without quantum memory for the min- and max-entropy. In the case of the von Neumann entropy the inequality is saturated for the EPR state with infinite squeezing.

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