ZETA FUNCTIONS ON INFINITE EXTENSIONS

ANUP B. DIXIT

Abstract. For an infinite extension $\mathcal{K}/\mathbb{Q}$, various attempts have been made to define an appropriate zeta-function, which encapsulates vital arithmetic information over $\mathcal{K}$, such as the prime splitting behaviour. In this article, we discuss a few such zeta-functions. We explore implication of a hypothesis by Kumar Murty in relation to a problem of Malle and Roberts. Additionally, we address a gap in the proof of "basic inequality" due to Tsfasman-Vlăduţ in the theory of asymptotically exact families. Finally, we introduce a new zeta-function associated to $\mathcal{K}/\mathbb{Q}$ and highlight its connections to several important themes in number theory.

1. Introduction

For a finite extension $K/\mathbb{Q}$, the Dedekind zeta-function is defined on $\Re(s) > 1$ as

$$\zeta_K(s) := \sum_{0\neq\mathfrak{a}\subset\mathcal{O}_K} \frac{1}{N\mathfrak{a}^s},$$

where $\mathfrak{a}$ runs over all non-zero integral ideals of K. This series is absolutely convergent on the half plane $\Re(s) > 1$. The unique factorization of fractional ideals of K into prime ideals can be reformulated as the Euler product for $\zeta_K(s)$ on $\Re(s) > 1$ as

$$\zeta_K(s) = \prod_{\mathfrak{p}\subset\mathcal{O}_K} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1},$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of $\mathcal{O}_K$. The function $\zeta_K(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$. It encodes important arithmetic properties over K. For instance, Landau's prime ideal theorem [12], which states that

$$\#\left\{\mathfrak{p} \subseteq \mathcal{O}_K \,:\, N\mathfrak{p} \leq x\right\} \sim li(x) := \int_2^x \frac{dt}{\log t}$$

is a consequence of the fact that $\zeta_K(s)$ has a simple pole at $s = 1$ and $\zeta_K(1 + it) \neq 0$ for all $t \in \mathbb{R}^*$. Thus, remarkably the analytic study of $\zeta_K(s)$ helps unveil profound arithmetic data, especially about the distribution of prime ideals. One of the key features of $\zeta_K(s)$, owing to its Euler product, is that it fully captures the splitting behavior of primes in the extension $K/\mathbb{Q}$. In fact, the simple pole of $\zeta_K(s)$ at $s = 1$ immediately establishes that the set of primes that split completely in $K/\mathbb{Q}$ has Dirichlet density $1/[K : \mathbb{Q}]$.

For infinite extension $\mathcal{K}/\mathbb{Q}$, the analogous theory is fundamentally quite different. In this article, we explore this theme and introduce zeta-functions associated to infinite extensions over $\mathbb{Q}$. Any infinite extension $\mathcal{K}/\mathbb{Q}$ can be realized as

$$\mathcal{K} = \ldots \supset K_n \supset K_{n-1} \supset \ldots \supset K_1 = \mathbb{Q}$$

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a tower of number fields such that $\mathcal{K} = \bigcup_n K_n$. Note that $\lim_{n\to\infty} \zeta_{K_n}$ does not exist as a function on $\mathbb{C}\backslash\{1\}$. Indeed, the order of zero of $\zeta_K(s)$ at $s = 0$ is the rank of the unit group, which tends to infinity as the degree of K_n tends to infinity. Thus, if the limit $\lim_{n\to\infty} \zeta_{K_n}(s)$ existed, it would be identically 0. On the other hand, from the Dirichlet series representation, we have $\zeta_{K_n}(2) > 1$ for all n. So, the naive approach of taking limits fails to define a meaningful function. This has inspired numerous attempts in the literature to define a zeta-function for infinite extensions. In this article, we shall explore this theme in detail.

2. The Frobenius class

Let $K/\mathbb{Q}$ be a Galois extension with Galois group G of order n_K. The group G naturally acts on the prime ideals of K. For a rational prime p, suppose its factorization is given by

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r},$$

where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r \subset \mathcal{O}_K$ are the prime ideals in $\mathcal{O}_K$ lying above p. Let $F_{p^{f_i}} = \mathcal{O}_K/\mathfrak{p}_i$ denote the residue field corresponding to $\mathfrak{p}_i$. We call e_i and f_i the ramification and inertia degrees of $\mathfrak{p}_i$ over p. Since the action of G preserves $p\mathcal{O}_K$ and acts transitively on $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, we have uniform ramification and inertia degrees, i.e., $e_1 = e_2 = \cdots = e_r$ and $f_1 = f_2 = \cdots = f_r$. Consequently, $n_K = efr$.

The stabilizer of $\mathfrak{p}_i$ in G is called its *decomposition group*

$$D_{\mathfrak{p}_i/p} = \{\sigma \in G : \sigma(\mathfrak{p}_i) = \mathfrak{p}_i\}.$$

Since G acts transitively on $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$, on varying $\mathfrak{p}_i$, the subgroups $D_{\mathfrak{p}_i/p}$ are conjugates of each other.

Fixing a prime ideal $\mathfrak{p} = \mathfrak{p}_i$, any automorphism $\sigma \in D_{\mathfrak{p}/p}$ induces an automorphism on the residue field $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_q$ with $q = p^f$. Since σ acts trivially on $\mathbb{Z}/p\mathbb{Z} \simeq \mathbb{F}_p$, this gives rise to a surjective homomorphism

$$D_{\mathfrak{p}/p} \twoheadrightarrow \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p).$$

The kernel of this map, known as the *inertia group* $I_{\mathfrak{p}/p}$, consists of elements in $D_{\mathfrak{p}/p}$ that act trivially on the residue field. This leads to the short exact sequence

$$0 \to I_{\mathfrak{p}/p} \to D_{\mathfrak{p}/p} \to \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \to 0,$$

with $|I_{\mathfrak{p}/p}| = e$ and $|\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)| = f$.

When the inertia group $I_{\mathfrak{p}/p}$ is trivial, the prime p is said to be unramified, and in this case $D_{\mathfrak{p}/p} \cong \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. Since $\mathbb{F}_q/\mathbb{F}_p$ is a finite field extension, its Galois group is cyclic, generated by the Frobenius automorphism given by

$$Frob : x \mapsto x^p.$$

The preimage of $Frob$ under the projection $D_{\mathfrak{p}/p} \to \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is called the Frobenius element $Frob_{\mathfrak{p}/p}$. Moreover, its conjugacy class in $\mathrm{Gal}(K/\mathbb{Q})$ remains unchanged regardless of the choice of $\mathfrak{p}$ and we denote it by $Frob_p$.

If p is ramified, the definition of $Frob_{\mathfrak{p}/p}$ is determined up to multiplication by $I_{\mathfrak{p}/p}$.

Since $\mathrm{Gal}(K/\mathbb{Q})$ is finite, it contains only finitely many conjugacy classes. Meanwhile, all but finitely many rational primes are unramified in K. This naturally leads to the question: how are the infinitely many $Frob_p$ distributed among the finitely many conjugacy classes in G. The answer to this question is the famous Chebotarev density theorem.

Theorem 2.1 (Chebotarev [24]). *Let $K/\mathbb{Q}$ be a finite Galois extension with Gal$(K/\mathbb{Q}) = G$. Let $\mathfrak{C}$ be a conjugacy class of G. Then, the set*

$$\left\{p \text{ prime} : p \text{ is unramified in } K \text{ and } Frob_p = \mathfrak{C}\right\}$$

has natural density $\frac{|\mathfrak{C}|}{|G|}$.

More precisely, suppose $\pi(x)$ denotes the number of rational primes $\leq x$ and $\pi_{\mathfrak{C}}(x, K/\mathbb{Q})$ counts the primes $p \leq x$ for which $Frob_p = \mathfrak{C}$, then the Chebotarev density theorem asserts that

$$\frac{\pi_{\mathfrak{C}}(x, K/\mathbb{Q})}{\pi(x)} \sim \frac{|\mathfrak{C}|}{|G|} \quad \text{as} \quad x \to \infty.$$

In other words, the Frobenius classes are equidistributed with respect to the uniform discrete measure on G. Let $\mathfrak{C}$ be the disjoint union of conjugacy classes $\mathfrak{C}_i$, then we call $\mathfrak{C}$ as a conjugacy set. Clearly

$$\pi_{\mathfrak{C}}(x, K/\mathbb{Q}) = \sum_i \pi_{\mathfrak{C}_i}(x, K/\mathbb{Q}).$$

When $G = \text{Gal}(K/\mathbb{Q})$ is abelian, the Frobenius classes $Frob_p$'s are singletons. A key example of an abelian extension is the cyclotomic field $K = \mathbb{Q}(\zeta_n)$, where ζ_n is the primitive n-th root of unity. This extension is abelian because any automorphism $\sigma \in \text{Gal}(K/\mathbb{Q})$ is uniquely determined by its action on ζ_n, which must also be mapped to another primitive n-th root of unity. Specifically, if $\sigma(\zeta_n) = \zeta_n^r$. Then, the map $\sigma \mapsto r(\sigma)$ defines an isomorphism

$$\text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*.$$

For a prime $p \nmid n$, p is unramified in $\mathbb{Q}(\zeta_n)$. In this case, the Frobenius element is defined as

$$Frob_p(\zeta_n) = \zeta_n^p.$$

Thus, for any r coprime to n, we obtain the equivalence

$$p \nmid n,\ Frob_p = r \in (\mathbb{Z}/n\mathbb{Z})^* \Leftrightarrow p \equiv r \bmod n.$$

Thus, Chebotarev density theorem for $\mathbb{Q}(\zeta_n)$ recovers the classical Dirichlet's theorem for primes in arithmetic progressions. Hence, for general Galois extensions, Chebotarev can be viewed as a generalization of Dirichlet's theorem.

Let $K/\mathbb{Q}$ be a Galois extension of degree n. A prime p splits completely in K,

$$p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$$

if and only if the decomposition group $D_{\mathfrak{p}/p} = \{id\}$. Equivalently, the Frobenius class $Frob_p = \{id\}$. Thus, Chebotarev density theorem implies that the set of primes which split completely in K has natural density $1/n$ and therefore Dirichlet density $1/n$.

The Dirichlet density of split primes can be deduced rather easily from the fact that $\zeta_K(s)$ has a simple pole at $s = 1$. Suppose S_K is the set of primes which split completely in K. Then, its Dirichlet density is given by

$$\delta(S_K) := \lim_{s\to 1^+} \frac{\sum_{p\in S_K} \frac{1}{p^s}}{\sum_p \frac{1}{p^s}} = \lim_{s\to 1^+} \frac{\sum_{p\in S_K} \frac{1}{p^s}}{\log \frac{1}{s-1}}.$$

Using the Euler product for $\zeta_K(s)$, for $\Re(s) > 1$, we obtain

$$\log \zeta_K(s) = -\sum_{\mathfrak{p}\subset\mathcal{O}_K} \log\left(1 - \frac{1}{N\mathfrak{p}^s}\right) = n\sum_{p\in S_K} \frac{1}{p^s} + O\left(\sum_{p\notin S_K} \frac{1}{p^{fs}}\right) = n\sum_{p\in S_K}\frac{1}{p^s} + O(1).$$

Since both $\zeta_K(s)$ and $\zeta(s)$ have simple poles at $s = 1$, we have

$$\lim_{s\to 1^+} \frac{\log \zeta_K(s)}{\log \zeta(s)} = 1.$$

Thus the Dirichlet density of S_K is $1/n$. It is important to emphasize that the Dirichlet density is determined by the behaviour of $\zeta_K(s)$ at $s \to 1^+$, whereas for natural density one needs to understand $\zeta_K(s)$ in the larger region near the line $\Re(s) = 1$.

3. **Zeta function for conjugacy sets**

It is often advantageous to work with the logarithmic derivative of the zeta-function. From the Euler product of the Dedekind zeta-function, one can write for $\Re(s) > 1$

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{q=p^k} \mathcal{N}_q(K)\,\frac{\log q}{q^s - 1} = \sum_{q=p^k}\sum_{m=1}^{\infty} \mathcal{N}_q(K)\,\frac{\log q}{q^{ms}},$$

where $\mathcal{N}_q(K)$ denotes the number of prime ideals in K with norm q. When $K/\mathbb{Q}$ is Galois of degree n,

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{p \text{ prime}}\sum_{m=1}^{\infty} \left(\frac{n}{ef}\right)\frac{\log q}{q^{ms}} = \sum_{p,m}\left(\frac{n}{e}\right)\frac{\log p}{p^{fms}}.$$

Since only finitely many primes ramify in K, we have

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{p\,\text{splits}} \frac{n\log p}{p^s} + O(1)$$

for $\Re(s) > 1$. This is similar to the statement that the Dirichlet density of split primes is $1/n$.

Let $G = \mathrm{Gal}(K/\mathbb{Q})$ and $\mathfrak{C}$ be a conjugacy set, i.e., union of conjugacy classes in G. For a prime p, which is unramified in K, let $\sigma = Frob_{\mathfrak{p}/p} \in \mathrm{Gal}(K/\mathbb{Q})$ denote the Frobenius element. Let $Frob_p$, also denoted by the Artin symbol $\langle p, K/\mathbb{Q}\rangle$ denote the conjugacy class containing $Frob_{\mathfrak{p}/p}$. Note that the conjugacy class of σ^n is also independent of $\mathfrak{p}$ and hence we define $\langle p^n, K/\mathbb{Q}\rangle$ to be the conjugacy class containing σ^n. Define the zeta function corresponding to $\mathfrak{C}$ on $\Re(s) > 1$ as

$$Z(K,\mathfrak{C},s) := \sum_{\substack{p \text{ prime}, m\geq 1\\ \langle p^m, K/\mathbb{Q}\rangle \subset \mathfrak{C}}} \frac{\log p}{(p^f)^{ms}}. \tag{1}$$

In particular, if $\mathfrak{C} = \{1\}$, then $Frob_p = \mathfrak{C}$ if and only if p splits completely in K. In this case,

$$Z(K,\mathfrak{C},s) = \sum_{p\,\text{split}}\sum_{m=1}^{\infty}\frac{\log p}{p^{ms}} = \sum_{p \text{ split}} \frac{\log p}{p^s - 1} = -\frac{1}{n}\frac{\zeta_K'}{\zeta_K}(s) + O(1)$$

for $\Re(s) > 1$.

4. **Topology and measures on infinite extensions**

An infinite algebraic extension $\mathcal{K}/\mathbb{Q}$ can be realized as a tower of number fields, namely

$$\mathcal{K} = \ldots \supsetneq K_n \supsetneq K_{n-1} \supsetneq \ldots \supsetneq K_1 = \mathbb{Q},$$

where K_n's are number fields. This representation is not unique. Moreover, $\mathcal{K}/\mathbb{Q}$ is Galois if and only if there exists a tower of number fields $\{K_n\}$ such that $\mathcal{K} = \bigcup_n K_n$ and $K_n/\mathbb{Q}$ is a finite Galois extension for all n.

For instance $\mathcal{K} = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)/\mathbb{Q}$ is an infinite Galois extension with

$$\mathrm{Gal}(\mathcal{K}/\mathbb{Q}) \cong \prod_{i=1}^{\infty} (\mathbb{Z}/2\mathbb{Z}).$$

Another example is the infinite cyclotomic extension $\mathbb{Q}(\zeta_{p^\infty}) := \bigcup_{i=1}^{\infty} \mathbb{Q}(\zeta_{p^i})$. This is a Galois extension over $\mathbb{Q}$ with

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.

Observe that the Galois groups above $\prod_{i=1}^{\infty} (\mathbb{Z}/2\mathbb{Z})$ or $\mathbb{Z}(p-1)\mathbb{Z} \times \mathbb{Z}_p$ are uncountable as sets. In fact, the Galois group of an infinite Galois extension is always uncountable. The topology on $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, called the Krull topology, is defined with the idea that two elements $\sigma, \pi \in \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ are "close" if they agree on a large finite extension $K_n/\mathbb{Q}$ with $K_n \subset \mathcal{K}$. For $\sigma \in \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, the basic open sets are given by the cosets $\sigma\,\mathrm{Gal}(\mathcal{K}/K_n)$, where $\mathcal{K} = \bigcup_n K_n$ and $K_n/\mathbb{Q}$ are finite Galois extensions. With this topology, $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ forms a compact, totally disconnected and Hausdorff topological group.

Let μ be the Haar measure on $G = \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$, with $\mu(G) = 1$. Then

$$\mu = \lim_n \mu_{K_n},$$

where μ_{K_n} is the discrete probability measure on $\mathrm{Gal}(K_n/\mathbb{Q})$. This limit is interpreted in the sense of weak convergence.

One can think of a prime in $\mathcal{K}$ as a system

$$\{\mathfrak{p}_n : K_n/\mathbb{Q} \text{ is finite Galois}\}$$

such that $\mathfrak{p}_n$ is a prime ideal in K_n and for $m > n$, the ideal $\mathfrak{p}_m$ lies over $\mathfrak{p}_n$. Now, the Frobenius element associated with a prime in $\mathcal{K}$ is defined as a sequence

$$\left\{ Frob_{\mathfrak{p}_n} : K_n/\mathbb{Q} \text{ is finite Galois} \right\}.$$

The conjugacy class of all sequences with $\mathfrak{p}_1 = p$ is denoted by $Frob_p$. With respect to the Krull topology, these Frobenius elements are dense in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$. But note that $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ is uncountable, whereas the number of Frobenius classes are countable. Thus, it is possible to have elements in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ which are not Frobenius elements. For example, if $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ is abelian, then its conjugacy classes are singletons $\{\sigma\} \in \mathcal{K}/\mathbb{Q}$ and hence there must be elements, which are not Frobenius elements.

Let $\mathfrak{C}$ be a conjugacy class in the infintie Galois group $G = Gal(\mathcal{K}/\mathbb{Q})$. Let $\pi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q})$ be the number of primes $p \leq x$, unramified in $\mathcal{K}$, such that the Frobenius class $Frob_p$ lies in $\mathfrak{C}$. Serre [21] showed that if the boundary of $\mathfrak{C}$ has measure zero, then

$$\pi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q}) \sim \mu(\mathfrak{C})\, li(x)$$

as $x \to \infty$. In other words, the Chebotarev density theorem holds for conjugacy classes with positive Haar measure, i.e., having positive density in G. This raises the question of whether we can predict asymptotics when $\mu(\mathfrak{C}) = 0$, but $\mathfrak{C}$ is "sufficiently large". To quantify this, we recall the notion of Minkowski dimension following Serre [21].

Let $\mathcal{K}/\mathbb{Q}$ be an infinite Galois extension with $G = \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$. Let $\mathcal{K} = \bigcup_n K_n$ be a tower, where $K_n/\mathbb{Q}$ is Galois with Galois group G_n for each n. Let $\mathfrak{C} \subset G$ be a conjugacy set and $\mathfrak{C}_n$ be the projection of $\mathfrak{C}$ on G_n. We say that the Minkowski dimension $\dim_{\mathcal{K}}(\mathfrak{C}) \leq \alpha$ if for each $K_n/\mathbb{Q}$,

$$|\mathfrak{C}_n| \ll_{\mathcal{K}} |G_n|^{\alpha}.$$

If $\alpha < 1$, then $\mu(\mathfrak{C}) = 0$. A natural problem is to estimate $\pi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q})$ when $\dim_{\mathcal{K}}(\mathfrak{C}) = \alpha > 0$. One way to think about this problem is to associate a zeta function on the conjugacy set $\mathfrak{C}$ in the infinite group G, analogous to (1), as introduced by Kumar Murty in [16]. Write

$$Z(\mathfrak{C}, \mathcal{K}, s) := \sum_{\langle p^m, \mathcal{K}/\mathbb{Q}\rangle \subset \mathfrak{C}} \frac{\log p}{(p^f)^{ms}}, \tag{2}$$

where p runs over all rational primes. It is possible to develop the theory of the above zeta-function for $\mathcal{K}/L$, infinite Galois extension over any number field L. For simplicity, we restrict ourselves to $\mathcal{K}/\mathbb{Q}$.

In [16], Kumar Murty showed that for $\mathcal{K}/\mathbb{Q}$, if a conjugacy set $\mathfrak{C} \subset \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ has Minkowski dimension $\alpha > 0$, then under GRH, the Dirichlet series $Z(\mathcal{K}, \mathfrak{C}, s)$ converges for $\Re(s) > \frac{1+\alpha}{2}$. Additionally, assuming Artin's holomorphy conjecture, the region of convergence extends to $\Re(s) > \frac{1}{2} + \frac{\alpha}{2(2+\alpha)}$. Based on these observations along with heuristics from Lang-Trotter conjecture, Kumar Murty [16] hypothesized that $Z(\mathcal{K}, \mathfrak{C}, s)$ must have analytic continuation to $\Re(s) > 1/2$. Since its Dirichlet coefficients are positive real numbers, by Landau's theorem, it has a singularity at the abscissa of convergence.

Hypothesis 1 (Kumar Murty)**.** *Let $\mathcal{K}/\mathbb{Q}$ be an infinite Galois extension and $\mathfrak{C}$ be a conjugacy set of its Galois group with* $\dim_{\mathcal{K}} \mathfrak{C} = \alpha > 0$. *Then, the Dirichlet series $Z(\mathcal{K}, \mathfrak{C}, s)$ is convergent on $\Re(s) > 1/2$ and has analytic continuation to $\Re(s) \geq 1/2$ with a simple pole at $s = 1/2$.*

One of the goals of this article is to discuss ramifications of this hypothesis. Towards this, we first remind the reader of various effective versions of the Chebotarev density theorem.

5. **Effective Chebotarev density theorem**

Let L/K be a finite Galois extension with Galois group G and $\mathfrak{C}$ be a union of conjugacy classes in G. Define $\pi_{\mathfrak{C}}(x, L/K)$ as the number of prime ideals $\mathfrak{p}$ of K, unramified in L with $N\mathfrak{p} \leq x$, such that the Frobenius class $Frob_{\mathfrak{p}}$ lies in $\mathfrak{C}$. The Chebotarev density theorem asserts that

$$\pi_{\mathfrak{C}}(x, L/K) \sim \frac{|\mathfrak{C}|}{|G|}\, li(x)$$

as $x \to \infty$. In 1977, Lagarias and Odlyzko [11] established an effective version of this result, later refined by Serre [22]. Under the assumption of GRH, they proved that

$$\left| \pi_{\mathfrak{C}}(x, L/K) - \frac{|\mathfrak{C}|}{|G|} \, li(x) \right| \ll \frac{|\mathfrak{C}|}{|G|} \, x^{1/2} \left(\log d_K + n_K \log x \right), \tag{3}$$

where d_K is the absolute discriminant $|disc(K/\mathbb{Q})|$ and n_K is the degree $[K : \mathbb{Q}]$. Additionally, assuming Artin's holomorphy conjecture(AC), Ram Murty, Kumar Murty and Saradha [15] established a stronger version, namely,

$$\left| \pi_{\mathfrak{C}}(x, L/K) - \frac{|\mathfrak{C}|}{|G|} \, li(x) \right| \ll |\mathfrak{C}|^{1/2} \, x^{1/2} \, n_K \left(\log \mathcal{M}_{L/K} \, x \right), \tag{4}$$

where

$$\mathcal{M}_{L/K} := [L : K] \, d_K^{1/n_K} \prod_{p \in P(L/K)} p.$$

Here $P(L/K)$ denotes the set of rational primes p such that there is a prime ideal $\mathfrak{p}$ in K above p which ramifies in L. An even sharper bound was recently established by Ram Murty, Kumar Murty and P. J. Wong in [17], assuming GRH, AC and the pair correlation conjecture (PCC) for Artin L-functions. Under these assumptions, they showed that

$$\left| \pi_{\mathfrak{C}}(x, L/K) - \frac{|\mathfrak{C}|}{|G|} \, li(x) \right| \ll |\mathfrak{C}|^{1/2} \, x^{1/2} \left(\frac{|G^{\#}|}{|G|} \right)^{1/2} n_K^{1/2} \left(\log \mathcal{M}_{L/K} \, x \right), \tag{5}$$

where $G^{\#}$ is the set of all conjugacy classes of G.

In [16], Kumar Murty used the bounds (3) and (4) to obtain convergence of the zeta function $Z(\mathcal{K}, \mathfrak{C}, s)$ in the regions $\Re(s) > (1+\alpha)/2$ (under GRH) and $\Re(s) > \frac{1}{2} + \frac{\alpha}{2(2+\alpha)}$ (under GRH and AC). However, on employing (5), one can obtain better regions of convergence under GRH, AC and PCC, at least in specific cases. We demonstrate this with an explicit example. For small conjugacy classes, the bound in (5) is comparable to (4). On the other hand, if the conjugacy classes are larger, then the number of conjugacy classes is smaller and the bound in (5) is significantly more effective. This opens the possibility of constructing infinite extensions with interesting properties.

Example. Consider the infinite Galois extension $\mathcal{K}/\mathbb{Q}$ given by a tower $\mathcal{K} = \bigcup_n K_n$ with Galois group G, constructed as follows. We begin with $K_1 = \mathbb{Q}$ and take K_2/K_1 to be a quadratic extension. Next, let $L_3/\mathbb{Q}$ be a Galois extension with Galois group S_3, chosen so that L_3 and K_2 are linearly disjoint over $\mathbb{Q}$, i.e., $L_3 \cap K_2 = \mathbb{Q}$. Define $K_3 := K_2 \cdot L_3$, their compositum. Inductively, we construct $L_n/\mathbb{Q}$ a Galois extension with Galois group S_n such that $L_n \cap K_{n-1} = \mathbb{Q}$. Such extensions can always be constructed. Defining $K_n = K_{n-1} \cdot L_n$, we obtain the Galois group

$$\mathrm{Gal}(K_n/\mathbb{Q}) = G_n \cong S_n \times S_{n-1} \times \ldots \times S_2$$

and $G = \varprojlim G_n$ is the inverse limit.

It is easy to compute the number of conjugacy classes of G_n. Indeed, the number of conjugacy classes of S_n is precisely given by the number of partitions of n, denoted by p_n. By the famous Hardy-Ramanujan [8] asymptotic formula for p_n, we have

$$|S_n^{\#}| \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}} \right)$$

as $n \to \infty$. Therefore, the number of conjugacy classes of G_n is given by

$$|G_n^{\#}| = |S_n^{\#}| \times |S_{n-1}^{\#}| \times \cdots \times |S_2^{\#}| \sim \prod_{l=2}^{n} \frac{1}{4l\sqrt{3}} \exp\left(\pi\sqrt{\frac{2l}{3}}\right)$$

as $n \to \infty$. Hence,

$$\frac{|G_n^{\#}|}{|G_n|} \ll \prod_{l=2}^{n} \frac{e^{\beta\sqrt{l}}}{l \cdot l!},$$

where $\beta = \pi\sqrt{2/3} > 0$ is an absolute constant. Let $\mathfrak{C} \subset G$ be a conjugacy set with Minkowski dimension $\alpha > 0$, meaning that its projection $\mathfrak{C}_n$ onto G_n satisfies $|\mathfrak{C}_n| \ll |G_n|^{\alpha}$. Applying (5), we deduce that for any $x > 2$

$$\begin{aligned} \left|\pi_{\mathfrak{C}_n}(x, K_n/\mathbb{Q}) - \frac{|\mathfrak{C}_n|}{|G_n|} li(x)\right| &\ll |\mathfrak{C}_n|^{1/2} x^{1/2} \left(\frac{|G_n^{\#}|}{|G_n|}\right)^{1/2} \left(\log \mathcal{M}_{K_n/\mathbb{Q}} x\right) \\ &\ll |\mathfrak{C}_n|^{1/2} x^{1/2} \left(\prod_{l=1}^{n} \frac{e^{\beta\sqrt{l}}}{l \cdot l!}\right)^{1/2} \left(\log \mathcal{M}_{K_n/\mathbb{Q}} x\right). \end{aligned} \tag{6}$$

Denote by

$$\psi_{\mathfrak{C}}(x, K/\mathbb{Q}) := \sum_{\substack{p^m < x \\ (p^m, K/\mathbb{Q}) \subset \mathfrak{C}}} \log p.$$

Using partial summation, (6) gives

$$\psi_{\mathfrak{C}_n}(x, K_n/\mathbb{Q}) = \frac{|\mathfrak{C}_n|}{|G_n|} x + O\left(|\mathfrak{C}_n|^{1/2} x^{1/2} \left(\prod_{l=1}^{n} \frac{e^{\beta\sqrt{l}}}{l \cdot l!}\right)^{1/2} \left(\log \mathcal{M}_{K_n/\mathbb{Q}} x\right) \log x\right).$$

Suppose there exists a finite set of primes S such that $\mathcal{K}/\mathbb{Q}$ is unramified over all rational primes outside S. Then,

$$\log \mathcal{M}_{K_n/\mathbb{Q}} = \log n_{K_n} + \sum_{p \in S} \log p = \log n_{K_n} + O(1).$$

Choosing

$$x = x_n \sim |G_n|^{1-\alpha}$$

and using the facts that $|G_n| = \prod_{l \le n} l!$ and $\prod_{l \le n} e^{\beta\sqrt{l}} \ll e^{\beta n^{3/2}} \ll \log x_n$, we deduce that

$$\psi_{\mathfrak{C}_n}(x, K_n/\mathbb{Q}) \ll (\log x_n)^3,$$

where the implied constants are absolute and independent of K_n. By definition,

$$\psi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q}) \le \psi_{\mathfrak{C}_n}(x, K_n/\mathbb{Q})$$

for all $x > 1$. Since $x_n \to \infty$ as $n \to \infty$, we get that $\psi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q}) \ll (\log x)^3$ for sufficiently large x. Consequently the integral

$$\int_1^{\infty} \frac{\psi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q})}{x^{s+1}} \, dx$$

converges for $\Re(s) > 0$. Since

$$Z(\mathfrak{C}, \mathcal{K}, s) = s \int_1^{\infty} \frac{\psi_{\mathfrak{C}}(x, \mathcal{K}/\mathbb{Q})}{x^{s+1}} \, dx,$$

we conclude that $Z(\mathcal{K}, \mathfrak{C}, s)$ is absolutely convergent for $\Re(s) > 0$.

This is partially compatible and also partially contradictory to Hypothesis 1. On the one hand, by convergence of the Dirichlet series, $Z(\mathfrak{C}, \mathcal{K}, s)$ has analytic continuation to $\Re(s) > 0$. On the other hand, this also implies that the associated zeta-function does not have a pole at $s = 1/2$.

Although this example is conditional on GRH, AC and PCC, it does not necessarily suggest that the Hypothesis 1 is incompatible with GRH, AC or PCC. We elaborate on this below.

In the example above, two conditions are implicitly imposed on the infinite extension $\mathcal{K}/\mathbb{Q}$. The first is that its Galois group G has a conjugacy set with positive Minkowski dimension and the second is that it is unramified over all rational primes outside a finite set S. It is easy to see that the first condition is satisfied. Indeed, consider the conjugacy class $\mathfrak{C}_n \subset S_n$ consisting of product of disjoint $[\frac{n}{2}]$ number of 2-cycles. In other words, the conjugacy class in S_n containing the element $(12)(23)\cdots((2[\frac{n}{2}]-1)(2[\frac{n}{2}]))$. Then, $|\mathfrak{C}_n| = [\frac{n}{2}]!$. Clearly, $|\mathfrak{C}_n| \leq |S_n|^{1/2}$ because

$$\binom{n}{[n/2]} = \frac{n!}{[n/2]!(n-[n/2])!} \sim \frac{n!}{([n/2]!)^2} \geq 1.$$

Furthermore, using Stirling's approximation one can verify that for any $\epsilon > 0$ and sufficiently large n,

$$(n!)^{1/2-\epsilon} \ll \left[\frac{n}{2}\right]!.$$

Taking the conjugacy set $\mathfrak{C}'_n := \mathfrak{C}_2 \times \mathfrak{C}_3 \times \cdots \times \mathfrak{C}_n$ of G_n, one can obtain a conjugacy set $\mathfrak{C}$ of G with Minkowski dimension 1/2.

The second condition which claims that all but finitely many primes are unramified in $\mathcal{K}$ is rather subtle. Recall that the inverse Galois problem asks for a number field $K/\mathbb{Q}$ with a Galois group G. This problem remains open in general. For certain Galois groups, such as S_n (or A_n), it is easy to construct such extensions over any given number field K. However, if one puts the additional condition that the extension must be unramified outside a prescribed finite set of places S, then it becomes a rather difficult question. This problem is raised by Malle and Roberts in [14].

Question of Malle-Roberts: For any positive integer $n \geq 2$ and a finite set of primes S, is it possible to always construct a Galois extension over $\mathbb{Q}$ with Galois group S_n which is unramified outside S.

Towards this, they produce number fields which are ramified only over primes 2 and 3 and have Galois group S_n for $n = 9, 10, 11, 12, 17, 18, 25, 28, 30, 33$. If such Galois extensions could be constructed for all n, it would imply that the existence of an infinite Galois extension as shown in the example above. But this would violate Hypothesis 1 because the corresponding zeta-function does not have a pole at $s = 1/2$. In other words, Hypothesis 1 has the following implication on the problem of Malle-Roberts.

Hypothesis 1 implies that for a finite set of primes S, there exists an integer $N \geq 1$ such that for any $n \geq N$, there are no Galois extensions over $\mathbb{Q}$ with Galois group S_n, which are unramified outside S.

For certain groups, one can construct extensions with a prescribed finite set of ramifying primes. For instance, if $S = \{p, l\}$ for distinct primes p and l, then it is possible to prove, using modular forms, that there are infinitely many fields with Galois group of the form $PGL_2(\mathbb{F}_{l^n})$ which are unramified outside S (see [19]).

6. **Tsfasman-Vlăduţ zeta-functions**

Another interesting zeta-function for infinite extensions arises from the work of Tsfasman and Vlăduţ in [25]. In fact, they introduce the zeta-function for any family of number fields $\{K_i\}$, with a smoothness condition on the splitting of primes. Such families are called asymptotically exact families. In particular, towers of number fields are asymptotically exact. We define it more precisely below.

For a number field K and a rational prime power q, recall that $\mathcal{N}_q(K)$ denotes the number of prime ideals in K with norm q and d_K, the absolute discriminant $|disc(K/\mathbb{Q})|$. A family of number fields $\mathcal{K} := \{K_n\}_n$ is said to be asymptotically exact if the following limits exist:

$$\phi_{\mathbb{R}}(\mathcal{K}) = \lim_{n\to\infty} \frac{r_1(K_n)}{\log d_{K_n}}, \ \ \phi_{\mathbb{C}}(\mathcal{K}) = \lim_{n\to\infty} \frac{r_2(K_n)}{\log d_{K_n}}, \ \ \phi_q(\mathcal{K}) := \frac{\mathcal{N}_q(K_n)}{\log d_{K_n}},$$

for all prime powers $q = p^k$. Here $r_1(K)$ and $r_2(K)$ denote the number of real and complex embeddings (upto conjugation) of K respectively. Call $n_K := [K : \mathbb{Q}]$. Clearly, $r_1(K) + 2r_2(K) = n_K$ and $\mathcal{N}_q(K) \leq n_K$. By Minkowski's bound, $n_K/\log\sqrt{d_K}$ is bounded above by an absolute constant ≤ 1. Thus, all the above limits $\phi_{\mathbb{R}}, \phi_{\mathbb{C}}$ and ϕ_q, if exist, are also absolutely bounded. Now, Tsfasman and Vlăduţ define the zeta-function for $\mathcal{K} = \{K_n\}_n$ on $\Re(s) > 1$ as

$$\xi_{\mathcal{K}}(s) := \prod_{q=p^k} \left(1 - \frac{1}{q^s}\right)^{-\phi_q}. \tag{7}$$

Since ϕ_q's are real numbers, we have to choose the principal branch of logarithm to define the above product. The logarithmic derivative of $\xi_{\mathcal{K}}(s)$ is given by

$$-\frac{\xi'_{\mathcal{K}}}{\xi_K}(s) = \sum_{q=p^k} \frac{\phi_q(\mathcal{K}) \log q}{q^s - 1} \tag{8}$$

for $\Re(s) > 1$.

An infinite extension $\mathcal{K}/\mathbb{Q}$ can be realized as a tower $\mathcal{K} = \bigcup_n K_n$. It is easy to see that any tower of number fields $\mathcal{K} = \bigcup_n K_n$ is an asymptotically exact family. Here, the associated zeta-function is non-trivial if $\phi_q > 0$ for some prime power $q = p^k$. Such extensions are called *asymptotically good towers*. Note that if $\phi_q > 0$, then for sufficiently large n,

$$0 < \frac{\mathcal{N}_q(K_n)}{\log\sqrt{d_{K_n}}} \leq \frac{n_{K_n}}{\log\sqrt{d_{K_n}}} = \frac{r_1(K_n) + 2r_2(K_n)}{\log\sqrt{d_{K_n}}}.$$

Thus, for asymptotically good towers, we also have $\phi_{\mathbb{R}}(\mathcal{K}) + 2\phi_{\mathbb{C}}(\mathcal{K}) > 0$. Here, the zeta-function captures the splitting of primes in the infinite extension. Note that we do not require the extensions to be Galois in this case. There is however one limitation. Even if a prime p splits completely in $\mathcal{K}/\mathbb{Q}$, the invariant ϕ_{p^k} may still be 0 for all k. Meanwhile, for special infinite extensions such as the Hilbert class field towers, where $\frac{n_{K_n}}{\log d_{K_n}}$ is a constant, this zeta-function effectively captures the splitting of primes.

Applying Weil's explicit formula, Tsfasman-Vlăduţ [25, p. 20] proved that the series in (8) is convergent on $\Re(s) \geq 1$ and under GRH, the region of convergence can be extended to $\Re(s) \geq 1/2$. However, their proof has certain inaccuracies, which can be fixed with minor modifications. We discuss the correct argument below.

Recall that for a number field $K/\mathbb{Q}$, the logarithmic derivative of the Dedekind zeta-function is defined on $\Re(s) > 1$ by

$$-\frac{\zeta'_K}{\zeta_K}(s) := \sum_{q=p^k} \mathcal{N}_q(K) \frac{\log q}{q^s - 1}.$$

This series is convergent only on $\Re(s) > 1$ and has a simple pole at $s = 1$ owing to the pole of $\zeta_K(s)$ at $s = 1$. However, when we consider the limiting zeta-function as in (7) over a tower of number fields, the pole at $s = 1$ disappears! This fact appears in [25, GRH Theorem A], stated in Proposition 6.2. But, there is a gap in the original argument, which we shall fix below.

The proof of this statement is an application of Weil's explicit formula. We use the formulation of Guinand (see [18, p 122]).

Theorem 6.1 (Guinand-Weil explicit formula)**.** *Let $F(x)$ be a differentiable, even, positive function defined on the whole real line $\mathbb{R}$ such that $F(0) = 1$ and there exist positive constants c and ϵ such that*

$$F(x), F'(x) \le c\, e^{-(1/2+\epsilon)|x|}$$

as $|x| \to \infty$. Define

$$\phi(s) := \int_{-\infty}^{\infty} F(x) e^{(s-1/2)x}\, dx.$$

Let $K/\mathbb{Q}$ be a number field. Denote by n_K and d_K the degree and absolute discriminant of K over $\mathbb{Q}$. Let r_1 and r_2 denote the number of real and complex embedding (upto conjugation) of K. Then, we have

$$\begin{aligned}\log d_K = r_1 \frac{\pi}{2} + n_K(\gamma + \log 8\pi) - n_K \int_0^{\infty} \frac{1 - F(x)}{2\sinh x/2}\, dx \\ - r_1 \int_0^{\infty} \frac{1 - F(x)}{2\cosh x/2}\, dx - 4\int_0^{\infty} F(x)\cosh x/2\, dx + \sum_{\rho}{}' \phi(\rho) \\ + 2\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} N(\mathfrak{p})^{-m/2} F(m \log N(\mathfrak{p})) \log N(\mathfrak{p}),\end{aligned}$$

where in the first sum ρ runs over all zeros of the Dedekind zeta function $\zeta_K(s)$ in the critical strip, where ρ and $\overline{\rho}$ are grouped together, $\mathfrak{p}$ runs over the non-zero prime ideals of K and $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$.

We are now ready to state and prove the result of Tsfasman-Vlăduţ.

Proposition 6.2 (GRH Theorem A,[25])**.** *For any asymptotically exact family $\mathcal{K} = \{K_n\}_n$, the zeta-function $\xi_{\mathcal{K}}(s)$ given by (7) is convergent for $\Re(s) \ge 1/2$ under the assumption of GRH.*

Proof. Assume GRH holds. For $y > 0$, taking

$$F(x) = e^{-yx^2}.$$

and applying Theorem 6.1, we obtain

$$\begin{aligned}1 = \frac{r_1(K)}{\log d_K}\frac{\pi}{2} + \frac{n_K}{\log d_K}(\gamma + \log 8\pi) - \frac{n_K}{\log d_K}\int_0^{\infty} \frac{1 - e^{-yx^2}}{2\sinh x/2}\, dx - \frac{r_1(K)}{\log d_K}\int_0^{\infty} \frac{1 - e^{-yx^2}}{2\cosh x/2}\, dx \\ - \frac{4}{\log d_K}\int_0^{\infty} e^{-yx^2}\cosh x/2\, dx + \frac{1}{\log d_K}\Re \sum_t \int_{-\infty}^{\infty} e^{itx - yx^2}\, dx \\ + \frac{2}{\log d_K}\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} N(\mathfrak{p})^{-m/2} e^{-ym^2 \log^2 N(\mathfrak{p})} \log N(\mathfrak{p}),\end{aligned}$$

where the first sum runs over all real t such that $\zeta_K(1/2 + it) = 0$. In a tower of number fields $\mathcal{K} = \bigcup_n K_n$, all the terms in the RHS above are either > 0 or tend to 0 as n tends to infinity (see [25] for details). Therefore, only considering the last term, we deduce that

$$\frac{1}{\log d_K}\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} N(\mathfrak{p})^{-m/2} e^{-ym^2 \log^2 N(\mathfrak{p})} \log N(\mathfrak{p}) \ll 1. \tag{9}$$

The partial sum of the above summation is

$$\sum_{q\le z}\frac{\mathcal{N}_q(K)}{\log d_K}\sum_{m=1}^{\infty}q^{-m/2}e^{-ym^2\log^2 q}\,\log q \le \sum_{q\le z}\frac{\mathcal{N}_q(K)}{\log d_K}\sum_{m=1}^{\infty}q^{-m/2}\,\log q = \sum_{q\le z}\frac{\mathcal{N}_q(K)}{\log d_K}\frac{\log q}{\sqrt{q}-1}. \tag{10}$$

For the lower bound, using the fact that $e^{-ym^2\log^2 q}\ge 1-ym^2\log^2 q$ and choosing $y=(\log\log d_K)^{-1}$, we get

$$\begin{aligned}\sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\sum_{m=1}^{\infty}q^{-m/2}e^{-ym^2\log^2 q}\,\log q &\ge \sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\sum_{m\le\log^{1/4}(\log d_K)}q^{-m/2}(1-ym^2\log^2 q)\,\log q\\ &\ge\left(1-\frac{(\log\log\log d_K)^2}{\sqrt{\log\log d_K}}\right)^2\sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\sum_{m\le\log^{1/4}(\log d_K)}q^{-m/2}\,\log q\\ &=\left(1-\frac{(\log\log\log d_K)^2}{\sqrt{\log\log d_K}}\right)^2\sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\frac{\log q}{\sqrt{q}-1}\left(1+O(q^{-\log^{1/4}(\log d_K)})\right).\end{aligned}$$

In other words, we have shown that

$$\begin{aligned}(1-\epsilon_K)\sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\frac{\log q}{\sqrt{q}-1}&\le\sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\sum_{m=1}^{\infty}q^{-m/2}e^{-ym^2\log^2 q}\,\log q\\ &\le\sum_{q\le\log\log d_K}\frac{\mathcal{N}_q(K)}{\log d_K}\frac{\log q}{\sqrt{q}-1},\end{aligned}$$

where $\epsilon_K\to 0$ as $d_K\to\infty$. Now, by (9), the summation in the RHS is uniformly bounded independent of K. Therefore, we can take limits and conclude that

$$\sum_{q=p^k}\frac{\phi_q(\mathcal{K})\log q}{\sqrt{q}-1}$$

is bounded. This completes the proof that the zeta function (7) is convergent on $\Re(s)\ge 1/2$ under GRH. □

The proof of the same statement due to Tsfasman-Vlăduţ [25] follows along the same lines, but has two minor issues. The upper bound as in (10) is given in terms of a infinite sum, which is divergent. The more serious issue is the interchange of limits and summation, which requires uniform convergence. Although it was shown [25, Lemma 2.3] that for any extension L/K,

$$\sum_{q\le x}\frac{\mathcal{N}_q(K)\,\log q}{\log d_K}\ge\sum_{q\le x}\frac{\mathcal{N}_q(L)\,\log q}{\log d_L},$$

it does not imply that

$$\sum_{q\le x}\frac{\mathcal{N}_q(K)}{\log d_K}\frac{\log q}{\sqrt{q}}\ge\sum_{q\le x}\frac{\mathcal{N}_q(L)}{\log d_L}\frac{\log q}{\sqrt{q}}.$$

Hence, monotonicity does not justify the convergence of the series and one has to allude to uniform convergence, as shown in our argument above.

A similar modification to Tsfasman-Vlăduţ's argument can be undertaken to show the convergence, and hence the analytic continuation of the zeta-function (7) on $\Re(s)\ge 1$ unconditionally.

Several interesting open questions arise from the study of $\xi_{\mathcal{K}}(s)$. Since the Dirichlet coefficients are all positive, by Landau's theorem, there is a singularity on the real point at its abscissa of absolute convergence. It is still not known whether one can find an infinite extension $\mathcal{K}/\mathbb{Q}$ such that the abscissa of absolute convergence of $\xi_{\mathcal{K}}(s)$ is $> -\infty$. In other words, an extension where infinitely many $\phi_q(\mathcal{K})$ are positive.

7. Zeta function over splitting primes

We now introduce another zeta-function attached to an infinite extension, which more precisely captures the splitting behaviour of primes, in a similar spirit as the Tsfasman-Vlăduţ zeta-functions. Let $\mathcal{K}/\mathbb{Q}$ be an infinite extension given by a tower $\mathcal{K} = \bigcup_{n=1}^{\infty} K_n$. For all prime powers $q = p^k$, we define the invariants

$$\psi_q(\mathcal{K}) := \lim_{n\to\infty} \frac{\mathcal{N}_q(K_n)}{n_{K_n}}.$$

This limit exists and is well-defined, meaning it is independent of the chioce of tower $\{K_n\}$ and satisfies $0 \le \psi_q(\mathcal{K}) \le 1$. For instance, if a prime p splits completely in $\mathcal{K}$, then $\psi_p(\mathcal{K}) = 1$. The existence of this limit can be deduced from the following inequality (see [5, Lemma 4.1]).

Let L/K be an extension of number fields. Then for any $x > 0$ and any prime p,

$$\sum_{p^k\le x} \frac{\mathcal{N}_{p^k}(K)\,\log p^k}{n_K} \ge \sum_{p^k\le x} \frac{\mathcal{N}_{p^k}(L)\,\log p^k}{n_L}.$$

Define the zeta-function associated to the infinite extension $\mathcal{K}/\mathbb{Q}$ as

$$\zeta_{\mathcal{K}}(s) := \prod_{q=p^k} \left(1 - \frac{1}{q^s}\right)^{-\psi_q(\mathcal{K})} \tag{11}$$

on $\Re(s) > 1$. Here, exponents are defined by choosing the principal branch of logarithm. Now, the logarithmic derivative is given by

$$Z_{\mathcal{K}}(s) := -\frac{\zeta_{\mathcal{K}}'}{\zeta_{\mathcal{K}}}(s) = \sum_{q=p^k} \frac{\psi_q(\mathcal{K})\log q}{q^s - 1} \tag{12}$$

for $\Re(s) > 1$. Suppose $\mathcal{K}/\mathbb{Q}$ is a Galois extension with Galois group G, which is unramified outside a finite set of places S. Then,

$$Z_{\mathcal{K}}(s) = Z(\mathcal{K}, G, s)\,P(s),$$

where $P(s)$ is a Dirichlet polynomial and $Z(\mathcal{K}, G, s)$ is as in Section 5. Hence, $Z_{\mathcal{K}}(s)$ converges if and only if $Z(\mathcal{K}, G, s)$ converges. Thus, this zeta-function is closely related to the zeta-function associated to the conjugacy set given by the entire group G.

Furthermore, ψ_q precisely captures the splitting nature of the primes in the infinite extension. For instance, in a Galois extension $\mathcal{K}/\mathbb{Q}$, suppose a prime factorizes as $p\mathcal{O}_{K_n} = \mathfrak{p}_1^e\mathfrak{p}_2^e\cdots\mathfrak{p}_g^e$ in K_n and splits completely thereafter, then $\psi_q(\mathcal{K}) = \frac{g}{n}$, where $q = N\mathfrak{p}_i$. Conversely, for a Galois extension $\mathcal{K}/\mathbb{Q}$ if $\psi_{p^k}(\mathcal{K}) > 0$, then it implies that there exists a number field K_n such that all the primes above p in K_n split completely in $\mathcal{K}$. Hence, the zeta-function $Z_{\mathcal{K}}(s)$ precisely captures the splitting nature of primes, even better than the Tsfasman-Vlăduţ zeta-function. In the later case, it is possible for a prime to split completely, and yet the invariants $\phi_{p^k} = 0$ for all k.

The natural question to consider is whether the above zeta-function has an analytic continuation to a region beyond the half-plane $\Re(s) > 1$. It is possible to construct infinite extensions where the series (12) diverges at $s = 1$. Such an extension was constructed by S. Checcoli and A. Fehm [4]. They showed that for any family of finite solvable groups G_i, there exists a totally real infinite Galois extension $\mathcal{K}/\mathbb{Q}$ with Galois group $\prod_{i=1}^{\infty} G_i$, such that it has a finite local degree over all primes p and the sum

$$\sum_q \frac{\psi_q(\mathcal{K}) \log q}{q+1}$$

diverges. This implies that the sum in (12) diverges. Thus, unlike the earlier zeta-functions in (2) and (7), analytic continuation for $\zeta_{\mathcal{K}}(s)$ beyond the region $\Re(s) > 1$ will not follow from the the convergence of the Dirichlet series.

The behaviour of $\zeta_{\mathcal{K}}(s)$ at $s = 1$ provides valuable information about the infinite extension $\mathcal{K}$. A concrete instance of this is in obtaining lower bounds on the Weil height of elements in $\mathcal{K}$. Recall that for an algebraic number $\alpha \in \overline{\mathbb{Q}}$, the logarithmic Weil height is defined as follows. Let $\alpha \in K^*$. Then

$$h(\alpha) = \sum_{v \in M_K} \log^+ |\alpha|_v,$$

where M_K is the set of all places of K, $\log^+ x = \max(0, \log x)$ and $|\alpha|_v$ is the normalized valuation on α defined as:

$$|\alpha|_v := \begin{cases} (N\mathfrak{p})^{-\frac{ord_{\mathfrak{p}}(\alpha)}{[K:\mathbb{Q}]}}, & \text{if } v \text{ is non-archimedean corresponding to the prime ideal } \mathfrak{p}, \\ |\sigma(\alpha)|^{\frac{[K_\nu:\mathbb{R}]}{[K:\mathbb{Q}]}} & \text{if } v \text{ is archimedean corresponding to the embedding } \sigma \text{ of } K. \end{cases}$$

A well-known theorem of Kronecker [10] states that an algebraic number α satisfies $h(\alpha) = 0$ if and only if α is a root of unity. When α is not a root of unity, obtaining lower bounds for $h(\alpha)$ has been a long standing problem and the famous Lehmer's conjecture [13] states that for such non-zero $\alpha \in \overline{\mathbb{Q}}$

$$h(\alpha) \geq \frac{c}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$$

for an absolute constant $c > 0$. This problem still remains open. But it is interesting to find subsets $S \subset \overline{\mathbb{Q}}$, where a lower bound on $h(\alpha)$ for $\alpha \in S$ can be established. This inspires the definition of the Northcott property (N) and the Bogomolov property (B). A set $S \subseteq \overline{\mathbb{Q}}$ is said to satisfy the *Northcott property (N)* if for any $c > 0$, the set

$$\{\alpha \in S \,|\, \alpha \text{ non-zero and } h(\alpha) < c\}$$

is finite. For instance, Northcott proved that algebraic numbers with bounded degree have property (N). We say that a set $S \subset \overline{\mathbb{Q}}$ satisfies the *Bogomolov property (B)* if there exists a constant $c > 0$, such that

$$\{\alpha \in S \,|\, \alpha \text{ non-zero and not a root of unity and } h(\alpha) < c\}$$

is an empty set. If a set S satisfies property (N), then it clearly satisfies property (B).

These properties have been extensively studied for infinite extensions. For instance, F. Amoroso and R. Dvornicich [1] have proved that $\mathbb{Q}^{ab}$, the maximal abelian extension of $\mathbb{Q}$ satisfies property (B). This was generalized to K^{ab} by F. Amoroso and U. Zannier [2] for any number field K. Earlier, in 1973, A. Schinzel [20] obtained property (B) for the infinite extension $\mathbb{Q}^{tr}$, the field of totally real algebraic numbers. Another family of infinite extensions of $\mathbb{Q}$ for which we know the Bogomolov property are totally p-adic fields, i.e., infinite Galois extensions of $\mathbb{Q}$ with finite local degree over a prime p. This is the famous theorem of E. Bombieri and U. Zannier [3]. Recently, Habegger [7] has showed that for an elliptic curve E/K, the extension $\mathbb{Q}(E_{tor})$ generated by all the torsion points of

E satisfies property (B).

Suppose $\mathcal{K}/\mathbb{Q}$ is an infinite extension. In [5], the author and S. Kala discovered a peculiar relation between the behaviour of $\zeta_{\mathcal{K}}(s)$ at $s = 1$ and the above problem. They showed that if $\lim_{s\to 1^+}\zeta_{\mathcal{K}}(s)$ is non-zero, then $\mathcal{K}$ satisfies property (B) and if this limit tends to infinity, then $\mathcal{K}$ satisfies property (N). This is the first instance of infinite extensions, which are not necessarily Galois, having property (B) or (N). On specializing to Galois extensions, they retrieve the result of Bombieri-Zannier [3].

On another front, the zeros of $\zeta_{K_n}(s)$ near $s = 1/2$ hold information on the lower bounds of $h(\alpha)$ for $\alpha \in K_n$. This was established by the author and S. Kala in [6]. Under GRH, they proved the Lehmer's conjecture over infinite extensions $\mathcal{K}/\mathbb{Q}$ provided there are several zeros of small height for $\zeta_{K_n}(s)$ near $s = 1/2$.

8. **Relation to Euler-Kronecker constants**

The Euler-Mascheroni constant denoted by γ is defined as

$$\gamma := \lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right).$$

It can also be described as the constant term in the Laurent expansion of the Riemann zeta-function,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

The analogous notion for a number field was introduced by Y. Ihara [9] as follows. Let K be a number field and suppose the Laurent expansion of $\zeta_K(s)$ near $s = 1$ is given by

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1).$$

Then the Euler-Kronecker constant associated to K is defined as

$$\gamma_K := \frac{c_0}{c_{-1}}.$$

One could also view γ_K as the constant term in the Laurent expansion of the logarithmic derivative of $\zeta_K(s)$ at $s = 1$, i.e.,

$$-\frac{\zeta_K'}{\zeta_K}(s) = \frac{1}{s-1} - \gamma_K + O(s-1).$$

Recall the famous Stark's lemma [23], which states that for a number field $K/\mathbb{Q}$ and any $s \in \mathbb{C}$

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} - \frac{1}{s-1} + \sum_\rho \frac{1}{s-\rho} = \frac{1}{2}\log d_K + \left(\frac{1}{s} - \frac{n_K}{2}\log\pi\right) + \frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + r_2\left(\frac{\Gamma'}{\Gamma}(s) - \log 2\right), \quad (13)$$

where the summation is over the non-trivial zeros of $\zeta_K(s)$.

Taking $s \to 1^+$ in (13) and dividing by n_K, we obtain

$$-\frac{\gamma_K}{n_K} = \frac{\log|d_K|}{2n_K} - \frac{1}{n_K}\sum_\rho\frac{1}{\rho} + O(1), \quad (14)$$

where the error term is independent of K. On the other hand, for $\Re(s) > 1$

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_q \mathcal{N}_q(K)\sum_{m=1}^\infty \frac{\log q}{q^{ms}},$$

where q runs over all prime powers. Dividing by n_K, for $s = 1+\sigma > 1$, we have

$$-\frac{1}{n_K}\frac{\zeta_K'}{\zeta_K}(1+\sigma) = \sum_q \frac{\mathcal{N}_q(K)}{n_K}\sum_{m=1}^{\infty}\frac{\log q}{q^{ms}} = \sum_q \frac{\mathcal{N}_q(K)}{n_K}\frac{\log q}{q^{\sigma+1}-1}.$$

Now, by (13), for $\sigma > 0$

$$-\frac{1}{n_K}\frac{\zeta_K'}{\zeta_K}(1+\sigma) = \frac{1}{\sigma\, n_K} - \sum_\rho \frac{1}{\rho+\sigma} = \frac{\log|d_K|}{2n_K} + \left(\frac{1}{(1+\sigma)\,n_K} - \frac{\log \pi}{2}\right)$$
$$+ \frac{r_1}{2n_K}\frac{\Gamma'}{\Gamma}\left(\frac{1+\sigma}{2}\right) + \frac{r_2}{n_K}\left(\frac{\Gamma'}{\Gamma}(1+\sigma) - \log 2\right). \tag{15}$$

Let $\mathcal{K} = \bigcup_n K_n$ be an infinite tower over $\mathbb{Q}$. Putting $\sigma_n = 1/\sqrt{n_{K_n}}$ in (15) and taking limits, we obtain

$$\sum_q \psi_q \frac{\log q}{q} = \lim_{i\to\infty}\left(\frac{\log|d_{K_n}|}{2n_{K_n}} - \frac{1}{n_{L_i}}\sum_\rho \frac{1}{\rho + (n_{L_i})^{-1/2}}\right) + O(1), \tag{16}$$

where the implied constant is absolute.

Comparing (14) and (16), it is clear that the convergence of $\zeta_{\mathcal{K}}(s)$ at $s=1$ is intricately connected to the bounds on $-\gamma_K/n_K$. For instance, in the example of the infinite extension $\mathcal{K}/\mathbb{Q}$ constructed by S. Checcoli and A. Fehm [4, Theorem 1.2], the zeta-function $\zeta_{\mathcal{K}}(s)$ tends to infinity as $s \to 1^+$. In other words, the sum

$$\sum_q \psi_q \frac{\log q}{q+1}$$

diverges. Using (14) and (16), one can also conclude that

$$-\frac{\gamma_{K_i}}{n_{K_i}} \to \infty.$$

This should be compared with the known lower bounds on γ_K. In [9], Ihara proved that

$$\gamma_K \geq -\log|d_K|$$

for any number field K. He also demonstrated that

$$\liminf_K \frac{\gamma_K}{\log|d_K|} \leq C,$$

where $C = -0.16352\cdots$. Thus, infinite extensions $\mathcal{K} = \bigcup_n K_n$ where we can show that $|\gamma_{K_n}|$ does not tend to infinity faster than the degree n_K, there is hope in establishing the analytic continuation of $\zeta_{\mathcal{K}}(s)$ to $\Re(s) \geq 1$.

9. **Concluding remarks**

The splitting of primes over infinite extensions is a fundamental theme, several aspects of which remain mysterious. The study of zeta-functions over infinite extensions holds the key to unraveling this mystery and warrants careful investigation. In this paper, we discuss three such zeta-functions, one associated to the conjugacy set, second and third arising from the splitting behaviour of primes in infinite extensions. Unlike the Dedekind zeta-function, the zeta-function attached to a conjugacy set and the Tsfasman-Vlăduţ zeta-function exhibit the unusual property that they do not have a pole at $s=1$. Instead, they admit analytic continuation to $\Re(s) > \sigma_a$, where $\sigma_a < 1$ is the abscissa of their absolute convergence. However, we do not have a single example where this analytic continuation extends beyond the half plane $\Re(s) > \sigma_a$. Constructing such examples would provide vital new insights on existence of number fields with interesting prime splitting conditions.

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Institute of Mathematical Sciences (HBNI), CIT Campus Taramani, Chennai, India, 600113
Email address: anupdixit@imsc.res.in