ON THE SELBERG CLASS OF L-FUNCTIONS

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Abstract. The Selberg class of L-functions, $\mathbb{S}$, introduced by A. Selberg in 1989, has been extensively studied in the past few decades. In this article, we give an overview of the structure of this class followed by a survey on Selberg's conjectures and the value distribution theory of elements in $\mathbb{S}$. We also discuss a larger class of L-functions containing $\mathbb{S}$, namely the Lindelöf class, introduced by V. K. Murty. The Lindelöf class forms a ring and its value distribution theory surprisingly resembles that of the Selberg class.

1. Introduction

The most basic example of an L-function is the Riemann zeta-function, which was introduced by B. Riemann in 1859 as a function of one complex variable. It is defined on $\Re(s) > 1$ as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It can be meromorphically continued to the whole complex plane $\mathbb{C}$ with a pole at $s = 1$ with residue 1. The unique factorization of natural numbers into primes leads to another representation of $\zeta(s)$ on $\Re(s) > 1$, namely the Euler product

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The study of zeta-function is vital to understanding the distribution of prime numbers. For instance, the prime number theorem is a consequence of $\zeta(s)$ having a simple pole at $s = 1$ and being non-zero on the vertical line $\Re(s) = 1$.

In pursuing the analogous study of distribution of primes in an arithmetic progression, we consider the Dirichlet L-function,

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for $\Re(s) > 1$, where χ is a Dirichlet character modulo q, defined as a group homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ extended to $\chi : \mathbb{Z} \to \mathbb{C}$ by periodicity and setting $\chi(n) = 0$ if $(n, q) > 1$.

Attached to a number field $K/\mathbb{Q}$, we have the Dedekind zeta-function defined for $\Re(s) > 1$ as

$$\zeta_K(s) := \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{(N_{K/\mathbb{Q}}(\mathfrak{a}))^s},$$

where $\mathcal{O}_K$ denotes the ring of integers of K and $\mathfrak{a}$ runs over all non-zero ideals of $\mathcal{O}_K$.

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All the above L-functions capture valuable information about the underlying structure of the associated arithmetic objects. The general philosophy is to expect a relation between "motivic" L-functions and automorphic L-functions. Such relations are called reciprocity laws. One of the most significant reciprocity laws of today is the modularity theorem (formerly known as the Taniyama-Shimura conjecture), which associates to every elliptic curve over $\mathbb{Q}$, a modular form through an L-function. This must be viewed as a tip of the iceberg of the more challenging Langland's reciprocity conjecture. In an attempt to understand this theory, Selberg defined a class of L-functions, $\mathbb{S}$, which is expected to satisfy all familiar properties of an automorphic L-function. His motivation was to study the value distribution of linear combinations of L-functions in this class.

Since then, there has been significant progress in the study of the Selberg class. An overview of the recent results and conjectures regarding the structure of $\mathbb{S}$ can be found in several expositions, such as excellent surveys by A. Perelli [34], [33] and J. Kaczorowski [15]. In this article, we outline some results and highlight certain open problems and unexplored avenues for future study. The emphasis is on Selberg's conjectures and the value distribution theory of the Selberg class. The last section is devoted to the Lindelöf class of L-functions $\mathbb{M}$, defined by V. K. Murty [27]. This class $\mathbb{M}$ is closed under addition and enjoys a richer algebraic structure than $\mathbb{S}$. Moreover, the value distribution theory of $\mathbb{M}$ closely resembles that of $\mathbb{S}$.

2. The Selberg class

Definition 2.1. *The Selberg class $\mathbb{S}$ consists of meromorphic functions $F(s)$ satisfying the following properties.*

(1) **Dirichlet series** - *It can be expressed as a Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which is absolutely convergent in the region $\Re(s) > 1$. We also normalize the leading coefficient as $a_F(1) = 1$.

(2) **Analytic continuation** - *There exists a non-negative integer k, such that $(s-1)^k F(s)$ is an entire function of finite order.*

(3) **Functional equation** - *There exist real numbers $Q > 0$, $\alpha_i \geq 0$, complex numbers β_i for $0 \leq i \leq k$ and $w \in \mathbb{C}$, with $\Re(\beta_i) \geq 0$ and $|w| = 1$, such that*

$$\Phi(s) := Q^s \prod_i \Gamma(\alpha_i s + \beta_i) F(s) \tag{1}$$

satisfies the functional equation

$$\Phi(s) = w\overline{\Phi}(1-\overline{s}).$$

(4) **Euler product** - *There is an Euler product of the form*

$$F(s) = \prod_{p \text{ prime}} F_p(s), \tag{2}$$

where

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}$$

with $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$.

(5) **Ramanujan hypothesis** - *For any* $\epsilon > 0$,

$$|a_F(n)| = O_\epsilon(n^\epsilon). \tag{3}$$

The Euler product implies that the coefficients $a_F(n)$ are multiplicative, i.e., $a_F(mn) = a_F(m)a_F(n)$ when $(m,n) = 1$. Moreover, each Euler factor also has a Dirichlet series representation

$$F_p(s) = \sum_{k=0}^{\infty} \frac{a_F(p^k)}{p^{ks}},$$

which is absolutely convergent on $\Re(s) > 0$ and non-vanishing on $\Re(s) > \theta$, where θ is as in (2).

We mention a few examples of elements in $\mathbb{S}$.

i) The Riemann zeta-function $\zeta(s) \in \mathbb{S}$.
ii) Dirichlet L-functions $L(s,\chi)$ and their vertical shifts $L(s+i\theta,\chi)$ are in $\mathbb{S}$, where χ is a primitive Dirichlet character and $\theta \in \mathbb{R}$. Note that $\zeta(s+i\theta) \notin \mathbb{S}$ for $\theta \neq 0$, since it has a pole at $s = 1 - i\theta$.
iii) For a number field $K/\mathbb{Q}$, the Dedekind zeta functions $\zeta_K(s)$ is an element in $\mathbb{S}$.
iv) Let L/K be a Galois extension of number fields, with Galois group G. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of G. The associated Artin L-function is defined as

$$L(s,\rho,L/K) := \prod_{\mathfrak{p}\in K} \det\left(I - (N\mathfrak{p})^{-s}\rho(\sigma_\mathfrak{q})\big|_{V^{I_\mathfrak{q}}}\right)^{-1}$$

where $\mathfrak{q}$ is a prime ideal in L lying over prime ideal $\mathfrak{p}$ in K, $\sigma_\mathfrak{q}$ is the Frobenius automorphism associated to $\mathfrak{q}$ and $V^{I_\mathfrak{q}}$ is the complex vector space fixed by the inertia subgroup $I_\mathfrak{q}$.

A conjecture of Artin states that for non-trivial irreducible representation ρ of $Gal(L/K)$, the associated Artin L-function $L(s,\rho,L/K)$ is entire. If the Artin conjecture is true, then these functions lie in the Selberg class.
v) Let f be a holomorphic newform of weight k to some congruence subgroup $\Gamma_0(N)$. Suppose its Fourier expansion is given by

$$f(z) = \sum_{n=1}^{\infty} c(n)\exp(2\pi i n z).$$

Then its normalized Dirichlet coefficients are given by

$$a(n) := c(n)n^{(1-k)/2},$$

and the associated L-function given by $L(s,f) := \sum_{n=1}^{\infty} a(n)/n^s$ for $\Re(s) > 1$ is an element in the Selberg class. It is also believed that the normalized L-function associated to a non-holomorphic newform is an element in the Selberg class, but the Ramanujan hypothesis is yet to be proven in this case.
vi) The Rankin-Selberg L-function of any normalized eigenform is in the Selberg class.

3. Invariants in $\mathbb{S}$

The constants in the functional equation (1) depend on F, and although the functional equation may not be unique, we have some well-defined invariants, such as the degree d_F of F, which is defined as the finite sum

$$d_F := 2\sum_{i=1}^{k} \alpha_i.$$

The factor Q in the functional equation gives rise to another invariant referred to as the conductor q_F, which is defined as

$$q_F := (2\pi)^{d_F} Q^2 \prod_{i=1}^{k} {\alpha_i}^{2\alpha_i}. \tag{4}$$

A natural question in this context is to understand how unique the functional equation is for $F \in \mathbb{S}$. Given a gamma-factor for F in $\mathbb{S}$, one can produce new gamma-factors using the Gauss-Legendre multiplication formula for the Γ-function,

$$\Gamma(s) = m^{s-1/2}(2\pi)^{(1-m)/2} \prod_{k=0}^{m-1} \Gamma\left(\frac{s+k}{m}\right), \tag{5}$$

for any integer $m > 2$. One could also use the functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{6}$$

to produce new gamma-factors for F. It turns out that the functional equation of $F \in \mathbb{S}$ is unique up to the transformations (5) and (6) (see [17]).

It is an interesting conjecture that both the degree and the conductor for elements in the Selberg class are non-negative integers (see [9], [17]).

Conjecture 1. *If $F \in \mathbb{S}$, then d_F and q_F are non-negative integers.*

There is recent progress towards the degree conjecture. In 1993, it was shown by J. B. Conrey and A. Ghosh [9] that

Theorem 3.1 (Conrey-Ghosh)**.** *If $F(s) \in \mathbb{S}$, then $F = 1$ or $d_F \geq 1$.*

This was proved using the fact that any non-trivial element in the Selberg class must satisfy a certain growth on $\sigma + it$ for $\sigma < 0$ and t sufficiently large. This growth consequently is captured by the degree, which can be seen using the functional equation.

Conrey and Ghosh [9] also conjectured that the functions of degree one in the Selberg class are precisely given by the Riemann zeta-function $\zeta(s)$, Dirichlet L-functions $L(s, \chi)$ and their shifts $L(s + i\theta, \chi)$, where χ is non-principal primitive and $\theta \in \mathbb{R}$. This conjecture was later proved by Perelli and Kaczorowski [16]. No such classification is known for the higher degrees in the Selberg class.

However, there are known examples of elements in the Selberg class with higher degrees. Dedekind zeta-function attached to a number field $K/\mathbb{Q}$ has degree equal to the degree of the field extension $[K : \mathbb{Q}]$. L-functions associated to holomorphic newforms (see Example v) have degree 2. Moreover, L-functions associated to non-holomorphic newforms, if in the Selberg class, would also have degree 2. The Rankin-Selberg L-function of normalized eigenforms are elements of the Selberg class of degree 4.

For elements $F \in \mathbb{S}$ with $d_F > 1$, it is significantly more difficult to show that d_F is an integer. In this direction, Kaczorowski and Perelli [20] established the following.

Theorem 3.2 (Kaczorowski-Perelli)**.** *For $F \in \mathbb{S}$, if $1 \leq d_F < 2$ then $d_F = 1$.*

The key ingredient in this result is the study of non-linear twists of L-functions. The standard non-linear twist of a Dirichlet series $F(s) = \sum_{n\geq 1} a_n/n^s$ is defined as

$$F_d(s,\alpha) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e(-n^{1/d}\alpha),$$

where $e(x) = e^{2\pi i x}$ and $\alpha > 0$ is a real number. Kaczorowski and Perelli studied the generalization of such non-linear twists, replacing α with a real vector-valued function $f(\vec{\alpha})$. They showed that these non-linear twists can be written as a linear combination of some familiar holomorphic functions to establish their result.

In general, we are far from showing any partial result on the elements of $\mathbb{S}$ with degree > 2. We also do not know the complete classification of elements of $\mathbb{S}$ with degree 2.

4. Growth and number of zeros

For $F \in \mathbb{S}$, the Euler product ensures that $F(s)$ has no zeros on the right half plane $\Re(s) > 1$. Using the functional equation, one gets a sequence of zeroes in the left half plane $\Re(s) < 0$ corresponding to the poles arising from the Γ-factors. The more interesting case is to understand the zero-distribution in the strip $0 < \Re(s) < 1$. This region is called the critical strip of an L-function in $\mathbb{S}$. From the discussion above, it is clear that for $F \in \mathbb{S}$, the zeros of $F(s)$ are concentrated in the critical strip. Due to the symmetric nature of the functional equation, Riemann conjectured that all the zeros of the ζ-function must lie on the 1/2-line. This is known as the famous Riemann hypothesis and is considered to be one of the most challenging open questions in number theory. The same statement is also expected to hold for elements in $\mathbb{S}$. This is often referred to as the generalized Riemann hypothesis or the grand Riemann hypothesis.

Conjecture 2 (Generalized Riemann hypothesis)**.** *Let $F \in \mathbb{S}$. If $F(s) = 0$ for $0 < \Re(s) < 1$, then $\Re(s) = 1/2$.*

Although we are far from proving the Riemann hypothesis, a lot is known about the number of zeros of functions in $\mathbb{S}$ in the critical strip. In this direction, it is important to discuss the growth of an L-function in vertical strips. For any analytic function, counting the number of zeros in a region is often tackled by its values on the boundary using Jensen's theorem. Therefore, in order to capture the number of zeros of $F(s) \in \mathbb{S}$ in the strip $0 < \Re(s) < 1$ and $|\Im(s)| < T$, we need to understand the growth of $F(\sigma + it)$ for σ fixed and t growing large. For $F(s) \in \mathbb{S}$, define

$$\mu_F(\sigma) := \limsup_{|t|\to\infty} \frac{\log F(\sigma + it)}{\log |t|}.$$

We clearly have $\mu_F(\sigma) = 0$ for $\sigma > 1$. Moreover, on the left half plane $\sigma < 0$, $\mu_F(\sigma)$ is obtained using the functional equation

$$F(s) = \frac{\gamma(1-\overline{s})}{\gamma(s)} \overline{F(1-\overline{s})},$$

where the gamma-factor is given by

$$\gamma(s) = Q^s \prod_{j=1}^{k} \Gamma(\alpha_j s + \beta_j).$$

Applying Stirling's formula, we get for $t \geq 1$, uniformly in σ,

$$\frac{\gamma(1-\overline{s})}{\gamma(s)} = \left(\alpha Q^2 t^{d_F}\right)^{1/2-\sigma-it} \exp\left(itd_F + \frac{i\pi(\beta - d_F)}{4}\right)\left(\omega + O\left(\frac{1}{T}\right)\right), \tag{7}$$

where

$$\alpha := \prod_{j=1}^{k} \alpha_j^{2\alpha_j} \text{ and } \beta := 2\sum_{j=1}^{k}(1-2\beta_j).$$

Recall the Phragmén-Lindelöf theorem given by

Theorem 4.1 (Phragmén-Lindelöf)**.** *Let $f(s)$ be analytic in the strip $\sigma_1 \leq \Re(s) \leq \sigma_2$ with $f(s) \ll \exp(\epsilon|t|)$. If*

$$|f(\sigma_1 + it)| \ll |t|^{c_1} \text{ and}$$
$$|f(\sigma_2 + it)| \ll |t|^{c_2},$$

then

$$|f(\sigma + it)| \ll |t|^{c(\sigma)},$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, where $c(\sigma)$ is linear in σ with $c(\sigma_1) = c_1$ and $c(\sigma_2) = c_2$.

Using the Phragmén-Lindelöf theorem and (7), we get the following upper bounds on the growth of an element in $\mathbb{S}$.

Proposition 4.2. *Let $F \in \mathbb{S}$. Uniformly in σ, as $|t| \to \infty$,*

$$F(\sigma + it) \sim |t|^{(1/2-\sigma)d_F} |F(1-\sigma+it)|,$$

where d_F denotes the degree of F. We also have

$$\mu_F(\sigma) \leq \begin{cases} 0 & \text{if } \sigma > 1, \\ \frac{1}{2}d_F(1-\sigma) & \text{if } 0 \leq \sigma \leq 1, \\ d_F(\frac{1}{2} - \sigma) & \text{if } \sigma < 0. \end{cases}$$

Using the functional equation, it is possible to show that for $F \in \mathbb{S}$,

$$d_F = \limsup_{\sigma<0} \frac{\mu_F(\sigma)}{1/2 - \sigma}. \tag{8}$$

This gives a characterization of degree in terms of the growth of $F(s)$ in the left half plane $\Re(s) < 0$.

Lindelöf conjectured that the order of growth of the Riemann zeta-function is much smaller than what the Phragmén-Lindelöf theorem gives. In fact, he predicted that $\zeta(s)$ is bounded on $\sigma > 1/2$ (see [24]). This statement is known to be false. But, a weaker version would state that $\mu_\zeta(1/2) = 0$. In other words,

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \ll |t|^{\epsilon},$$

for any $\epsilon > 0$. This is known as the Lindelöf hypothesis. Note that, the Phragmén-Lindelöf theorem only implies that $|\zeta(1/2 + it)| \ll_\epsilon |t|^{1/4+\epsilon}$ for any $\epsilon > 0$. Any improvement on the constant $1/4$ is called the phenomena of "breaking convexity". The best known improvement on this constant is replacing $1/4$ with $9/56$. This is due to E. Bombieri and H. Iwaniec [8] using Weyl's method of estimating exponential sums, which was earlier incorporated by G.

H. Hardy and J. E. Littlewood to attack the same problem.

A more general statement of the Lindelöf hypothesis on the Selberg class is given by

Conjecture 3 (Generalized Lindelöf hypothesis)**.** *For $F \in \mathbb{S}$ and any $\epsilon > 0$,*

$$\left|F\left(\frac{1}{2}+it\right)\right| \ll |t|^{\epsilon}.$$

It is known due to Littlewood that the Riemann hypothesis implies the Lindelöf hypothesis. By the same argument, one can show that the generalized Riemann hypothesis implies the generalized Lindelöf hypothesis. Moreover, the Lindelöf hypothesis itself has many interesting consequences. The most prominent one is in the context of value distribution of L-functions.

For $F \in \mathbb{S}$, let $N_F(\sigma, T)$ denote the number of zeros of $F(s)$ in the region

$$\left\{s \in \mathbb{C} : \Re(s) > \sigma, |\Im(s)| < T\right\}.$$

The Lindelöf hypothesis for Riemann zeta-function implies the density hypothesis, which states that for $\sigma > 1/2$,

$$N_{\zeta}(\sigma, T) \ll T^{2(1-\sigma)}.$$

In case of the Selberg class, the generalized Lindelöf hypothesis implies a statement regarding the zero-distribution of L-functions, which we call the zero hypothesis. The classical result on zero density estimate due to Bohr and Landau [6] states that most of the zeroes of $\zeta(s)$ are clustered near the 1/2-line, i.e., they showed that

$$N_{\zeta}(\sigma, T) \ll T^{4\sigma(1-\sigma)+\epsilon}, \tag{9}$$

for $\sigma > 1/2$. More recently, we have the following density theorem due to Kaczorowski and Perelli [19] for the Selberg class.

Theorem 4.3 (Density theorem)**.** *For $F \in \mathbb{S}$,*

$$N_F(\sigma, T) \ll_{\epsilon} T^{c(1-\sigma)+\epsilon},$$

for $\sigma > 1/2$ and $c = 4d_F + 12$.

The above zero-density estimate suggests that the number of zeros close to the vertical line $\Re(s) = 1$ is very small. In general, we formulate the zero hypothesis, which claims that for $F \in \mathbb{S}$ all the zeros are clustered near the 1/2-line.

Conjecture 4 (Zero hypothesis)**.** *For $F \in \mathbb{S}$, there is a positive constant c such that for $\sigma > 1/2$,*

$$N_F(\sigma, T) \ll T^{1-c(\sigma-1/2)+\epsilon}.$$

Using Riemann-von Mongoldt-type formula, it is possible to count the number of zeros of $F \in \mathbb{S}$ more precisely. (see [35])

Proposition 4.4. *For $F \in \mathbb{S}$, we have*

$$N_F(0, T) = \frac{d_F}{\pi} T \log T + c_F T + O(\log T), \tag{10}$$

where d_F is the degree of F and c_F is a constant depending on F.

Thus, we get another characterization of degree for $F \in \mathbb{S}$ using the number of zeros of F in the critical strip. We have for $F(s) \in \mathbb{S}$

$$d_F = \limsup_{T\to\infty} \frac{N_F(0,T)}{T\log T}\pi. \tag{11}$$

In the above proposition, we could replace couting zeros with counting any a-value and get the exact same result. Define

$$N(F,a,T) := \#\{F(s) = a : 0 < \Re(s) < 1, |\Im(s)| < T\},$$

counted with multiplicity. Then, we have

$$N(F,a,T) = \frac{d_F}{\pi} T\log T + c_F T + O(\log T).$$

5. Selberg's Conjectures

The elements in the Selberg class are not closed under linear combination. But, the Selberg class is closed under multiplication and forms a semi-group with respect to multiplication i.e., if $F, G, H \in \mathbb{S}$, then $FG \in \mathbb{S}$ and $F(GH) = (FG)H$. The fundamental elements with respect to multiplication in $\mathbb{S}$ are called the primitive elements.

Definition 5.1. *$F \in \mathbb{S}$ is said to be a primitive element if any factorization $F = F_1F_2$ with $F_1, F_2 \in \mathbb{S}$ implies that either $F_1 = 1$ or $F_2 = 1$.*

In other words, an element in $\mathbb{S}$ is primitive if it cannot be further factorized into non-trivial elements in $\mathbb{S}$. Using the characterization of degree in (11), we have that if $F \in \mathbb{S}$ has a factorization $F = F_1F_2$, with $F_1, F_2 \in \mathbb{S}$, then

$$N(T,F) = N(T,F_1) + N(T,F_2).$$

Taking $T \to \infty$, we conclude that

$$d_F = d_{F_1} + d_{F_2}.$$

We also know from Theorem 3.1 that non-trivial elements in $\mathbb{S}$ cannot have degree < 1. Therefore, we cannot factorize an element $F \in \mathbb{S}$ indefinitely. So,

Proposition 5.2. *Every element $F \in \mathbb{S}$ can be factorized into primitive elements in $\mathbb{S}$.*

It is still unknown whether the above factorization is unique.

Conjecture 5 (Unique factorization in $\mathbb{S}$)**.** *Every element $F \in \mathbb{S}$ can be uniquely factorized into primitive elements.*

From the above discussion, it is clear that every element $F \in \mathbb{S}$ with degree $d_F = 1$ is a primitive element. Thus, the Riemann zeta-function and Dirichlet L-functions are all primitive elements in the Selberg class. We know very little about the primitive elements of higher degrees. In [29], M. R. Murty showed that if π is an irreducible cuspidal representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$, then $L(s,\pi)$ is primitive if the Ramanujan conjecture is true.

Selberg's conjectures claim that distinct elements in $\mathbb{S}$ do not interact with each other. Vaguely speaking, distinct primitive elements are orthogonal to each other.

Conjecture 6 (Selberg's conjectures)**.** *In* [35]*, Selberg made the following conjectures.*

(1) Conjecture A - Let $F \in \mathbb{S}$. There exists a constant n_F such that

$$\sum_{p\le x} \frac{|a_F(p)|^2}{p} = n_F \log\log x + O(1). \tag{12}$$

(2) Conjecture B - Let $F, G \in \mathbb{S}$ be primitive elements. Then

$$\sum_{p\le x} \frac{a_F(p)\overline{a_G(p)}}{p} = \begin{cases} \log\log x + O(1), & \text{if } F = G, \\ O(1) & \text{otherwise.} \end{cases}$$

Conjecture B is known as the Selberg's orthogonality conjecture.

It is easy to verify Conjecture A in particular cases. For instance, Conjecture A clearly holds for the Riemann zeta-function and Dirichlet L-functions. Conjecture B can also be verified in the case of Dirichlet L-functions using the orthogonality relations for characters.

In view of Proposition 5.2, it is easy to see that Conjecture B implies Conjecture A. Indeed, if $F \in \mathbb{S}$ has a factorization into primitive elements given by

$$F(s) = F_1(s)F_2(s)\cdots F_m(s),$$

where $F_k(s)$ is primitive for all $1 \le k \le m$, then,

$$\sum_{p\le x} \frac{|a_F(p)|^2}{p} = \sum_{1\le j\le k\le m} \sum_{p\le x} \frac{a_{F_j}(p)\overline{a_{F_k}(p)}}{p}.$$

By Conjecture B, the above sum is of the form

$$m\log\log x + O(1),$$

where m is the number of factors in the factorization of $F(s)$ into primitive elements.

Selberg [35] noted that there are connections between these conjectures and several other conjectures like the Sato-Tate conjecture, Langlands conjectures etc. It is not difficult to see that Conjecture B implies unique factorization in $\mathbb{S}$. This was perhaps known to Selberg, but was shown in the work of J. B. Conrey and A. Ghosh [9].

Proposition 5.3. *Conjecture B implies that every element $F \in \mathbb{S}$ has unique factorization into primitive elements.*

Proof. Suppose $F \in \mathbb{S}$ has two different factorizations into primitives, say,

$$F(s) = \prod_{j=1}^{m} F_j(s) = \prod_{k=1}^{r} G_k(s).$$

We can further assume that no F_j is same as G_k. Since

$$\sum_{j=1}^{m} a_{F_j}(p) = \sum_{k=1}^{r} a_{G_k}(p),$$

multiplying both sides by $\overline{a_{F_1}(p)}/p$ and summing over $p \le x$, we get

$$\sum_{j=1}^{m} \sum_{p\le x} \frac{a_{F_j}(p)\overline{a_{F_1}(p)}}{p} = \sum_{k=1}^{r} \sum_{p\le x} \frac{a_{G_k}(p)\overline{a_{F_1}(p)}}{p}. \tag{13}$$

Now, Conjecture B implies that the LHS of (13) is unbounded where as the RHS is bounded as x tends to infinity, which leads to a contradiction. □

By a similar argument as above, we also conclude the following.

Proposition 5.4. *An element $F \in \mathbb{S}$ is a primitive element if and only if $n_F = 1$, where n_F is given by* (12).

In [28], M.R. Murty proved that Conjecture B implies Artin's conjecture. In particular, he showed the following.

Theorem 5.5 (M. R. Murty)**.** *For any irreducible representation ρ of $Gal(L/K)$ of degree n, the Artin L-function $L(s,\rho,L/K)$ is entire if Conjecture B holds.*

In fact, he showed something stronger. Langland's reciprocity conjecture states that for any irreducible representation ρ of $Gal(L/K)$ of degree n, there exists an irreducible cuspidal automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$, such that $L(s,\rho,L/K) = L(s,\pi)$. Since $L(s,\pi)$ are known to be entire, Artin's conjecture is a consequence of this statement. In [28], M. R. Murty showed that if $K/\mathbb{Q}$ is solvable, then Conjecture B implies Langlands reciprocity conjecture.

In this direction, M. R. Murty [29] initiated the study of Selberg's conjectures over number fields. For any number field K, the idea is to consider functions, given by

$$F(s) = \sum_{\mathfrak{n}\subset\mathcal{O}_K} \frac{a_{\mathfrak{n}}}{\mathbb{N}(\mathfrak{n})^s} \tag{14}$$

on $\Re(s) > 1$, where $\mathfrak{n}$ runs over all non-zero integral ideals of K. The expected functional equation and the Euler product were modified analogously. This new class of functions denoted $\mathbb{S}_K$ could be considered as the Selberg class over a number field K. It is not difficult to see that $\mathbb{S}_K$ is a subset of $\mathbb{S}$. He introduced the notion of K-primitives in $\mathbb{S}_K$ analogous to the primitive elements in $\mathbb{S}$ and made conjectures analogous to the Selberg's conjectures for $\mathbb{S}_K$ discussing its applications to Langland's conjectures (see [29]). This front of study seems to have a lot of potential for future exploration.

There are many more interesting consequences of Conjecture B. Using a similar argument as in Proposition 5.3, one can prove that the Conjecture B implies that if $F \in \mathbb{S}$ has a pole at $s = 1$, it must come from the Riemann-zeta function. More precisely,

Lemma 5.6. *If $F(s) \in \mathbb{S}$ has a pole of order m at $s = 1$, then Conjecture B implies that $F(s) = \zeta(s)^m L(s)$, where $L \in \mathbb{S}$.*

Proof. Since Conjecture B implies unique factorization into primitive elements in $\mathbb{S}$, it suffices to show that if $F \in \mathbb{S}$ is a primitive element with a pole at $s = 1$, then it is $\zeta(s)$. From Proposition 5.4 we know that $n_\zeta = 1$ and $n_F = 1$. If $F \neq \zeta$, then Conjecture B implies that

$$\sum_{p\leq x} \frac{a_F(p)}{p} \ll 1,$$

which is a contradiction. □

This expectation that every pole comes from $\zeta(s)$ can be thought of as the amelioration of Dedekind's conjecture, which states that every Dedekind zeta-function $\zeta_K(s)$ must factorize through $\zeta(s)$.

The Selberg class is designed to model the class of L-functions satisfying the Riemann hypothesis. So, one might ask whether the analogue of prime number theorem is true for the elements in $\mathbb{S}$. Recall that the prime number theorem for natural numbers follows from the fact that $\zeta(s)$ does not vanish on the vertical line $\Re(s) = 1$. It was shown by Kaczorowski and Perelli [19] that prime number theorem for any $F \in \mathbb{S}$ is equivalent to the non-vanishing

of $F(s)$ on $\Re(s) = 1$. Thus, one can formulate the prime number theorem in the Selberg class as follows.

Conjecture 7 (Generalized prime number theorem)**.** *If $F \in \mathbb{S}$, then $F(s) \neq 0$ for $s = 1 + it$ for any $t \in \mathbb{R}$.*

This is still open. But, the above conjecture can be shown assuming Conjecture B. In fact, Kaczorowski and Perelli [19] proved the Conjecture 7 with an assumption weaker than Conjecture A. This weaker assumption is often called the normality conjecture, which is similar to Conjecture A, but with a weaker error term. Here, we present an argument showing that Conjecture B implies Conjecture 7. We use the following lemma.

Lemma 5.7. *If $F \in \mathbb{S}$ has a pole or a zero at $s = 1 + i\theta$ for $\theta \in \mathbb{R}$, then*

$$\sum_{p \leq x} \frac{a_F(p)}{p^{1+i\theta}}$$

is unbounded as x tends to ∞.

Proof. If $F(s)$ has a pole or zero of order $m \neq 0$ at $1 + i\theta$, then we have

$$F(s) \sim c(s - (1 + i\theta))^m,$$

near $1 + i\theta$. Writing $s = \sigma + it$ and taking log, we get

$$\log F(s) \sim m \log(\sigma - 1)$$

near $s = 1 + i\theta$. Moreover, from the Euler product, we have for $\sigma > 1$,

$$\log F(s) = \sum_p \frac{a_F(p)}{p^s} + O(1).$$

Thus, we get

$$\sum_p \frac{a_F(p)}{p^s} \sim m \log(\sigma - 1),$$

as $\sigma \to 1^+$. Assume the function

$$S(x) = \sum_{p \leq x} \frac{a_F(p)}{p^{1+i\theta}}$$

is bounded. Then, we have

$$\begin{aligned} \sum_p \frac{a_F(p)}{p^s} &= \int_1^\infty x^{1-\sigma} dS(x) \\ &= (\sigma - 1) \int_1^\infty S(x) x^{-\sigma}\, dx \ll 1, \end{aligned}$$

which is a contradiction. □

We are now ready to prove the following proposition.

Proposition 5.8. *Conjecture B implies Conjecture 7.*

Proof. Since Conjecture B implies unique factorization, it is enough to show the non-vanishing of $F(s)$ on $\Re(s) = 1$ for primitive elements $F \in \mathbb{S}$. Since $\zeta(s)$ does not vanish on $\Re(s) = 1$, using Lemma 5.6, we can further assume that $F(s)$ is entire. This implies that $F(s + i\alpha) \in \mathbb{S}$ for any $\alpha \in \mathbb{R}$.

Now, if F has a zero at $s = 1 + i\theta$, Lemma 5.7 implies that

$$\sum_{p\leq x}\frac{a_F(p)}{p^{1+i\theta}}$$

is unbounded as $x \to \infty$. But the Conjecture B applied to $\zeta(s)$ and $F(s+i\theta)$ yields

$$\sum_{p\leq x}\frac{a_F(p)}{p^{1+i\theta}} \ll 1,$$

which leads to a contradiction. □

It was observed by Selberg in [35] and Bombieri-Hejhal in [7] that distinct elements in the Selberg class are linearly independent. For an explicit argument, the reader may refer to [10, Lemma 3.5.5]. A natural question that arises is whether distinct primitive elements in $\mathbb{S}$ are algebraically independent. G. Molteni [26] showed that this is a consequence of Conjecture B.

Proposition 5.9. *Conjecture B implies that distinct primitive elements in* $\mathbb{S}$ *are algebraically independent.*

Proof. Selberg's orthonormality conjecture implies that the factorization into primitive elements in the Selberg class is unique. Suppose, $F_1, F_2, ..., F_n$ are distinct primitive elements in $\mathbb{S}$ satisfying a polynomial $P(x_1, x_2, ..., x_n) \in \mathbb{C}[x_1, x_2, ..., x_n]$. By linear independence of distinct elements in $\mathbb{S}$, we conclude that not all terms in the polynomial expansion of $P(F_1, ..., F_n)$ are distinct. Thus, we have relations of the form

$$F_1^{a_1}F_2^{a_2}...F_n^{a_n} = F_1^{b_1}F_2^{b_2}...F_n^{b_n}, \tag{15}$$

where not all the a_i's are the same as the b_i's. But, both the left hand side and the right hand side in (15) are elements in the Selberg class. This contradicts the unique factorization. □

6. Uniqueness results for elements in $\mathbb{S}$

Selberg's orthogonality conjecture implies that for $F, G \in \mathbb{S}$, if $a_F(p) = a_G(p)$ for all but finitely many primes p, then $F = G$. Such uniqueness results are called strong multiplicity one theorems for the Selberg class. Unconditionally, it was shown by M. R. Murty and V. K. Murty [30] that

Theorem 6.1 (Murty-Murty)**.** *For $F, G \in \mathbb{S}$, if $a_F(p) = a_G(p)$ and $a_F(p^2) = a_G(p^2)$ for all but finitely many primes p, then $F = G$.*

As an immediate consequence, we have that if $F, G \in \mathbb{S}$ satisfy the property that the Euler factors $F_p(s) = G_p(s)$ for all but finitely many primes p, then $F = G$. It is expected that the condition $a_F(p) = a_G(p)$ for all but finitely many primes p uniquely characterizes the function in $\mathbb{S}$. But a proof of this fact is still unknown. However, if we further impose the condition that $F(s)$ and $G(s)$ have polynomial Euler product, i.e. an Euler product of the form

$$F(s) = \prod_p\prod_{j=1}^{k}\left(1-\frac{\alpha_p(j)}{p^s}\right)^{-1},$$

with $|\alpha_p(j)| < 1$, then it was shown by J. Kaczorowski and A. Perelli [18] that for $F, G \in \mathbb{S}$ if $a_F(p) = a_G(p)$ for all but finitely many p, then $F = G$. It is worth noting that the elements in the Selberg class are expected to have polynomial Euler product. As a token of evidence, note that the Riemann zeta-function, Dedekind zeta-functions, Hecke L-functions, L-functions attached to holomorphic cusp forms and in fact all automorphic L-functions have polynomial

Euler product.

Another aspect to the uniqueness of elements in the Selberg class arises from the a-value distribution. If $F, G \in \mathbb{S}$ take the same value at sufficiently many points in the critical strip, then $F = G$.

For any two meromorphic functions f and g, we say that they share a value 'a' ignoring multiplicity if $f^{-1}(a)$ is same as $g^{-1}(a)$ as sets. We further say that f and g share a value 'a' counting multiplicity if the zeroes of $f(x) - a$ and $g(x) - a$ are the same with multiplicity. Nevanlinna theory [32] establishes that any two meromorphic functions of finite order sharing five values ignoring multiplicity must be the same. Moreover, if they share four values counting multiplicity, then one must be a Möbius transform of the other. The numbers four and five are the best possible for meromorphic functions.

One can get much stronger results for L-functions. For $F, G \in \mathbb{S}$, define

$$D_{F,G}(T) = \sum_{\rho} |m_F(\rho) - m_G(\rho)|,$$

where ρ runs over all the non-trivial zeroes of F and G with $|\Im(\rho)| < T$ and $m_F(\rho)$ denotes the order of the zero of F at ρ. Then, M. R. Murty and V. K. Murty [30] showed that if $D_{F,G}(T) = o(T)$, then $F = G$. In other words, if F, G share sufficiently many zeros counting multiplicity, then they must be the same. It is possible to show the above result for any a-values.

Proposition 6.2. *For $F, G \in \mathbb{S}$, if F, G share a complex value 'a' counting multiplicity for all but finitely many points, then $F = G$.*

Proof. Since F and G have only one possible pole at $s = 1$, we define H as

$$H := \frac{F - a}{G - a} Q,$$

where $Q(s) = (s-1)^k p(s)$ is a rational function and $p(s)$ a polynomial such that H has no poles or zeros. Since, F and G have complex order 1, we conclude that H has order at most 1 and hence is of the form

$$H(s) = e^{ms+n}.$$

This immediately leads to $m = 0$, since F and G are absolutely convergent on $\Re(s) > 1$ and taking $s \to \infty$, $F(s)$ and $G(s)$ approach their leading coefficient 1. Similarly, we also get $Q(s) = 1$. This forces

$$F(s) = cG(s) + d.$$

for some constants $c, d \in \mathbb{C}$. Since, F and G have leading coefficient 1, we conclude that $F = G$. □

It is possible to prove stronger results than above using similar techniques used by M. R. Murty and V. K. Murty in [30] to show that if $F, G \in \mathbb{S}$ satisfy $D_{F-a,G-a}(T) = o(T)$, then $F = G$.

In this context, a natural question of interest would be to investigate how many values can two distinct elements in $\mathbb{S}$ share ignoring multiplicity. Clearly, $F(s)$ and $F^2(s)$ share zeros ignoring multiplicity. So the best one could expect is that $F, G \in \mathbb{S}$ sharing two distinct values ignoring multiplicity must be the same. J. Steuding [36] proved this with some extra conditions. In 2010, B. Q. Li [23] gave a proof dropping the additional conditions.

Theorem 6.3 (B. Q. Li). *Let a, b be two distinct complex numbers. If $F, G \in \mathbb{S}$ share values a and b ignoring multiplicity, then $F = G$.*

The main idea in such uniqueness results was to introduce Nevanlinna theory to the study of value distribution theory. In a previous paper, B. Q. Li [22] also showed the following.

Theorem 6.4 (B. Q. Li). *Let $F \in \mathbb{S}$ and f be a meromorphic function with finitely many poles. Suppose F and f share a value a counting multiplicity and another value b ignoring multiplicity, then $F = f$.*

For stronger versions of the above results, the reader may refer to [11]. One can show all the above results by dropping the Euler product and the Ramanujan hypothesis. The question still remains of how large can the error $D_{F,G}(T)$ be. When sharing values ignoring multiplicity, there is no known satisfactory answer to this question.

7. Limit theorems and universality

In the early twentieth century, Harald Bohr introduced geometric and probabilistic methods to the study of the value distribution of the Riemann zeta-function. In this section, the probabilistic methods will be of significance.

For the Riemann zeta-function $\zeta(s)$, we know that if $\sigma_0 > 1$, then

$$|\zeta(s)| \leq \zeta(\sigma_0)$$

in the right half plane $\Re(s) \geq \sigma_0$. In other words, $\zeta(s)$ is bounded on any right half plane $\Re(s) > 1 + \epsilon$. The natural question to consider is what happens as σ_0 approaches 1 from the right. In this regard, Bohr [2] proved that in any strip $1 < \Re(s) < 1+\epsilon$, $\zeta(s)$ takes any non-zero complex value infinitely often. The main tool used by Bohr was the Euler product of $\zeta(s)$. Similar study in the critical strip is much more difficult. To tackle this problem, Bohr studied truncated Euler products

$$\zeta_M(s) := \prod_{p \leq M} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The functions $\zeta_M(s)$ do not converge in the critical strip as M tends to ∞. However, Bohr showed that in the critical strip, for large M, $\zeta_M(s)$ approximates $\zeta(s)$ well in the following sense.

$$\int_T^{2T} \iint_D \left| \frac{\zeta(s + i\tau)}{\zeta_M(s + i\tau)} - 1 \right|^2 d\sigma\, dt\, d\tau \ll \epsilon T, \quad \text{for all } \epsilon > 0,$$

where $D := \{s = \sigma + it : 1/2 + \delta < \sigma \leq 2, |t| \leq 1\}$. This remarkable idea plays a key role in many interesting discoveries of Bohr.

In [3], Bohr showed that for any $\sigma_0 \in (1/2, 1)$, the image of the vertical line $\{\Re(s) = \sigma_0\}$ given by

$$\left\{\zeta(s) : s = \sigma_0 + it, t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}$. Later, Bohr and Jessen [4], [5] improved these results using probabilistic methods to prove the following limit theorem.

Theorem 7.1 (Bohr, Jessen). *Let R be any rectangle in $\mathbb{C}$ with sides parallel to the real and imaginary axis. Let G be the half plane $\{\Re(s) > 1/2\}$ except for points $z = x + iy$ such that there is a zero of $\zeta(s)$ given by $\rho = \alpha + iy$ with $x \le \alpha$. For any $\sigma > 1/2$, the limit*

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{\tau \in [0,T] : \sigma + i\tau \in G,\ \log\zeta(\sigma+i\tau) \in R\right\}$$

exists.

Here the measure is the usual Lebesgue measure. Later, Hattori and Matsumoto [13] identified the probability distribution given by the above limit theorem. It is reasonable to hope that Bohr-Jessen type results can be shown for general automorphic L-functions.

In 1972, Voronin [38] proved the following generalization of Bohr's limit theorem.

Theorem 7.2 (Voronin). *For any fixed distinct numbers $s_1, s_2, \cdots, s_n$ with $1/2 < \Re(s_j) < 1$ for $1 \le j \le n$, the set*

$$\left\{(\zeta(s_1+it), \cdots, \zeta(s_n+it)) : t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}^n$. Moreover, for any fixed number s with $1/2 < \Re(s) < 1$,

$$\left\{(\zeta(s+it), \zeta'(s), \cdots, \zeta^{(n-1)}(s+it)) : t \in \mathbb{R}\right\}$$

is dense in $\mathbb{C}^n$.

Analogous limit and density theorems for other L-functions were obtained by Matsumoto [25], Laurinčikas [21], Šleževičienė [40] et al.

It is interesting to note that despite the density theorems, we do not understand the value distribution of $\zeta(s)$ on $\Re(s) = 1/2$. A folklore, yet unsolved conjecture is that the set of values of $\zeta(s)$ on $\Re(s) = 1/2$ is dense in $\mathbb{C}$. In this direction, Selberg showed that "up to some normalization" of $\zeta(s)$, the values on the 1/2-line satisfy the Gaussian distribution (see Joyner [14]).

In 1975, Voronin [39] proved a fascinating theorem for the Riemann zeta-function, which roughly says that any non-vanishing analytic function is approximated uniformly by shifts of the zeta-function in the critical strip. This is called the Voronin's universality theorem. More precisely,

Theorem 7.3 (Voronin). *Let $0 < r < \frac{1}{4}$ and suppose that $g(s)$ is a non-vanishing continuous function on the disc $\{s : |s| \le r\}$, which is analytic in its interior. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\ meas\left\{|\tau| < T : \max_{|s|<r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right) - g(s)\right| < \epsilon\right\} > 0.$$

After the result of Voronin, Bagchi [1] gave a proof of universality for the Riemann zeta-function $\zeta(s)$ and some other L-functions using probabilistic methods. Using Bagchi's technique, the universality property for many L-functions has been established, mainly due to the work of Laurančikas, Matsumoto, Steuding et al. In particular, we know that the universality property holds for elements in the Selberg class $\mathbb{S}$ satisfying a condition analogous to the prime number theorem (see [31]).

Theorem 7.4 (Steuding, Nagoshi). *Let $L(s) \in \mathbb{S}$ with degree d_L satisfying the condition*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |a_L(p)|^2 = \kappa_L,$$

where κ_L is a constant depending on L. Let K be a compact subset of the strip

$$1 - \frac{1}{2d_L} < \Re(s) < 1,$$

with connected complement. Suppose $g(s)$ is any non-vanishing continuous function on K, which is analytic in the interior of K. Then, for any $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\, meas\left\{|\tau| < T : \max_{s\in K} |L(s+i\tau) - g(s)| < \epsilon\right\} > 0$$

It is important to note that the L-functions for which the universality property has been established is much larger than the Selberg class. In fact, L-functions such as the Hurwitz zeta-function, Lerch zeta-function or Matsumoto zeta-functions are all known to be universal in a certain strip. In view of this, Linnik and Ibragimov conjectured the following.

Conjecture 8 (Linnik, Ibragimov). *Let $F(s)$ have a Dirichlet series representation, absolutely convergent on $\Re(s) > 1$ and suppose $F(s)$ can be analytically continued to $\mathbb{C}$ except for a possible pole at $s = 1$ satisfying some "growth conditions", then $F(s)$ is universal in a certain strip.*

Although the universality property for elements in $\mathbb{S}$ is conditionally known, the study is far from complete. In particular, for $F \in \mathbb{S}$, the strip for which the universality property has been established is given by $1 - 1/2d_F < \Re(s) < 1$. But the expected strip of universality is $1/2 < \Re(s) < 1$ (see [37], [10]). This is, in fact a consequence of the Lindelöf hypothesis.

Another front to investigate is the following: for a given non-vanishing analytic function $g(s)$ on a compact subset K inside the strip of universality and a given $\epsilon > 0$, for what value of T_0 is the universality property realized? In other words, how large must T_0 be such that for any $T > T_0$, $F(s)$ approximates $g(s)$ up to ϵ, δT number of times, where $\delta > 0$. Unfortunately, there are no known results in this direction. It would be interesting to explicitly describe T when g is a polynomial or a Dirichlet polynomial.

8. Lindelöf class: A generalization

Despite its generality, the Selberg class has several limitations. For instance, it is not closed under addition. This is because of the rigidity of functional equation and the Euler product. Thus, the zero distribution of linear combination of L-functions in the Selberg class, which appears in the work of Bombieri and Hejhal [7] is not addressed by studying the value distribution theory of elements in $\mathbb{S}$. Moreover, some naturally occurring L-functions such as the Hurwitz zeta-function or Lerch zeta-function are not members of the Selberg class. Furthermore, functions such as the Epstein zeta-function, which satisfy a functional equation of the Riemann-type may not always have an Euler product and hence are not members of the Selberg class. This motivated V. K. Murty [27] to introduce a larger class of L-functions $\mathbb{M}$ which contains $\mathbb{S}$, is closed under linear combination and also captures many familiar L-functions, which are not in $\mathbb{S}$. This new class $\mathbb{M}$ forms a ring and the value distribution of elements in $\mathbb{M}$ is very similar to that of the Selberg class. In order to define $\mathbb{M}$, we start by introducing some growth parameters.

Let $F(s)$ be an entire function of order ≤ 1, which is given by the Dirichlet series $F(s) = \sum_n a_n/n^s$ on $\Re(s) > 1$. Define $\mu_F(\sigma)$ as

$$\mu_F(\sigma) := \begin{cases} \inf\left\{\lambda \in \mathbb{R} : |F(s)| \leq (|s|+2)^\lambda, \text{ for all } s \text{ with } \Re(s) = \sigma\right\}, \\ \\ \infty, \text{ if the infimum does not exist.} \end{cases} \tag{16}$$

Also define:

$$\mu_F^*(\sigma) := \begin{cases} \inf\left\{\lambda \in \mathbb{R} : |F(\sigma+it)| \ll_\sigma (|t|+2)^\lambda\right\}, \\ \\ \infty, \text{ if the infimum does not exist.} \end{cases} \tag{17}$$

If $F(s)$ has a pole of order k at $s = 1$, consider the function

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s). \tag{18}$$

Now define, $\mu_F(\sigma) := \mu_G(\sigma)$ and $\mu_F^*(\sigma) := \mu_G^*(\sigma)$. Intuitively, $\mu_F^*(\sigma)$ does not see how $F(s)$ behaves close to the real axis. It is only dependent on the growth of $F(s)$ on $\Re(s) = \sigma$ and $\Im(s) \gg T$ for arbitrary large T. On the other hand, $\mu_F(\sigma)$ captures an absolute bound for $F(s)$ on the entire vertical line $\Re(s) = \sigma$. It follows from the definition that

$$\mu_F^*(\sigma) \leq \mu_F(\sigma)$$

for any σ.

Definition 8.1. The class $\mathbb{M}$. *Define the class $\mathbb{M}$ (see* [27, sec.2.4]*) to be the set of functions $F(s)$ satisfying the following conditions.*

(1) **Dirichlet series** - *$F(s)$ is given by a Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which is absolutely convergent in the right half plane $\Re(s) > 1$.

(2) **Analytic continuation** - *There exists a non-negative integer k such that $(s-1)^k F(s)$ is an entire function of order ≤ 1.*

(3) **Growth condition** - *The quantity $\frac{\mu_F(\sigma)}{(1-2\sigma)}$ is bounded for $\sigma < 0$.*

(4) **Ramanujan hypothesis** - *$|a_F(n)| = O_\epsilon(n^\epsilon)$ for any $\epsilon > 0$.*

Examples of elements in $\mathbb{M}$ include Dirichlet polynomials, all Dirichlet series which are convergent on the whole complex plane, all elements in the Selberg class and their linear combinations, translates of Epstein zeta-functions etc. From the observation (8), we define the following invariants for $\mathbb{M}$, which would play the role of degree in $\mathbb{S}$.

Definition 8.2. *For $F \in \mathbb{M}$, define*

$$c_F := \limsup_{\sigma<0} \frac{2\mu_F(\sigma)}{1-2\sigma},$$
$$c_F^* := \limsup_{\sigma<0} \frac{2\mu_F^*(\sigma)}{1-2\sigma}.$$

By the growth condition, c_F and c_F^* are well-defined in $\mathbb{M}$. Furthermore, these invariants satisfy an ultrametric inequality. For $F, G \in \mathbb{M}$,

$$c_{FG} \leq c_F + c_G \quad \text{and} \quad c_{F+G} \leq \max(c_F, c_G).$$

Similarly,

$$c_{FG}^* \leq c_F^* + c_G^* \quad \text{and} \quad c_{F+G}^* \leq \max(c_F^*, c_G^*).$$

In fact, if $c_F > c_G$ (resp. $c_F^* > c_G^*$), then

$$c_{F+G} = c_F \text{ (resp. } c_{F+G}^* = c_F^*).$$

This ensures that $\mathbb{M}$ is closed under addition.

If $F \in \mathbb{S}$, then $c_F = c_F^* = d_F$. Since the degree in the Selberg class is conjectured to be a non-negative integer, one may wonder if the same is expected to be true for the invariants c_F and c_F^* in $\mathbb{M}$. It turns out that c_F can take non-integer values. In fact, one can manufacture functions in $\mathbb{M}$ with any arbitrary non-negative value c_F. However, we expect c_F^* to take non-negative integer values. In this direction, we have the following partial result (see [27], [12]).

Proposition 8.3. *Suppose $F(s) \in \mathbb{M}$. Then $c_F^* < 1$ implies $c_F^* = 0$.*

It is also possible to classify all elements with $c_F^* = 0$. These are essentially given by all Dirichlet series, which are convergent on the whole of $\mathbb{C}$. There are many more interesting algebraic properties of $\mathbb{M}$. For instance, $\mathbb{M}$ is non-Noetherian. This is interesting because $\mathbb{C}[\mathbb{S}]$ is a subring of $\mathbb{M}$ and Selberg's conjectures imply that $\mathbb{C}[\mathbb{S}]$ is non-Noetherian. Furthermore, the uniqueness result 6.3 and 6.4, and a weaker version of the universality theorem 7.4 can be established for the class $\mathbb{M}$. We refer the reader to [10] for details.

One may wonder if there is some underlying topology on $\mathbb{M}$. Perhaps, understanding the geometry and learning to interpolate between L-functions may hold the key to new discoveries in this fascinating field of mathematics.

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