THE LINDELÖF CLASS OF L-FUNCTIONS, II

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Abstract. In 2002, the second author [7] introduced a class of L-functions $\mathbb{M}$, which contains the Selberg class and forms a ring. In this article, we study this class and prove that the invariant c_F^*, which is the generalization of degree in the Selberg class cannot take non-integer values between 0 and 1. We also study the ring structure of $\mathbb{M}$ showing that it is non-Noetherian.

1. Introduction

In 1989, A. Selberg [9] introduced a class of L-functions $\mathbb{S}$ satisfying properties similar to that of the Riemann zeta-function. The Selberg class can be regarded as a model for L-functions coming from arithmetic and geometry. Many naturally occurring L-functions such that the Riemann zeta-function, the Dirichlet L-functions and Dedekind zeta-functions are members of the Selberg class. Since then, the Selberg class has been extensively studied and many interesting properties on the structure of this class have been discovered. In [9], Selberg made two key conjectures on this class which vaguely claim that distinct L-functions in $\mathbb{S}$ do not interact with each other. These conjectures have far reaching consequences. As shown by M. Ram Murty [8], the Selberg's orthogonality conjecture implies the strong Artin's holomorphy conjecture. Despite its generality, the Selberg class has many limitations. It is not closed under addition and many naturally occurring L-functions such as the Hurwitz zeta-function, Lerch zeta-function or Epstein zeta-function are not members of the Selberg class.

This motivated the second author [7] to introduce a class of L-functions $\mathbb{M}$, which is defined based on growth conditions. This class $\mathbb{M}$ contains the Selberg class and forms a ring. In this article, we study this class by introducing an invariant which generalizes the notion of degree in the Selberg class and prove that it does not take non-integer values between 0 and 1. We also introduce a method to construct non-trivial ideals of $\mathbb{M}$ and prove that $\mathbb{M}$ is non-Noetherian.

2. The Selberg Class

The Selberg class $\mathbb{S}$ consists of meromorphic functions $F(s)$ satisfying the following properties.

(1) **Dirichlet series-** F can be expressed as a Dirichlet series
$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$
which is absolutely convergent in the region $\Re(s) > 1$. We also normalize the leading coefficient as $a_F(1) = 1$.

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(2) **Analytic continuation** - There exists a non-negative integer k, such that $(s-1)^k F(s)$ is an entire function of finite order.
(3) **Functional equation** - There exist real numbers $Q > 0$ and $\alpha_i > 0$, complex numbers β_i and $w \in \mathbb{C}$, with $\Re(\beta_i) \geq 0$ and $|w| = 1$, such that

$$\Phi(s) := Q^s \prod_i \Gamma(\alpha_i s + \beta_i) F(s) \tag{1}$$

satisfies the functional equation

$$\Phi(s) = w\bar{\Phi}(1 - \bar{s}).$$

(4) **Euler product** - There is an Euler product of the form

$$F(s) = \prod_{p \text{ prime}} F_p(s), \tag{2}$$

where

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}$$

with $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$.
(5) **Ramanujan hypothesis** - For any $\epsilon > 0$,

$$|a_F(n)| = O_\epsilon(n^\epsilon). \tag{3}$$

The constants in the functional equation (1) depend on F. Although the functional equation may not be unique, because of the duplication formula of Γ-function, we have some well-defined invariants, such as the degree d_F of F, which is defined as

$$d_F := 2 \sum_i \alpha_i.$$

The factor Q in the functional equation gives rise to another invariant referred to as the conductor q_F, which is defined as

$$q_F := (2\pi)^{d_F} Q^2 \prod_i {\alpha_i}^{2\alpha_i}. \tag{4}$$

It is an interesting conjecture that both the degree and the conductor associated to elements of the Selberg class are non-negative integers.

3. The class $\mathbb{M}$

In [7], the second author defined a class of L-functions based on growth conditions. We start by defining two growth parameters μ and μ^*.

Definition 3.1. The class $\mathbb{T}$. *Define the class $\mathbb{T}$ to be the set of meromorphic functions $F(s)$ satisfying the following conditions.*

(1) **Dirichlet series** - *For $\Re(s) > 1$, $F(s)$ is given by the absolutely convergent Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}.$$

(2) **Analytic continuation** - *There exists a non-negative integer k, such that $(s-1)^k F(s)$ is an entire function of order ≤ 1.*
(3) **Ramanujan hypothesis** - *$|a_F(n)| = O_\epsilon(n^\epsilon)$ for any $\epsilon > 0$.*

Definition 3.2. *Let $F \in \mathbb{T}$ be entire. Define $\mu_F(\sigma)$ as*

$$\mu_F(\sigma) := \begin{cases} \inf\left\{\lambda \in \mathbb{R} : |F(s)| \le (|s|+2)^{\lambda},\ \text{for all } s \text{ with } \Re(s) = \sigma\right\}, \\ \\ \infty,\ \text{if the infimum does not exist.} \end{cases} \tag{5}$$

Also define:

$$\mu_F^*(\sigma) := \begin{cases} \inf\left\{\lambda \in \mathbb{R} : |F(\sigma+it)| \ll_{\sigma} (|t|+2)^{\lambda}\right\}, \\ \\ \infty,\ \text{if the infimum does not exist.} \end{cases} \tag{6}$$

In the definition of $\mu_F^*(\sigma)$, the implied constant depends on both F and σ, whereas in $\mu_F(\sigma)$, the constant only depends on F and is independent of σ.

We further extend the definition of μ_F and μ_F^* to all the elements in the class $\mathbb{T}$ as follows. Suppose $F \in \mathbb{T}$ has a pole of order k at $s = 1$. Consider the function

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s). \tag{7}$$

Clearly, $G(s)$ is an entire function and belongs to $\mathbb{T}$. We define

$$\mu_F(\sigma) := \mu_G(\sigma),$$
$$\mu_F^*(\sigma) := \mu_G^*(\sigma).$$

Intuitively, $\mu_F^*(\sigma)$ does not see how $F(s)$ behaves close to the real axis. On the other hand, $\mu_F(\sigma)$ captures an absolute bound for $F(s)$ on the entire vertical line $\Re(s) = \sigma$.

It follows from the definition that

$$\mu_F^*(\sigma) \le \mu_F(\sigma)$$

for any σ. From the above definition, we immediately conclude the following.

Proposition 3.3. *Let $F \in \mathbb{T}$. For $\sigma > 1 + \epsilon$,*

$$\mu_F^*(\sigma) = 0$$
$$\mu_F(\sigma) \ll_{F,\epsilon} 1.$$

for any $\epsilon > 0$.

Proof. Since $F \in \mathbb{T}$, it is given by a Dirichlet series $F(s) = \sum_n a_n/n^s$, which is absolutely convergent for $\sigma > 1$ and hence bounded in the region $\sigma \ge 1 + \epsilon$, with the bound depending on F and ϵ, but independent of σ. Hence, we have the proposition. □

Note that $\mu_F(\sigma)$ and $\mu_F^*(\sigma)$ are always non-negative for all σ. This is because $\mu_F^*(\sigma) = 0$ for $\sigma > 1$ by Proposition 3.3. Hence, if $\mu_F(\sigma_1) < 0$, then by Phragmén-Lindelöf theorem, $\mu_F(\sigma) \le 0$ in the strip $\sigma_1 \le \Re(s) \le \sigma$. Thus, we get a vertical strip where $F(s)$ is bounded and tends to 0 as $\Im(s) \to \infty$. This is a contradiction. By a similar argument, we also conclude that $\mu_F(\sigma)$ is always non-negative.

If $F \in \mathbb{S}$, by the functional equation (1), using Stirling's formula, we have (see [7], Sec.2.1])

$$\mu_F^*(\sigma) \le \frac{1}{2} d_F(1 - 2\sigma) \text{ for } \sigma < 0. \tag{8}$$

Using the Phragmén-Lindelöf theorem, we deduce that

$$\mu_F^*(\sigma) \le \frac{1}{2} d_F (1-\sigma) \text{ for } 0 < \sigma < 1.$$

The same results hold for μ_F up to a constant depending on F. To see this, we use the functional equation for F,

$$F(s) = \Delta_F(s) \overline{F(1-\overline{s})},$$

where

$$\Delta_F(s) := \omega Q^{1-2s} \prod_{j=1}^{k} \frac{\Gamma(\alpha_j (1-s) + \overline{\beta}_j)}{\Gamma(\alpha_j s + \beta_j)}.$$

Using Stirling's formula, we get

Lemma 3.4. *For $F \in \mathbb{S}$ and $t \ge 1$, uniformly in σ,*

$$\Delta_F(\sigma + it) = \left(\alpha Q^2 t^{d_F}\right)^{1/2-\sigma-it} \exp\left(itd_F + \frac{i\pi(\beta - d_F)}{4}\right)\left(\omega + O\left(\frac{1}{T}\right)\right),$$

where

$$\alpha := \prod_{j=1}^{k} \alpha_j^{2\alpha_j} \text{ and } \beta := 2\sum_{j=1}^{k} (1 - 2\beta_j).$$

By Lemma 3.4, we conclude that

$$\mu_F(\sigma) \le \frac{1}{2} d_F (1 - 2\sigma) + O(1) \text{ for } \sigma < 0. \tag{9}$$

and

$$\mu_F(\sigma) \le \frac{1}{2} d_F (1-\sigma) + O(1) \text{ for } 0 < \sigma < 1.$$

Thus, for $F \in \mathbb{S}$, these parameters $\mu_F(\sigma)$ and $\mu_F^*(\sigma)$ are well-defined (i.e., $\mu_F(\sigma), \mu_F^*(\sigma) < \infty$). We use this behaviour of μ and μ^* to introduce a growth condition. This leads to the definition of class $\mathbb{M}$.

Definition 3.5. The class $\mathbb{M}$. *Define the class $\mathbb{M}$ (see* [7, sec.2.4]*) to be the set of meromorphic functions $F(s)$ satisfying the following conditions.*

(1) **Dirichlet series** - *$F(s)$ is given by a Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which is absolutely convergent in the right half plane $\Re(s) > 1$.

(2) **Analytic continuation** - *There exists a non-negative integer k such that $(s-1)^k F(s)$ is an entire function of order ≤ 1.*

(3) **Growth condition** - *The quantity $\frac{\mu_F(\sigma)}{(1-2\sigma)}$ is bounded for $\sigma < 0$.*

(4) **Ramanujan hypothesis** - *$|a_F(n)| = O_\epsilon(n^\epsilon)$ for any $\epsilon > 0$.*

Notice that in the condition of analytic continuation, we have to force the complex order to be ≤ 1. In case of the Selberg class, this condition is implicit due to the functional equation.

We now define some invariants for $\mathbb{M}$, which generalize the notion of degree in $\mathbb{S}$.

Definition 3.6. *For $F \in \mathbb{M}$, define*

$$c_F := \limsup_{\sigma<0} \frac{2\mu_F(\sigma)}{1-2\sigma},$$
$$c_F^* := \limsup_{\sigma<0} \frac{2\mu_F^*(\sigma)}{1-2\sigma}.$$

By the growth condition, c_F and c_F^* are bounded for $F \in \mathbb{M}$. Moreover, since $\mu_F^*(\sigma) \le \mu_F(\sigma)$ for all $\Re(s) = \sigma$, we have

$$c_F^* \le c_F.$$

Note that c_F and c_F^* are ≥ 0. Using the Phragmén-Lindelöf theorem, we have

$$\mu_F(\sigma) \le \frac{1}{2} c_F (1-\sigma),$$
$$\mu_F^*(\sigma) \le \frac{1}{2} c_F^* (1-\sigma)$$

for $0 < \sigma < 1$. We mention a few examples below.

Example 1. *Any Dirichlet polynomial F belongs to $\mathbb{M}$ and $c_F = c_F^* = 0$.*

Example 2. *Any Dirichlet series F which is absolutely convergent on the whole complex plane has $c_F^* = 0$.*

Example 3. *The Riemann zeta-function $\zeta(s)$ is in $\mathbb{M}$ and $c_\zeta = c_\zeta^* = 1$.*

Example 4. *If F is an element in the Selberg class, then F also belongs to $\mathbb{M}$. Moreover, $c_F = c_F^*$ and is given by the degree of F.*

The proof of Examples 3 and 4 will follow from Proposition 3.12.

Example 5. *Linear combinations of elements in the Selberg class $\mathbb{S}$ are in $\mathbb{M}$.*

We shall see this later, when we prove that $\mathbb{M}$ forms a ring.

Another example of L-functions in $\mathbb{M}$, which are not constructed from linear combination of elements in $\mathbb{S}$ are the translates of Epstein zeta-functions.

Example 6. *For a given real positive definite $n \times n$-matrix T, the Epstein zeta-function is defined as (see* [3], [4]*)*

$$\zeta(T,s) := \sum_{\boldsymbol{0} \ne \boldsymbol{v} \in \mathbb{Z}^n} (\boldsymbol{v}^t T \boldsymbol{v})^{-s}.$$

This series is absolutely convergent for $\Re(s) > n/2$. It can be analytically continued to $\mathbb{C}$ except for a simple pole at $s = n/2$ with residue

$$\frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det T}}.$$

Moreover, it satisfies a functional equation. Let

$$\psi(T,s) := \pi^{-s/2}\Gamma(s)\zeta(T,s).$$

Then,

$$\psi(T,s) = (\det T)^{-1/2}\psi\left(T^{-1}, \frac{n}{2} - s\right).$$

Thus, the function $\zeta(T, s + n/2 - 1)$ is an element in $\mathbb{M}$. The growth condition is satisfied because of the functional equation and further we have $c_{\zeta(T,s+n/2-1)} = c^*_{\zeta(T,s+n/2-1)} = 2$.

Example 7. *If $F(s)$ belongs to $\mathbb{M}$, then all its translates given by $F(s)+a$ also belong to $\mathbb{M}$. If $F(s)$ is analytic at $s=1$, then the scalar shifts $F(rs)$ also belong to $\mathbb{M}$ for $a \in \mathbb{C}$ and real $r \geq 1$. Furthermore, if F is analytic at $s=1$, then for $\Re(a) \geq 0$, $F(s+a)$ is also in $\mathbb{M}$. In all the above cases, they have the same values of c_F and c_F^*.*

In order to ensure that $\mathbb{M}$ is closed under addition, we need to establish that these invariants c_F and c_F^* in fact satisfy an ultrametric inequality.

Proposition 3.7 ([7], Prop. 1)**.** *For $F, G \in \mathbb{M}$,*

$$c_{FG} \leq c_F + c_G \quad \text{and} \quad c_{F+G} \leq \max(c_F, c_G).$$

Similarly,

$$c_{FG}^* \leq c_F^* + c_G^* \quad \text{and} \quad c_{F+G}^* \leq \max(c_F^*, c_G^*).$$

In fact, if $c_F > c_G$ (resp. $c_F^* > c_G^*$), then

$$c_{F+G} = c_F \ (\text{resp. } c_{F+G}^* = c_F^*).$$

Proof. The proof of the above inequalities for c_F^* follows immediately from the definition of μ_F^*. This is because, for any fixed σ and $|t|>1$, we have

$$|F(\sigma+it)| \ll_\sigma |t|^{\mu_F^*(\sigma)+\epsilon},$$

for any $\epsilon > 0$. Therefore, for any $F, G \in \mathbb{M}$, we have

$$|FG(\sigma+it)| \ll_\sigma |t|^{\mu_F^*(\sigma)+\mu_G^*(\sigma)+\epsilon}.$$

Similarly,

$$\begin{aligned} |(F+G)(\sigma+it)| &\ll_\sigma |t|^{\mu_F^*(\sigma)+\epsilon} + |t|^{\mu_G^*(\sigma)+\epsilon} \\ &\ll_\sigma |t|^{\max(\mu_F^*(\sigma), \mu_G^*(\sigma))+\epsilon}. \end{aligned}$$

Incorporating this into the definition of c_F^*, we are done. By a similar argument, we also get that $c_{FG} \leq c_F + c_G$.

We are left to prove $c_{F+G} \leq \max(c_F, c_G)$. For a fixed σ, without loss of generality, assume $\mu_F(\sigma) \geq \mu_G(\sigma)$. For $s = \sigma + it$ and any $\epsilon > 0$, we have

$$\begin{aligned} |(F+G)(s)| &\leq |F(s)| + |G(s)| \\ &\leq (|s|+2)^{\mu_F(\sigma)+\epsilon} + (|s|+2)^{\mu_G(\sigma)+\epsilon} \\ &\leq 2(|s|+2)^{\mu_F(\sigma)+\epsilon}. \end{aligned}$$

Hence, from the definition of μ_F we get,

$$\mu_{F+G}(\sigma) \leq \mu_F(\sigma) + \frac{\log 2}{\log(|s|+2)}.$$

Using the above inequality in the definition of c_F, we get

$$c_{F+G} \leq \max(c_F, c_G).$$

□

It follows from the above Proposition 3.7 that $\mathbb{M}$ forms a ring.

The degree conjecture for the Selberg class claims that the degree of any element in $\mathbb{S}$ must be a non-negative integer. The question arises if we can make similar claims about these invariants c_F and c_F^*. As it turns out, c_F can take non-integer values. In fact, one can manufacture functions in $\mathbb{M}$ with any arbitrary positive real value of c_F, as we shall see in (14). But, we expect the analogue of the degree conjecture to be true for the invariant c_F^*.

Conjecture 1. *If $F \in \mathbb{M}$, then c_F^* is a non-negative integer.*

In this direction, following the argument of Conrey-Ghosh [2], in which they showed that the degree d_F in the Selberg class cannot take non-integer values between 0 and 1, we obtain the following result.

Proposition 3.8 (Corrected version of [7], Prop. 3)**.** *Suppose $F \in \mathbb{M}$. Then $c_F < 1$ implies $c_F^* = 0$.*

Proof. Let $F \in \mathbb{M}$. For $\sigma > 1$, let $F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$. Then we know that,

$$f(x) := \sum_{n=1}^{\infty} a_F(n) e^{-nx} = \frac{1}{2\pi i} \int_{(2)} F(s)\Gamma(s) x^{-s} ds,$$

where the integration is on the line $\Re(s) = 2$.

By the growth condition and convexity, F has a polynomial growth in $|t|$ in vertical strips. Thus, moving the line of integration to the left and taking into account the possible pole at $s = 1$ of $F(s)$ and poles of $\Gamma(s)$ at $s = 0, -1, -2, \cdots$, we get that

$$f(x) = \frac{P(\log x)}{x} + \sum_{n=0}^{\infty} \frac{F(-n)}{n!} (-x)^n, \tag{10}$$

where P is a polynomial. By the definition of c_F, we have

$$|F(-n)| \ll n^{\frac{1}{2} c_F (1+2n) + \epsilon}.$$

Using Stirling's formula, i.e,

$$n! \sim n^{n-1/2} e^{-n} (2\pi)^{-1/2},$$

we get

$$\left| \frac{F(-n)}{n!} (-x)^n \right| \ll n^{\frac{c_F+1}{2} + \epsilon} \left(\frac{e|x| n^{c_F}}{n} \right)^n.$$

If $c_F < 1$, then the series in equation (10) converges absolutely for all values of x. Hence, the function $f(x)$ is analytic in $\mathbb{C} \backslash \{x \le 0 : x \in \mathbb{R}\}$. But, this function is also periodic with period $2\pi i$ and so it converges for all x. Thus, the function

$$f(z) = \sum_{n=1}^{\infty} a_F(n) e^{-nz}$$

is entire. Taking $z = -1$, we get that

$$\sum_{n=1}^{\infty} a_F(n) e^n$$

is convergent and thus, $a_F(n) = o(e^{-n})$. So, the coefficients have exponential decay and therefore, $a_F(n) n^k \ll 1$ for all $k \ge 1$. Hence, we have that

$$\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

is absolutely convergent for all values of s. Therefore, $\mu_F^*(\sigma) = 0$ for all σ and hence $c_F^* = 0$. $\square$

We can completely characterize all $F \in \mathbb{M}$ such that $c_F^* = 0$. These are precisely all the functions, which when multiplied by a suitable Dirichlet polynomial give a Dirichlet series which is convergent on the whole complex plane. We invoke the following theorem of Landau to prove this result.

Theorem 3.9 (Landau, [5] Chapter VII, sec. 10, Thm. 51). *Let $F(s)$ be an entire function. Suppose $F(s)$ has a Dirichlet series representation*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

which is absolutely convergent for $\Re(s) > 1$. Also, suppose that

$$a_n = O(n^{\epsilon})$$

for all positive ϵ. If

$$F(s) = O(|t|^{\beta}) \quad (\beta > 0)$$

uniformly in the half plane $\Re(s) > \eta$, then the Dirichlet series is convergent in the half plane $\Re(s) > \eta_1$, where

$$\eta_1 = \begin{cases} \frac{\eta+\beta}{1+\beta}, & \text{if } \eta + \beta > 0 \\ \eta + \beta, & \text{if } \eta + \beta < 0. \end{cases}$$

Using the above Theorem 3.9, we get the following result classifying all elements in $\mathbb{M}$ with $c_F^* = 0$.

Proposition 3.10. *Suppose $F \in \mathbb{M}$ and let*

$$H(s) = 1 - \frac{2}{2^s}.$$

If $c_F^* = 0$, then the Dirichlet series given by

$$H(s)^k F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

is absolutely convergent on the whole complex plane, where k is the order of the possible pole of $F(s)$ at $s = 1$.

Proof. Let $F \in \mathbb{M}$. Suppose $\sigma_0(F)$ is the abscissa of absolute convergence for the Dirichlet series associated with F. If the Dirichlet series is not convergent on the whole complex plane, then we have $\sigma_0(F) > -\infty$. Note that

$$G(s) := F(s + \sigma_0(F) - 1)$$

is in $\mathbb{M}$ whose abscissa of absolute convergence is $\sigma_0(G) = 1$. Therefore, without loss of generality we assume that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n/n^s$ has abscissa of absolute convergence $\sigma_0(F) = 1$.

Since $c_F^* = 0$, we can choose a $\sigma_1 < 0$ such that

$$F(\sigma_1 + it) = O((|t| + 2)^{\epsilon(1/2 - \sigma_1)}),$$

for any $\epsilon > 0$. Using the Phragmén-Lindelöf theorem, we get

$$F(\sigma + it) = O((|t| + 2)^{\epsilon\left(\frac{1}{2} - \sigma_1\right)\left(\frac{1-\sigma}{1-\sigma_1}\right)} \tag{11}$$

for $\sigma_1 < \sigma < 1$. Let σ be fixed. For any $\epsilon_1 > 0$, choose $\epsilon = \frac{2\epsilon_1(1-\sigma_1)}{(1-2\sigma_1)(1-\sigma)}$ in (11) to get

$$|F(\sigma + it)| = O((|t| + 2)^{\epsilon_1}).$$

Suppose $F(s)$ has a pole of order k at $s = 1$. Define

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s).$$

$G(s)$ is analytic on the whole complex plane and $G \in \mathbb{M}$ with the Dirichlet series representation

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with abscissa of absolute convergence at $\sigma_0(G) = 1$. Also $c_G^* = c_F^* = 0$ by the definition of μ_F^*.

Moreover, for $\sigma_1 \ll 0$ and any $\epsilon > 0$, we have

$$|G(\sigma + it)| = O(|t|^{\epsilon})$$

uniformly for $\sigma > \sigma_1$. By Theorem 3.9, we conclude that the Dirichlet series representation of

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

converges in the half plane $\Re(s) > \sigma_1 + \epsilon$. Therefore, the abscissa of absolute convergence is $\sigma_0(G) < \sigma_1 + 1 + \epsilon < 1$, which contradicts the assumption that $\sigma_0 = 1$.

Hence $\sigma_0(G) = -\infty$ and the Dirichlet series representation of

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

is convergent on the whole complex plane and the proposition follows. □

Next, we show that the analogue of the degree conjecture given by Conjecture 1 holds between 0 and 1, i.e., c_F^* does not take non-integer values between 0 and 1.

Theorem 3.11. *If $F(s) \in \mathbb{M}$, then $c_F^* < 1$ implies $c_F^* = 0$.*

Proof. Let $c_F^* < 1$. If the Dirichlet series representation of $F(s) = \sum_n a_n/n^s$ is convergent on the whole complex plane, then $c_F^* = 0$. Now, suppose $\sum_n a_n/n^s$ absolutely converges in the half plane $\Re(s) > a$. Since $F \in \mathbb{M}$, $a \leq 1$. If $a < 1$, we can consider the shift $F(s + a - 1)$ such that its half-plane of absolute convergence is $\Re(s) > 1$. Hence, without loss of generality, we assume

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

has abscissa of absolute convergence $\sigma_0(F) = 1$. If F has a pole of order k at $s = 1$, we consider

$$G(s) := \left(1 - \frac{2}{2^s}\right)^k F(s).$$

As discussed in the proof of Proposition 3.10,

$$G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

also has abscissa of absolute convergence at $\sigma_0(G) = 1$. Moreover, $c_G^* = c_F^*$.

Hence, for any $\epsilon > 0$, we have a $\sigma_1 < 0$, such that

$$|G(\sigma_1 + it)| = O\left((|t| + 2)^{c_F^*(1/2 - \sigma_1) + \epsilon}\right). \tag{12}$$

Using Phragmén-Lindelöf theorem on the strip $\sigma_1 < \Re(s) < 1$, we get

$$|G(\sigma + it)| = O\left((|t| + 2)^{c_F^*\left(\frac{1}{2} - \sigma_1\right)\left(\frac{1-\sigma}{1-\sigma_1}\right) + \epsilon}\right) \tag{13}$$

for $\sigma > \sigma_1$. By Theorem 3.9, we conclude that the Dirichlet series of $G(s)$ converges in the half plane

$$\Re(s) > \sigma + c_F^* \left(\frac{1}{2} - \sigma_1\right)\left(\frac{1-\sigma}{1-\sigma_1}\right) + \epsilon_1$$

for $\sigma_1 < \sigma < 0$ and $\epsilon_1 > 0$. Since $c_F^* < 1$, by (12), picking σ_1 highly negative and choosing $\sigma \ll 0$ such that $\sigma - c_F^*\sigma < -2$, we have

$$\begin{aligned}\sigma + c_F^* \left(\frac{1}{2} - \sigma_1\right)\left(\frac{1-\sigma}{1-\sigma_1}\right) &= (\sigma - c_F^*\sigma) + c_F^* - \frac{c_F^*(1-\sigma)}{2(1-\sigma_1)} \\ &< -1.\end{aligned}$$

Therefore, the Dirichlet series of $G(s)$ converges on $\Re(s) > -1$. Since the abscissa of absolute convergence for the Dirichlet series of $G(s)$ is $\sigma_0(G) = 1$, we know that the abscissa of convergence $\sigma_c(G) \geq 0$, which leads to a contradiction. □

We finally show that if $F \in \mathbb{S}$, then c_F and c_F^* in fact coincide with the degree of F in the Selberg class.

Proposition 3.12. *Let $F \in \mathbb{M}$, and suppose that F has a functional equation of the Riemann type as in the Selberg class. Then $c_F = d_F$, where d_F is the degree of F in the sense of the Selberg class.*

Proof. For simplicity, assume F has only one Γ-factor. We have

$$Q^s \Gamma(as+b) F(s) = w Q^{1-\bar{s}} \Gamma(a(1-\bar{s})+b) F(1-\bar{s}),$$

where a, Q are non-negative real numbers, $b \in \mathbb{C}$ with $\Re(b) \geq 0$ and $|w| = 1$. We know from (8) that $c_F \leq d_F$. So we only need to show that c_F is at least d_F. We shall first show it for the Riemann zeta-function and use this template to prove it in general. Substituting $s = 1-2k$ for any integer $k > 0$ in the functional equation for $\zeta(s)$ gives

$$\pi^{-(1-2k)/2} \zeta(1-2k) \Gamma\left(\frac{1-2k}{2}\right) = \pi^{-k} \zeta(2k) \Gamma(k).$$

Thus, we get

$$\zeta(1-2k) = \frac{\pi^{1/2-2k} \zeta(2k) \Gamma(k)}{\Gamma(1/2-k)}.$$

Evaluating the right hand side, we have

$$\zeta(1-2k) = \frac{\pi^{1/2-2k} (k-1)!(2k)!}{(-4)^k k!} \zeta(2k).$$

Since $\zeta(2k)$ is bounded, using Stirling's formula we conclude that

$$|\zeta(1-2k)| \sim (2k)^{2k-\frac{1}{2}}\, A,$$

where $A \ll k^k$. Thus, we conclude that $c_\zeta = 1$ simply by considering the values taken by ζ on the real line.

Imitating this proof in general, substitute $s = 1-ck$ for any integer $k > 0$ and any positive constant c, such that $a(1-ck)+b$ is not an integer. The functional equation gives

$$Q^{1-ck}\, \Gamma(a+b-ack)\, F(1-ck) = w\, Q^{ck}\, \Gamma(ack+b)\, F(ck).$$

Thus, we get

$$F(1-ck) = \frac{w\, Q^{2ck-1}\, \Gamma(ack+b)\, F(ck)}{\Gamma(a+b-ack)}.$$

Repeatedly using the identity $\Gamma(s+1) = s\Gamma(s)$ and then using Stirling's formula, we get

$$|F(1-ck)| \sim |ack+b|^{2ack+c'}.A,$$

where $A \ll k^k$ and c' is a constant independent of k. Thus, we conclude that $c_F = 2a$, which is the degree of F. □

Moreover, if $F \in \mathbb{M}$ satisfies a functional equation of the Riemann type, then c_F^* is also the same as the degree d_F of F. This is an immediate consequence of Lemma 3.4.

In the absence of a functional equation, c_F and c_F^* can be very different. For $F \in \mathbb{M}$, it can happen that c_F^* is 0 and c_F is arbitrarily large. We exhibit examples below of Dirichlet series which are absolutely convergent on the whole complex plane and hence have $c_F^* = 0$, yet take large values of c_F.

Let

$$F(s) = \sum_{n=1}^{\infty} \frac{e^{-n}}{n^s}.$$

This Dirichlet series is absolutely convergent on the whole complex plane. But at the negative integers we have,

$$F(-k) = \sum_{n=1}^{\infty} e^{-n} n^k \sim \int_1^{\infty} e^{-t} t^k dt = \Gamma(k+1) + O(1).$$

Using Stirling's formula, we conclude that $c_F = 1$. Moreover, $F^r(s)$ is in $\mathbb{M}$ for any integer $r > 0$ and $c_{F^r}^* = 0$ and $c_{F^r} = r$. In fact, if we start with the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} e^{-n^r} n^{-s}, \tag{14}$$

by a similar argument we see that, for any real $r > 0$, $c_F = r$ but $c_F^* = 0$.

4. The Lindelöf class

Let $\mathbb{M}_0$ and $\mathbb{M}_0^*$ be the subsets of $\mathbb{M}$ with $c_F = 0$ and $c_F^* = 0$, respectively. Note that both $\mathbb{M}_0$ and $\mathbb{M}_0^*$ form subrings of $\mathbb{M}$, which follows immediately from Proposition 3.7. Therefore, all the non-zero elements of $\mathbb{M}_0$ (resp. $\mathbb{M}_0^*$) form a multiplicatively closed set. Call it $\mathbb{M}_{00}$ (resp. $\mathbb{M}_{00}^*$).

We define the Lindelöf class [7] by localizing at these sets,

$$\mathfrak{L} = \mathbb{M}_{00}^{-1}\mathbb{M}.$$

$$\mathfrak{L}^* = {\mathbb{M}_{00}^*}^{-1}\mathbb{M}.$$

Elements of the rings $\mathfrak{L}$ and $\mathfrak{L}^*$ satisfy growth conditions coming from $\mathbb{M}$. Moreover, after multiplication by an entire Dirichlet series, they are analytic on $\mathbb{C}$ and have Dirichlet series representation which is absolutely convergent on $\Re(s) > 1$. This is a consequence of Proposition 3.10. Since a Dirichlet polynomial is entire and has many zeroes, elements of these classes $\mathfrak{L}$ and $\mathfrak{L}^*$ may have many poles. But for any $F \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), one can always find a right half plane, where F does not have a pole. Note that $\mathbb{S} \subset \mathbb{M} \subset \mathfrak{L}^* \subset \mathfrak{L}$ and $\mathbb{S} \cap \mathbb{M}_{00} = \{1\}$. Further since $\mathbb{M}$ is a ring, the group ring $\mathbb{C}[\mathbb{S}]$ is contained in $\mathbb{M}$.

We extend the definition of the function c_F (resp. c_F^*) on $\mathfrak{L}$ (resp. $\mathfrak{L}^*$). If $G \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), then $G = F/\mu$, where $\mu \in \mathbb{M}_{00}$ (resp. $\mathbb{M}_{00}^*$) and $F \in \mathbb{M}$. We define $c_G = c_F$ (resp. $c_G^* = c_F^*$). Note that this definition is compatible with the definition of c_F based on the growth condition. We first check that the function c_F (resp. c_F^*) is well-defined on $\mathfrak{L}$ (resp. $\mathfrak{L}^*$). If we write $G = H/\nu$ for some other $H \in \mathbb{M}$ and $\nu \in \mathbb{M}_{00}$, and suppose $c_H > c_F$, then $c_{F\nu - H\mu} = c_H$. But, by definition $c_{F\nu - H\mu} = 0$, which leads to a contradiction. Thus, c_F is well-defined on $\mathfrak{L}$. Moreover, we also have that a non-zero F is a unit of $\mathfrak{L}$ if and only if $c_F = 0$. We use the same argument to show that c_F^* is well-defined on $\mathfrak{L}^*$.

5. Ring theoretic properties of $\mathbb{M}$, $\mathfrak{L}$ and $\mathfrak{L}^*$

The classes of L-functions $\mathbb{M}, \mathfrak{L}$ and $\mathfrak{L}^*$ form rings. Moreover, since they consist of meromorphic functions, they form integral domains. We further show that the ring $\mathbb{M}$ is non-Noetherian and non-Artinian.

Proposition 5.1. *$\mathbb{M}, \mathfrak{L}$ and $\mathfrak{L}^*$ are non-Artinian.*

Proof. Consider $F \in \mathbb{M}$ with $c_F^* > 0$. Let $\langle F \rangle$ be the ideal generated by F in $\mathbb{M}$, which is clearly a non-trivial proper ideal of $\mathbb{M}$. We have the following strictly decreasing sequence of ideals in $\mathbb{M}$ given by

$$\langle F \rangle \supsetneq \langle F^2 \rangle \supsetneq \langle F^3 \rangle \cdots. \tag{15}$$

Thus, $\mathbb{M}$ is not Artinian. Moreover, $\langle F \rangle \subset \mathbb{M}$ generates a non-trivial ideal $I \subset \mathfrak{L}$ and $I^* \in \mathfrak{L}^*$ and we have strictly decreasing sequence of ideals.

$$I \supsetneq I^2 \supseteq I^3 \cdots,$$

$$I^* \supsetneq (I^*)^2 \supsetneq (I^*)^3 \cdots.$$

Hence, we conclude that $\mathfrak{L}$ and $\mathfrak{L}^*$ are non-Artinian. □

Now we show that $\mathbb{M}$ is non-Noetherian.

Lemma 5.2. *If $F(s) \in \mathbb{M}$ and $F(s)$ is entire, then $F_1(s) := F(s + s_0) \in \mathbb{M}$ for $\Re(s_0) > 0$.*

Proof. Again, properties (1),(2) and (4) in the definition of $\mathbb{M}$ hold clearly. We only need to check the growth condition.

$$c_{F_2} = \limsup_{\sigma<0} \frac{2\mu_{F_2}(\sigma)}{1 - 2\sigma} = \limsup_{\sigma<0} \frac{2\mu_F(\sigma)}{1 - 2\sigma} = c_F.$$

□

Lemma 5.2 shows that shifts of entire functions in $\mathbb{M}$ also lie in $\mathbb{M}$. Recall that in $\mathbb{S}$, vertical shifts of entire functions in $\mathbb{S}$ lie in $\mathbb{S}$. In case of $\mathbb{M}$, we no longer have to restrict ourselves to only vertical shifts.

Proposition 5.3. *$\mathbb{M}$ is non-Noetherian.*

Proof. As earlier, we prove this by explicitly constructing a strictly increasing infinite chain of ideals. For $s \neq 1$, let

$$I_s := \{F \in \mathbb{M} : F(s) = 0\}.$$

Then, I_s forms an ideal of $\mathbb{M}$. Define

$$K_1 := \bigcap_{k=1}^{\infty} I_{-2k}.$$

We know that $L(s,\chi) \in K_1$, where χ is any even primitive Dirichlet character, because it vanishes at all negative even integers. By Lemma 5.2, $L(s+2n,\chi) \in \mathbb{M}$ and it vanishes on all even negative integers except $\{-2k : k \le n, k \in \mathbb{N}\}$. We define

$$K_i := \bigcap_{k=i}^{\infty} I_{-2k}.$$

Clearly, $K_1 \subseteq K_2 \subseteq K_3 \subseteq .....$ Since we have exhibited that the function $L(s+2n,\chi)$ belongs to K_{n+1} but not K_n. Thus, we get

$$K_1 \subsetneq K_2 \subsetneq K_3 \subsetneq ...$$

Therefore, $\mathbb{M}$ is non-Noetherian. □

We are not yet able to show that $\mathfrak{L}$ and $\mathfrak{L}^*$ are non-Noetherian, although we expect it to be true. Note that the ring $\mathbb{C}[\mathbb{S}]$ is a subring of $\mathbb{M}$. We expect that $\mathbb{C}[\mathbb{S}]$ is also non-Noetherian. Indeed we show below that this is a consequence of Selberg's orthonormality conjecture. This was already known due to Molteni [6].

Definition 5.4. *An element $F \neq 1 \in \mathbb{S}$ is said to be a primitive element if it cannot be further factorized in $\mathbb{S}$ i.e., if $F = F_1F_2$ with $F_i \in \mathbb{S}$, then either $F_1 = 1$ or $F_2 = 1$.*

Selberg's orthonormality conjecture states that:

Conjecture (Selberg's Orthonormality Conjecture). *Let $F, G \in \mathbb{S}$ be any two primitive elements, whose Dirichlet series expansion on $\mathfrak{R}(s) > 1$ is given by $F(s) = a_F(n)n^{-s}$ and $G(s) = a_G(n)n^{-s}$. Then,*

$$\sum_{p \le x} \frac{a_p(F)\overline{a_p(G)}}{p} = \begin{cases} \log\log x + O(1), & \text{if } F = G \\ O(1), & \text{if } F \neq G. \end{cases}$$

Proposition 5.5. *Selberg's orthonormality conjecture implies that $\mathbb{C}[\mathbb{S}]$ is non-Noetherian.*

Proof. It was observed by Selberg in [9] and Bombieri-Hejhal in [1] that distinct elements in the Selberg class are linearly independent over $\mathbb{C}$. We show that Selberg's conjecture implies that distinct primitive elements in the Selberg class are algebraically independent. Since there are infinitely many primitive elements in $\mathbb{S}$, we conclude that $\mathbb{C}[\mathbb{S}]$ is non-Noetherian. Selberg's orthonormality conjecture implies that the factorization into primitive elements in the Selberg class is unique, see [2]. Suppose, $F_1, F_2, ..., F_n$ are distinct primitive elements in $\mathbb{S}$ satisfying a polynomial $P(x_1, x_2, ..., x_n) \in \mathbb{C}[x_1, x_2, ..., x_n]$. By linear independence of distinct elements in $\mathbb{S}$, we conclude that not all terms in the polynomial expansion of $P(F_1, ..., F_n)$ are distinct. Thus, we have relations of the form

$$F_1^{a_1} F_2^{a_2} ... F_n^{a_n} = F_1^{b_1} F_2^{b_2} ... F_n^{b_n}, \tag{16}$$

where not all the a_i's are the same as the b_i's. But, both left hand side and right hand side in (16) are elements in the Selberg class. This contradicts the unique factorization. □

Since $\mathbb{C}[\mathbb{S}] \subseteq \mathbb{M}$ and we know that $\mathbb{M}$ is non-Noetherian, the above proposition can be thought of as some indicative evidence towards the validity of unique factorization into primitives in the Selberg class.

5.1. **Primitive elements in $\mathbb{M}$ and $\mathfrak{L}^*$.** The notion of primitive elements in $\mathbb{S}$ can be extended to the class $\mathbb{M}$ and $\mathfrak{L}^*$.

Definition 5.6. *We say that an element $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) is primitive if $c_F^* > 0$ and $F = F_1F_2$ implies that either $c_{F_1}^* = 0$ or $c_{F_2}^* = 0$.*

Note that every element $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) with $c_F^* = 1$ is primitive, which directly follows from Theorem 3.11. But this is not quite true for $\mathfrak{L}$. This is because we could have an entire Dirichlet series $F(s)$ with $c_F > 0$, which can be written as a product of infinitely many entire Dirichlet series $\prod_i F_i(s)$, each of which have $c_{F_i} > 0$ for each i. Hence, we avoid defining the notion of primitive elements for the class $\mathfrak{L}$.

We now show that every element in $\mathbb{M}$ and $\mathfrak{L}^*$ can be written as a product of primitive elements. We follow the same argument as in the case of the Selberg class $\mathbb{S}$.

Proposition 5.7. *Let $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$), then $F(s)$ can be be written as a finite product of primitive elements in $\mathfrak{L}^*$ (resp. $\mathbb{M}$).*

Proof. Let $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) and suppose

$$F = F_1F_2,$$

where $F_1, F_2 \in \mathfrak{L}^*$ (resp. $\mathbb{M}$). By properties of c_F^*, we know that $c_F^* \leq c_{F_1}^* + c_{F_2}^*$. If $c_{F_i}^* < 1$, then by Theorem 3.11, $c_{F_i}^* = 0$. Hence, we cannot factorize F indefinitely into non-units (i.e. elements with $c^* > 0$). Therefore, $F(s)$ has a factorization into primitive elements. □

With the above notion of primitivity, we may ask whether this factorization is unique.

Conjecture 2. *Every element $F \in \mathfrak{L}^*$ (resp. $\mathbb{M}$) can be uniquely factorized into primitive elements.*

Assuming unique factorization, we conclude the algebraic independence of distinct primitive elements in $\mathbb{M}$ and $\mathfrak{L}^*$.

Proposition 5.8. *Conjecture 2 implies that linearly independent primitive elements in $\mathfrak{L}^*$ (resp. $\mathbb{M}$) are algebraically independent.*

Proof. The proof follows similar approach of Proposition 5.5. Let $F, G \in \mathfrak{L}^*$ be linearly independent primitive elements satisfying a polynomial equation $P(F,G) = 0$. From Proposition 3.7, we conclude that if the terms in the polynomial with largest c^* value, say d, must cancel each other. In other words, if $P(x,y) = \sum_{m,n} a_{m,n}x^my^n$, then

$$\sum_{mc_F^* + nc_G^* = d} a_{m,n}F^mG^n = 0.$$

Since F and G are linearly independent, we also have that $d > 0$. Cancelling the common factors, we get an expression of the form

$$F^k = \sum_{m<k,n} b_{m,n}F^mG^n,$$

where each term in the RHS has a factor of G and hence $F^k/G \in \mathfrak{L}^*$ (resp. $\mathbb{M}$). This contradicts the unique factorization for F^k. Hence, F and G are algebraically independent. □

We now show that $\mathbb{M}, \mathfrak{L}$ and $\mathfrak{L}^*$ are closed under differentiation.

Proposition 5.9. *If $F \in \mathbb{M}$, then $F' \in \mathbb{M}$. This is also true if we replace $\mathbb{M}$ by $\mathfrak{L}$ or $\mathfrak{L}^*$. Moreover, $c_F \geq c_{F'}$ and $c_F^* \geq c_{F'}^*$.*

Proof. If $F \in \mathbb{M}$, then the properties (1),(2) and (4) clearly hold for F'. We only have to check the growth condition. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$. If f is bounded on the half plane $\{z : \Re(z) < -N_1\}$ by M, for some $N_1 > 0$, then $f'(z)$ is also bounded on the half-plane $\{z : \Re(z) < -N_2\}$, for some $N_2 > 0$. We can see this by using Cauchy's formula namely,

$$f'(a) = \frac{1}{2\pi i}\int_{C(\epsilon,a)} \frac{f(z)}{(z-a)^2}\,dz,$$

where $C(\epsilon, a)$ is the circle of radius ϵ centered at a and f is analytic in the interior of $C(\epsilon, a)$. Since $f(z)$ is bounded by M on $C(\epsilon, a)$, we get

$$|f'(a)| \leq \frac{M}{\epsilon}.$$

If we choose $N_2 > N_1 + 2$, for every point in $\{z : \Re(z) < -N_2\}$, we can set $\epsilon > 1$, and thus get $f'(z)$ is bounded by M in the region $\{z : \Re(z) < -N_2\}$.

Now, if $F(z) \in \mathbb{M}$, then by the growth condition we know that for any $\epsilon > 0$,

$$g(z) := \frac{F(z)}{z^{c_F(1-2\sigma)+\epsilon}}$$

is bounded in the half-plane $\{z : \Re(z) < -N_1\}$, for some $N_1 > 0$. Therefore, we have for some $N_2 > 0$, in the half-plane $\{z : \Re(z) < -N_2\}$,

$$\left|\left(\frac{F(z)}{z^{c_F(1-2\sigma)+\epsilon}}\right)'\right| = \left|\frac{F'(z)z^{c_F(1-2\sigma)+\epsilon} - \left(z^{c_F(1-2\sigma)+\epsilon}\right)' F(z)}{z^{2c_F(1-2\sigma)+2\epsilon}}\right| < M$$

$$\implies |F'(z)||z|^{c_F(1-2\sigma)+\epsilon} < M|z|^{2c_F(1-2\sigma)+2\epsilon} + \left|\left(z^{c_F(1-2\sigma)+\epsilon}\right)'\right| |F(z)|\,.$$

If we fix σ, then

$$|F'(z)||z|^{c_F(1-2\sigma)+\epsilon} < M'|z|^{2c_F(1-2\sigma)+2\epsilon}.$$

Hence,

$$|F'(z)| < M''|z|^{c_F(1-2\sigma)+\epsilon}.$$

Thus, we get the growth condition on $F'(z)$. Moreover, we also conclude that if $F \in \mathbb{M}$, then $c_{F'} \leq c_F$ and $c^*_{F'} \leq c^*_F$. The proof of the statement for $\mathfrak{L}$ (resp. $\mathfrak{L}^*$) follows by proving the fact that the derivative of a unit in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$) is also a unit in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$). Then, for any $F \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), we can find a unit $\nu \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$), such that $\nu F \in \mathbb{M}$, after which we can use the above argument to say $(\nu F)' \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$). But $(\nu F)' = \nu' F + \nu F'$ and thus $F' \in \mathfrak{L}$ (resp. $\mathfrak{L}^*$).

□

6. Ideals in $\mathfrak{L}$ and $\mathfrak{L}^*$

As a first step to understanding the ideal theory of $\mathfrak{L}$ (resp. $\mathfrak{L}^*$), we construct some non-trivial ideals of $\mathfrak{L}$ (resp. $\mathfrak{L}^*$.) We use the following proposition for the construction.

Proposition 6.1. *If $F(s)$ is a non-constant entire Dirichlet series, then $F(s)$ cannot have zeroes on any arithmetic progression,*

$$S = \{a, a-d, a-2d, \ldots\}, \tag{17}$$

where $d \in \mathbb{R}^+$ and $a \in \mathbb{C}$.

Proof. First, we show that if
$$F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$
is convergent on the whole complex plane, it cannot have zeroes on non-positive integers. In other words, it cannot vanish on $\{0,-1,-2,-3...\}$. Since, $\sum_{n=1}^{\infty}a_n n^{-s}$ is entire, we have
$$|a_n|\ll\frac{1}{n^k}. \tag{18}$$
for all $k\in\mathbb{N}$. Define
$$f(x):=\sum_{n=1}^{\infty}a_n x^n.$$
By the root test and using (18), we conclude that the power series $f(x)$ defines an analytic function on the whole complex plane. Consider the Taylor series expansion around $x=1$, given by,
$$f(x)=\sum_{n=0}^{\infty}b_n(x-1)^n, \tag{19}$$
where,
$$b_n=\frac{f^{(n)}(1)}{n!}.$$
Now, suppose $F(s)$ takes zeroes on all non-positive integers, we have
$$F(-k)=\sum_{n=1}^{\infty}a_n n^k=0,$$
for all $k\in\mathbb{N}\cup\{0\}$. In particular, we get $b_0=F(0)=0$ and $b_1=F(-1)=0$. In general,
$$b_k=\sum_{n=k}^{\infty}\binom{n}{k}a_n.$$
We write it as,
$$\sum_{n=k}^{\infty}\binom{n}{k}a_n=\sum_{n=1}^{\infty}\frac{n(n-1)...(n-k+1)}{k!}a_n, \tag{20}$$
where the first $(k-1)$-terms of the right hand side of (20) are 0. Moreover, each term in the right hand side is a polynomial in n of degree k. Since, $\sum_{n=1}^{\infty}a_n n^k$ is absolutely convergent for all k, we can rearrange the terms in the summation. Thus, we get,
$$b_k=\sum_{n=1}^{\infty}\binom{n}{k}a_n=c_0\sum_{n=1}^{\infty}a_n+c_1\sum_{n=1}^{\infty}na_n+...+c_k\sum_{n=1}^{\infty}n^k a_n,$$
where c_i's are some real constants. Therefore,
$$b_k=c_0F(0)+c_1F(-1)+...+c_kF(-k)=0,$$
for all k. Hence, $f(x)$ is identically zero, which leads to $F(s)$ being identically 0.

We deduce the general case of an arithmetic progression S as in the statement of the proposition, by considering
$$F_1(s):=F(sd+a)=\sum_{n=1}^{\infty}\frac{a_n}{n^a}\frac{1}{n^{sd}}.$$

This function is no longer a standard Dirichlet series but it converges for all $s \in \mathbb{C}$ and vanishes at all non-positive integers. Now, consider the series

$$f(x) := \sum_{n=1}^{\infty} \frac{a_n}{n^a} x^{n^d}.$$

Choosing the principal branch of log, the function $f(x)$ is well-defined and absolutely convergent on $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$. The rest of the proof is similar as above. □

Consider the ideal in $\mathbb{M}$ given by,

$$I_S = \bigcap_{s \in S} I_s,$$

where I_s is as in Proposition 5.3 and S is an arithmetic progression as in (17). I_S is non-empty since it contains elements in $\mathbb{S}$. By Proposition 6.1, I_S does not contain any non-zero entire Dirichlet series. By definition,

$$\mathfrak{L} \text{ (resp. } \mathfrak{L}^*) = \mathbb{M}_{00}^{-1}\mathbb{M} \text{ (resp. } {\mathbb{M}_{00}^*}^{-1}\mathbb{M}),$$

where $\mathbb{M}_{00}$ and $\mathbb{M}_{00}^*$ consist of Dirichlet series which are convergent on $\mathbb{C}$, up to a Dirichlet polynomial (by Proposition 3.10). Hence, I_S generates a non-trivial ideal in $\mathfrak{L}$ (resp. $\mathfrak{L}^*$).

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