ON LIMIT THEOREM FOR ZETA FUNCTIONS

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Dedicated to the memory of Prof. M. V. Subbarao

Abstract. The Riemann zeta-function is one of the most extensively studied functions in number theory. This is because its zero-distribution encapsulates important arithmetic information. Bohr initiated the investigation into the value distribution theory of this function. In 1929, Bohr and Jessen proved a probabilistic limit theorem for the values of Riemann zeta-function in the critical strip. Later, such limit theorems were established for several other L-functions. In this expository article, we motivate and discuss this theme.

1. **Introduction**

The intimate relation between arithmetic objects and their associated L-functions forms the backbone of modern analytic number theory. The prototypical example of an L-function is the Riemann zeta-function $\zeta(s)$, which is defined on the half-plane $\Re(s) > 1$ as the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

A priori, this looks like an innocent series made up of natural numbers. But the analytic study of $\zeta(s)$ gives astounding results on distribution of primes. By analytic continuation, it is possible to extend $\zeta(s)$ to the whole complex plane, except at $s = 1$. The zero-distribution of $\zeta(s)$ in this extended region, where the series does not converge, holds the key to significant arithmetic data on natural numbers. This motivates a careful study of distribution of values of $\zeta(s)$ and other L-functions. In this direction, one of the earliest breakthroughs came in the first half of 20th century, when Bohr and Jessen proved the asymptotic value distribution of $\zeta(s)$ on vertical lines in the region $1 \geq \Re(s) > 1/2$. This is called the Bohr-Jessen limit theorem. This result inspired decades of investigation into this topic, notably leading to the celebrated universality theorem due to Voronin in 1975.

There is vast literature studying different facets of the Bohr-Jessen limit theorem and its various generalizations. However, it is too technical for the uninitiated. Lately, significant progress on this topic has surfaced with the proof of the associated density theorem for automorphic L-functions and the limit theorem for periodic Hurwitz zeta-functions. Driven by this, we present this expository article, targeted at senior undergraduate and graduate students in mathematics, which gently introduces the theme of limit theorems and aims to capture the essential flavour of the topic.

Date: June 22, 2022.

2020 *Mathematics Subject Classification.* 11-06, 11M06, 11R47.

Key words and phrases. L-functions, value distribution, Bohr-Jessen limit theorem, Dedekind zeta-function.

The research of the author is supported by an Inspire Faculty fellowship.

2. The Riemann zeta-function

The Riemann zeta-function is defined on $\Re(s) > 1$ as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

Owing to the unique factorization of integers, $\zeta(s)$ has an Euler product representation on $\Re(s) > 1$, given by

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2}$$

This product representation is significant and has far reaching consequences. Euler studied the series (1) as a function of a real variable and by (2), he proved the infinitude of primes. He observed that as $\sigma \to 1^+$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1}$$

tends to infinity. However, if there were finitely many primes, the RHS would be finite, leading to a contradiction. Furthermore, his argument also yields that the series

$$\sum_p \frac{1}{p}$$

diverges, thus providing more information on the density of primes than merely their infinitude. This observation, along with astronomical developments in the theory of complex analysis, led Riemann to study $\zeta(s)$ as a function of a complex variable $s = \sigma + it$. His brilliance was in realizing the intimate connection between the distribution of prime numbers and the zero distribution of $\zeta(s)$.

2.1. **Functional equation and trivial zeros.** The series (1) of $\zeta(s)$ is absolutely convergent for $\Re(s) > 1$. In his pivotal paper, Riemann [32] showed that $\zeta(s)$ can be analytically continued to the whole complex plane except for a simple pole at $s = 1$ with residue 1. He also established a symmetry in the value distribution of $\zeta(s)$. More precisely, he proved that the function

$$\psi(s) = \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

satisfies the functional equation $\psi(s) = \psi(1-s)$.

Since the only pole of $\zeta(s)$ is at $s = 1$, $\psi(s)$ has simple poles at $s = 0$ and $s = 1$. From the Euler product, it is easily deduced that $\zeta(s)$ is non-vanishing on the half-plane $\Re(s) > 1$. Hence, $\psi(s)$ is non-vanishing in this region and by the functional equation, $\psi(s)$ is non-vanishing on $\Re(s) < 0$. Recall that $\Gamma(s)$ has simple poles at all non-positive integers. This implies that $\zeta(s)$ is non-vanishing on $\Re(s) < 0$ except on the set of negative even integers

$$\{-2, -4, -6, \cdots\}.$$

These are called the trivial zeros of $\zeta(s)$.

We are left to consider the vanishing of $\zeta(s)$ in the strip $0 \leq \Re(s) \leq 1$. Owing to the symmetry arising from the functional equation, Riemann predicted that all the non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$. This is the famous Riemann hypothesis, which still remains open and is one of the millennium problems in mathematics.

Conjecture 1 (Riemann hypothesis). *All the non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.*

2.2. **Prime number theorem.** Let $\pi(x)$ denote the number of primes $\leq x$. In 1849, in a communication with Encke, Gauss conjectured that

$$\pi(x) \sim li(x) = \int_2^x \frac{1}{\log t}\, dt \tag{3}$$

as $x \to \infty$. His conjecture was based on explicit calculations carried out as early as 1792-93. In [32], Riemann introduced the idea of applying methods in complex analysis to the study of the real function $\pi(x)$. Extending Riemann's ideas, two proofs of (3) were discovered independently by de la Vallée Poussin and Hadamard in 1896. This is famously known as the prime number theorem. The main step in the proof is to establish that for any $t \neq 0$,

$$\zeta(1 + it) \neq 0.$$

Thus, the non-vanishing of $\zeta(s)$ on the 1-line yields quite the stunning asymptotic formula for the distribution of primes, namely

$$\pi(x) = li(x) + O\left(x \exp(-c\sqrt{\log x})\right)$$

for a positive constant $c > 0$. One may wonder, if better error terms can be obtained from more knowledge on the non-vanishing of $\zeta(s)$ in the critical strip. Indeed, if one assumes the Riemann hypothesis, i.e., non-vanishing of $\zeta(s)$ in $\Re(s) > 1/2$, it is possible to show that

$$\pi(x) = li(x) + O\left(x^{1/2} \log x\right).$$

In fact, the non-vanishing of $\zeta(s)$ in $\Re(s) > 1 - \rho$, for some $\rho > 0$ improves the error term above to $O(x^{\rho+\epsilon})$ for any $\epsilon > 0$. We still do not know any such strip $\rho < \Re(s) < 1$ where $\zeta(s)$ does not vanish. De la Vallée Poussin proved that $\zeta(\sigma + it)$ does not vanish for

$$1 - \sigma \ll \frac{1}{\log |t|}.$$

for $|t| > 2$. This region was subsequently improved by many, significantly by Littlewood [25] in 1921 and Vinogradov [37] in 1958, who showed that $\zeta(\sigma + it)$ does not vanish when

$$1 - \sigma \ll \frac{1}{(\log |t|)^{2/3} (\log \log |t|)^{1/3}}.$$

for $|t| > 2$. Improving this zero-free region is an active area of research and every small improvement leads to further knowledge about the distribution of primes.

3. Value distribution: Denseness

Having realised the importance of the zero-free regions of $\zeta(s)$, it is natural to take a closer look and understand the distribution of values of $\zeta(s)$. In early 20th century, Bohr initiated the study of the value distribution of $\zeta(s)$ on subsets of $\mathbb{C}$.

From the series representation of $\zeta(s)$, it is clear that if we fix t, then $\zeta(\sigma + it) \to 1$ as $\sigma \to +\infty$. Thus, on horizontal lines in $\mathbb{C}$, $\zeta(s)$ always approaches 1 as we move to $+\infty$. However, the story is very different on vertical lines.

Proposition 3.1. *For a fixed $\sigma > 1$,*

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |\zeta(\sigma + it)| \leq \zeta(\sigma).$$

Proof. For $\sigma > 1$, $|\zeta(\sigma + it)| \leq \zeta(\sigma)$ as

$$|\zeta(\sigma + it)| = \left| \sum_{n=1}^{\infty} \frac{1}{n^{\sigma + it}} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} = \zeta(\sigma).$$

Recall that the Möbius function defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ product of distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_{d|n} \mu(d) = 0$ for $n > 1$, it is easily checked that for $\Re(s) > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Thus, for $\sigma > 1$

$$\left| \frac{1}{\zeta(\sigma + it)} \right| \leq \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{\sigma}} = \prod_p \left(1 + \frac{1}{p^{\sigma}} \right) = \frac{\prod_p \left(1 - \frac{1}{p^{\sigma}} \right)^{-1}}{\prod_p \left(1 - \frac{1}{p^{2\sigma}} \right)^{-1}} = \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

Therefore, $|\zeta(\sigma + it)| \geq \zeta(2\sigma)/\zeta(\sigma)$.

$\square$

With a careful application of the Kronecker approximation theorem (see [9]), it is possible to show that for $\sigma > 1$, the set $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ is dense in the annulus

$$\left\{ z : \frac{\zeta(2\sigma)}{\zeta(\sigma)} \leq |z| \leq \zeta(\sigma) \right\}. \tag{4}$$

For a fixed $\sigma > 1$, Bohr observed that the function $f(t) = \zeta(\sigma + it)$ has behaviour close to that of a periodic function. This phenomena is called *Bohr's almost periodicity theorem.*

Theorem 3.2 (Bohr's almost periodicity [5])**.** *Fix $\sigma_0 > 1$. For any $\epsilon > 0$, there exists $N > 0$ such that*

$$|\zeta(s + it) - \zeta(s)| < \epsilon$$

for some $t \in [0, N]$ holds for all s with $\Re(s) > \sigma_0$.

In fact, Bohr showed that the property of almost periodicity holds for all Dirichlet series $\sum_n a_n/n^s$ in its region of absolute convergence. Now, using (4), Theorem 3.2 and the open mapping theorem, one can deduce the following.

Proposition 3.3. *On the half plane $\Re(s) > \sigma_0 > 1$, $\zeta(s)$ takes all values in the annulus*

$$\left\{ z : \frac{\zeta(2\sigma_0)}{\zeta(\sigma_0)} < |z| < \zeta(\sigma_0) \right\}$$

infinitely often.

Taking $\sigma_0 \to 1^+$, the annulus extends to the region $\mathbb{C}\backslash\{0\}$. Thus, one obtains the following theorem, which is also due to Bohr.

Theorem 3.4 (Bohr [3])**.** *For any $\epsilon > 0$, $\zeta(s)$ takes all non-zero complex values infinitely often in the region $1 < \Re(s) < 1 + \epsilon$.*

3.1. **Idea of Bohr.** In the region of absolute convergence $\Re(s) > 1$, basic analysis already provides a good understanding of the value distribution of $\zeta(s)$. The more mysterious and less understood region is the critical strip $0 < \Re(s) < 1$. One of the main obstructions to study $\zeta(s)$ in the critical strip is that the Euler product does not converge. In this regard, the brilliant idea of Bohr was to consider partial Euler products

$$\zeta_M(s) = \prod_{p<M} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

As $M \to \infty$, this product does not converge in the critical strip. However, for large M, this partial Euler product well approximates $\zeta(s)$ in the "mean". To make it more precise, let $D \subset \{z : 1/2 < \Re(z) < 1\}$ be a compact set with open interior. Given a large M, for every $\epsilon > 0$, we can find a sufficiently large T depending on ϵ such that

$$\int_T^{2T} \iint_D \left| \frac{\zeta(s+it)}{\zeta_M(s+it)} - 1 \right|^2 d\mathbf{z}\, dt \ll_\epsilon T.$$

In other words, partial Euler products are close to $\zeta(s)$ in the critical strip "very often". By studying these partial Euler products, Bohr and Courant established the following result on vertical lines in the critical strip.

Theorem 3.5 (Bohr-Courant [4])**.** *For a fixed $\sigma \in (1/2, 1]$, the set $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}$.*

Multidimensional analogues of Theorem 3.5 were proved by Voronin [38] in the 1972. For $s, s_1, s_2, \cdots, s_k$ in the strip $1/2 < \Re(s), \Re(s_1), \cdots \Re(s_k) < 1$, he showed that the sets

$$\{(\zeta(s_1 + it), \zeta(s_2 + it), \cdots, \zeta(s_k + it)) : t \in \mathbb{R}\}$$

and

$$\left\{ \left(\zeta(s + it), \zeta'(s + it), \cdots, \zeta^{(k-1)}(s + it) \right) : t \in \mathbb{R} \right\}$$

are both dense in $\mathbb{C}^k$. All these results are indicative of the fact that both shifts as well as derivatives of $\zeta(s)$ are "independent" or "orthogonal" in nature.

We still do not know whether the set $\{\zeta(1/2+it) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}$, although it is believed to be true. The best known result regarding the value distribution on the 1/2-line is Selberg's central limit theorem [34], [35] which states that

$$\lim_{T\to\infty} \frac{1}{T}\, meas \left\{ t \in [0,T] : \frac{\log \zeta(1/2+it)}{\sqrt{\frac{1}{2}\log\log T}} \in R \right\} = \frac{1}{2\pi} \iint_R \exp\left(-\frac{x^2+y^2}{2}\right) dxdy,$$

where R is a fixed rectangle in $\mathbb{C}$.

4. Value distribution: limit theorem

Let $s = \sigma + it$ be a complex variable and R be a fixed rectangle in the complex plane $\mathbb{C}$, with sides parallel to the real and imaginary axes. The Lebesgue measure is denoted as *meas*. For $\sigma > 1/2$ and $T > 0$, define

$$V_\sigma(T, R; \zeta) := meas\, \{t \in [-T, T] : \log \zeta(\sigma + it) \in R\}.$$

Theorem 4.1 (Bohr-Jessen [6], [7]). *For $\sigma > 1/2$, the limit*

$$W_\sigma(R;\zeta) := \lim_{T\to\infty} \frac{1}{2T} V_\sigma(T, R; \zeta)$$

exists.

This is famously known as the *Bohr-Jessen limit theorem*. The value of the limit $W_\sigma(R;\zeta)$ is the probability with which $\log\zeta(\sigma+it)$ takes value in R as t runs over $\mathbb{R}$. This limiting distribution was explicitly computed by Hattori and Matsumoto in [14].

Moreover, Bohr and Jessen also constructed a continuous function $\mathcal{M}_\sigma(z,\zeta)$ on $\mathbb{C}$ taking non-negative real values such that

$$W_\sigma(R;\zeta) = \int_R \mathcal{M}_\sigma(z,\zeta)|dz|,$$

where $|dz| = \frac{1}{2\pi}\, dx\, dy$. This function $\mathcal{M}_\sigma(z;\zeta)$ is called the *density function* for $\zeta(s)$.

This classical limit theorem can be considered as the starting point of study of value distribution theory of $\zeta(s)$ and related functions. Since then, several alternate proofs of the limit theorem, using modern techniques of probability theory have been established. We refer the interested reader to Jessen-Wintner [16], Borchsenius-Jessen [8], Guo [12], Ihara-Matsumoto [15] and Laurinčikas [21], [22]. We outline the proof of the limit theorem following Laurinčikas below.

4.1. **Probabilistic methods and weak convergence.** Let $A \in \mathcal{B}(\mathbb{C})$ be a Borel set in $\mathbb{C}$. For a given $T > 0$, define the probability measure P_T on $\mathbb{C}$ as

$$P_T(A) = \frac{1}{T}\, meas\, \{t \in [0,T] : \zeta(s+it) \in A\}.$$

The Bohr-Jessen limit theorem states that P_T converges to a probability measure P as $T \to \infty$. In this section, we will describe this probability measure P.

More generally, we consider the function space analogue of the Bohr-Jessen limit theorem. The first limit theorem of this type is due to Bagchi [1]. For a concise survey of limit theorem on function spaces, the reader may refer to Laurinčikas [19].

Theorem 4.2 (Bagchi)**.** *Let $D = \{z : 1/2 < \Re(z) < 1\}$ and $\mathcal{H}(D)$ denote the space of analytic functions on D. $\mathcal{H}(D)$ comes equipped with a canonical metric (see* [10, Chapter VII]*) and is a complete separable metric space. Let $A \in \mathcal{B}(\mathcal{H}(D))$ be a Borel set in $\mathcal{H}(D)$. Define*

$$P_T(A) = \frac{1}{T} meas\,\{t \in [0,T] : \zeta(s+it) \in A\}\,.$$

Then, P_T converges weakly to a probability measure P as $T \to \infty$.

4.2. **Key elements in the proof.** We outline the main ideas in the proof of Theorem 4.2 as discussed in [19] and [36]. Consider the probability measure corresponding to absolutely convergent partial Dirichlet series,

$$F_N(s) := \sum_{n=1}^{N} \frac{1}{n^s}.$$

For $N > 0$ and $A \in \mathcal{B}(\mathcal{H}(D))$ as before, define the probability measure

$$P_{T,N}(A) = \frac{1}{T} meas\,\{t \in [0,T] : F_N(s+it) \in A\}\,. \tag{5}$$

Since F_N is a finite sum, it is possible to show that $P_{T,N}$ converges to P_N as $T \to \infty$. It is to be cautioned here that in the proof, one considers partial sums of the form $\sum_{n=1}^{N} \exp((-n/M)^\sigma/n^s$. However, in order to keep the exposition simple, we avoid such nuances. The reader may refer to [36, Chapter 4] for details.

The next step is to invoke certain ideas from probability theory. A family $\{P_n\}$ of probability measures on $(S, \mathcal{B}(S))$ is said to be *relatively compact* if every sequence of elements of $\{P_n\}$ contains a weakly convergent subsequence. A family $\{P_n\}$ of probability measures is called *tight* if for any $\epsilon > 0$, there exists a compact set K such that $P_n(K) > 1 - \epsilon$ for all $n \in \mathbb{N}$. A powerful tool in the theory of weak convergence of probability measures is Prokhorov's theorem (see Theorems 6.1 and Theorem 6.2 in Billingsley [2]).

Theorem 4.3 (Prokhorov)**.** *Let $(S, \mathcal{B}(S))$ be a measurable space.*

a. *If a family of probability measures is tight, then it is relatively compact.*
b. *Let S be separable (i.e., contains a countable dense subset) and complete. If a family of probability measures on $(S, \mathcal{B}(S))$ is relatively compact, it is tight.*

Consider the family $\{P_N\}$, where P_N is given by $\lim_{T\to\infty} P_{T,N}$ from (5). It is possible to show that this family of probability measures $\{P_N\}$ is tight. Invoking Theorem 4.3, we get that $\{P_N\}$ is relatively compact. Thus, there is a subsequence $\{P_{N_k}\}$ of probability measures which converges to some probability measure P. We are now left to show that $\{P_N\}$ itself converges to P. In order to do this, we use further tools from probability theory on separable spaces.

Theorem 4.4 (Theorem 4.2 of Billingsley [2])**.** *Let S be a separable metric space with metric ρ and, for $n \in \mathbb{N}$, let $Y_n, X_{1n}, X_{2n}, \cdots$ be S-valued random elements, all defined on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Suppose that*

$$X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k \ \ \textit{for each } k, \quad \textit{and} \ \ X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$

If for any $\epsilon > 0$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \boldsymbol{P}\{\rho(X_{kn}, Y_n) \geq \epsilon\} = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Since $\mathcal{H}(D)$ is a separable metric space, using Theorem 4.4 we ascertain that $\{P_N\}$ weakly converges to P. For details the reader may refer to Steuding [36, Chapter 3,4].

4.3. **Description of the limit.** Let

$$\Omega := \prod_p \gamma_p,$$

where p runs over all primes and $\gamma_p := \{z : |z| = 1\}$ denote the unit circle. Then, Ω is a compact topological group and hence there is a unique Haar measure $\mathfrak{m}$ on Ω, which can be described as

$$\mathfrak{m}(A) = \prod_p \mathfrak{m}_p(A_p),$$

where $A \subseteq \Omega$, A_p its projection on γ_p and $\mathfrak{m}_p$ the Haar measure on γ_p.

Let C be the set of completely multiplicative functions $g : \mathbb{N} \to \gamma$, where γ is the unit circle. Since completely multiplicative functions are fully determined by their values on primes, we can identify the space Ω with C.

For a random element $\omega \in \Omega$ (viewed as $\omega \in C$), define

$$\zeta(s, \omega) = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s}.$$

We first prove the following lemma.

Lemma 4.5. *For a random element* $\omega \in \Omega$, $\zeta(s,\omega)$ *is a* $\mathcal{H}(D)$*-valued random element.*

Proof. The proof hinges on the following fact from probability theory which states that if $X_1, X_2, \cdots$ are orthogonal independent random variables and

$$\sum_{n=1}^{\infty} E|X_n|^2 (\log n)^2 < \infty,$$

then the series $\sum_{n=1}^{\infty} X_n$ converges almost surely. Indeed, in our case, setting $\xi_n = \omega(n)/n^{\sigma}$ as the random variables, we have

$$E\xi_n\overline{\xi_m} = \frac{1}{(mn)^{\sigma}} \int_{\Omega} \omega(n)\overline{\omega(m)}\, dm = \begin{cases} \dfrac{1}{n^{2\sigma}} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, ξ_n's are orthogonal and

$$\sum_{n=1}^{\infty} E|\xi_n|^2 (\log n)^2 = \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^{2\sigma}} < \infty$$

for $\sigma > 1/2$. $\square$

We are now ready to describe the limit P.

Theorem 4.6. *Let $A \in \mathcal{B}(\mathcal{H}(D))$. Then,*

$$P(A) := \lim_{T\to\infty} \frac{1}{T}\, meas\{t \in [0,T] : \zeta(s+it) \in A\} = \mathfrak{m}\{\omega \in \Omega : \zeta(s,\omega) \in A\}.$$

This theorem is best understood through the lens of Ergodic theory. It can be shown that

$$\frac{1}{T}\, meas\{t \in [0,T] : \zeta(s+it,\omega) \in A\}$$

converges to P for almost all $\omega \in \Omega$ as $T \to \infty$ using standard techniques. Assuming this, we will show that

$$\frac{1}{T}\, meas\{t \in [0,T] : \zeta(s+it,\omega) \in A\} = \mathfrak{m}\{\omega \in \Omega : \zeta(s,\omega) \in A\} \tag{6}$$

for a random element $\omega \in \Omega$.

A one-parameter family $\{\phi_\tau : \tau \in \mathbb{R}\}$ of transformations on a compact topological group G is called a one-parameter group of transformations if

$$\phi_{\tau_1+\tau_2}(g) = \phi_{\tau_1}(\phi_{\tau_2}(g)) \text{ and } \phi_{-\tau}(g) = \phi_\tau^{-1}(g)$$

for real numbers τ, τ_1, τ_2 and $g \in G$. Let $\{\phi_\tau : \tau \in \mathbb{R}\}$ be a one-parameter group of measurable transformations on G. A set $A \in \mathcal{B}(G)$ is invariant with respect to the group $\{\phi_\tau : \tau \in \mathbb{R}\}$ if for each τ the sets A and $A_\tau := \phi_\tau(A_\tau)$ differ from one another by a set of measure 0. A one-parameter group is called *ergodic* if its σ-field of invariant sets consists only of the sets having Haar measure 0 or 1.

In our case, we consider the one-parameter family on Ω given by $a_\tau = \{p^{-i\tau} : p \text{ prime}\}$ for all $\tau \in \mathbb{R}$. Clearly, $\{a_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group. Define the one-parameter family $\{\phi_\tau : \tau \in \mathbb{R}\}$ of transformations on Ω as

$$\phi_\tau(\omega) := a_\tau \omega.$$

It is possible to show that $\{\phi_\tau : \tau \in \mathbb{R}\}$ is ergodic. For ergodic processes $X(\tau,\omega)$, we have a famous result of Birkhoff-Khintchine [36, Theorem 3.13] which states that "the average velocity of all particles at some given time is same as the average velocity of one particle over time."

Theorem 4.7 (Birkhoff-Khintchine)**.** *Let $X(\tau,\omega)$ be an ergodic process with $E|X(\tau,\omega)| < \infty$ and almost surely Riemann integrable. Then,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(\tau,\omega)\, d\tau = EX(0,\omega).$$

In our case, defining a random variable θ on $(\Omega, \mathcal{B}(\Omega))$ by

$$\theta(\omega) := \begin{cases} 1 & \text{if } \zeta(s,\omega) \in A \\ 0 & \textit{otherwise}, \end{cases}$$

we have $E\theta = \mathfrak{m}\{\omega : \zeta(s,\omega) \in A\}$. Since $\theta(\phi_\tau(\omega))$ is an ergodic process, as a consequence of Theorem 4.7 we obtain (6).

5. **Density theorem and the speed of convergence**

In addition to the limit theorem, Bohr and Jessen also proved the existence of a continuous non-negative density function $\mathcal{M}_\sigma(z,\zeta) : \mathbb{C} \to \mathbb{R}$ such that the limit

$$W_\sigma(R;\zeta) = \int_R \mathcal{M}_\sigma(z,\zeta)|dz|, \tag{7}$$

where $|dz| = \frac{1}{2\pi}\, dx\, dy$. This is called the *density theorem* for $\zeta(s)$. Although the limit theorem is known for a large class of L-functions, the density theorem is known only in few cases.

Let

$$G = \{s = \sigma + it : \sigma > 1/2\} - \bigcup_{\zeta(\rho)=0} \{s = \sigma + i\Im(\rho) : 1/2 < \sigma < \Re(\rho)\}.$$

By the Euler product, we can write

$$\log\zeta(s) = -\sum_{1}^{\infty} \log\left(1 - p_n^s\right)^{-1},$$

for $s \in G$. Setting $\theta_n := t(\log p_n)/2\pi$, each term in the summation above can be written as

$$f(\theta_n) = -\log\left(1 - p_n^{-\sigma}\exp(2\pi i\theta_n)\right).$$

The curve $\Gamma_n := \{f(\theta_n) : 0 \le \theta_n < 1\}$ describes a closed convex curve on $\mathbb{C}$. In general, such Euler products giving rise to a closed convex curve are called *convex Euler products*. Bohr and Jessen developed the geometric theory of infinite sums of convex curves, and used it in their original proof of the limit theorem. The convexity property of the curves is key to their argument. Later, the effect of convexity of curves was embodied in a certain inequality due to Jessen and Wintner [16, Theorem 13]. Using the Bohr-Jessen theory, Joyner [17] established the density theorem for Dirichlet L-functions and Matsumoto [27] proved the density theorem for Dedekind zeta-functions associated to Galois number fields. These generalizations were possible because the associated zeta-functions have a convex Euler product. But convexity of Euler products is not expected for general L-functions.

Instead of relying on weak convergence of probability measures and Prokhorov's result to prove the limit theorem (as highlighted in the earlier section), Matsumoto [27] introduced an alternate argument that involves using Levy's convergence theorem. This method, just as the earlier one, does not use convexity to establish the limit theorem. Therefore, it can be employed to prove the limit theorem for a large class of L-functions. The additional advantage of this method is that it is more suitable for studying density theorems for L-functions. Recently, adapting this technique, Matsumoto-Umegaki [31] established the density theorem for certain automorphic L-functions. In order to achieve this without having convexity of Euler product, they developed generalizations of Jessen-Wintner type inequalities. It is to be noted that all known proofs of density theorems either rely on convexity or a Jessen-Wintner type inequality, which also follows from convexity. We state the classical Jessen-Wintner inequality [16] below.

Theorem 5.1 (Jessen-Wintner)**.** *Let $\rho > 0$ and assume that the series*

$$F(z) = \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \in \mathbb{C}, a_1 \neq 0)$$

is absolutely convergent for $|z| < \rho$ *and* $a_n \in \mathbb{C}, a_1 \neq 0$. *Let* Γ *be the closed curve on the complex plane, defined by*

$$\Gamma = \Gamma(r) = \left\{ F\left(re^{2\pi i\theta}\right) | 0 \leq \theta < 1 \right\}$$

with $0 < r < \rho$. *Then,*

(1) *There exists* $\rho_0 = \rho_0(\rho, F)$ *with* $0 < \rho_0 < \rho$ *such that* $\Gamma(r)$ *is a closed convex curve for any* r *satisfying* $0 < \rho < \rho_0$.
(2) *There exists* $\rho_1 = \rho_1(\rho, F)$ *with* $0 < \rho_1 < \rho$ *such that*

$$\int_0^1 \exp\left(i\langle F\left(re^{2\pi i\theta}\right), \omega\rangle\right) d\theta = O\left(r^{-1/2}|\omega|^{-1/2}\right)$$

holds for any $\omega \in \mathbb{C}$ *and for any* r *satisfying* $0 < r \leq \rho_1$, *where the implied constant depends on* ρ *and* F.

We refer the reader to a significant generalization of this theorem due to Matsumoto [28, Lemma 2].

Another closely related question is that of quantitative results for the limit theorem. Refining arguments of Bohr-Jessen, in a series of papers Matsumoto [27] produced discrepancy estimates and thereby obtaining quantitative versions of Bohr-Jessen result, i.e., estimates for

$$\frac{V_\sigma(T, R; \zeta)}{T} - W_\sigma(R; \zeta),$$

which captures the speed of convergence in Theorem 4.1. The sharpest result in this context is due to Harman and Matsumoto [13], where they show that

$$\left|\frac{V_\sigma(T, R; \zeta)}{T} - W_\sigma(R; \zeta)\right| = O\left((meas(R) + 1)(\log T)^{-A(\sigma)+\epsilon}\right), \tag{8}$$

where

$$A(\sigma) = \begin{cases} \frac{\sigma-1}{3+2\sigma} & \text{if } \sigma > 1 \\ \\ \frac{4\sigma-2}{21+8\sigma} & \text{if } \frac{1}{2} < \sigma \leq 1. \end{cases} \tag{9}$$

Such quantitative results were also obtained by Matsumoto for Dedekind zeta-functions [28] and Hecke L-functions [30]. The Dedekind zeta-functions associated to Galois number fields have convex Euler-products similar to the case of Riemann zeta-function. Hence, a direct generalization leads to quantitative results in this case. For non-Galois number fields, the Euler product of the Dedekind zeta-function is no longer convex. However, it is still possible to use Artin-Chebotarev density theorem to reduce it to the convex case. This is precisely the approach adapted by Matsumoto in [28]. For a number field $K/\mathbb{Q}$ of degree n_K and any $\sigma > 1 - 1/n_K$, he proves that

$$\left|\frac{V_\sigma(T, R; \zeta)}{T} - W_\sigma(R; \zeta)\right| = O\left((meas(R) + 1)(\log T)^{-A(\sigma)+\epsilon}\right),$$

where $A(\sigma)$ is as in (9). Here, the implied constant depends on σ, ϵ and K. I believe one could remove the dependence on K by inserting a constant of the form $|\rho_K| n_K^2$ where ρ_K is the residue of $\zeta_K(s)$ at $s = 1$.

We briefly outline the ideas involved in establishing the speed of convergence for $\zeta(s)$. Let p_n denote the n-th prime. Define the partial Euler product

$$\zeta_N(s) = \prod_{n=1}^{N} \left(1 - \frac{1}{p_n^s}\right)^{-1}.$$

For $\sigma + it \in G$,

$$\log \zeta_N(\sigma + it) = -\sum_{n=1}^{N} \log\left(1 - \frac{1}{p^\sigma} e^{-it\log p_n}\right).$$

Given a rectangle R in $\mathbb{C}$, let

$$V_N(T, R) := \{t \in [0, T] : \log \zeta_N(\sigma + it) \in R\}.$$

Proof of the limit theorem would involve the existence of the limits

$$W_N(R) = \lim_{T\to\infty} \frac{V_N(T,R)}{T} \quad \text{ and } W(R) = \lim_{N\to\infty} W_N(R).$$

In order to prove speed of convergence of the limit theorem, we need to bound the following quantities.

$$\left| W_N(R) - \frac{V_N(T,R)}{T} \right|, \ |W_R - W_N(R)| \ \text{ and } \left| \frac{V_N(T,R)}{T} - \frac{V(T,R)}{T} \right|. \tag{10}$$

Define $S_N : [0,1]^N \to \mathbb{C}$ by

$$S_N(\theta_1, \cdots, \theta_N) = -\sum_{n=1}^{N} \log\left(1 - \frac{1}{p^\sigma}\, e^{2\pi i\theta_n}\right).$$

The Kronecker-Weyl theorem can now be used to show that $W_N(R) = meas(S_N^{-1}(R))$. This interpretation allows us to use tools from transcendental number theory such as in Waldschmidt [40] to produce bounds for the first quantity in (10). For bounding the other quantities, the main tools used are the Levy's inversion formula and a refined version of Carlson's theorem. We refer the readers to Matsumoto [27] for details.

6. **Generalizations**

The proof of the limit theorem outlined in Section 4.1 can be emulated to nicely behaved Dirichlet series with a finite Euler product representation. Indeed in [26], Matsumoto introduced a large class of L-functions $\mathcal{M}$ and established the limit theorem for this class. This class $\mathcal{M}$ is also called the class of Matsumoto zeta-functions. We describe it below.

Let $\mathbb{N}$ be the set of all positive integers, and $g(n) \in \mathbb{N}$, $f(j,n) \in \mathbb{N}\,(1 \leq j \leq g(n))$ and $a_n^{(j)} \in \mathbb{C}$. Denote by p_n the n-th prime number. We assume

$$g(n) \leq C_1 p_n^\alpha, \ |a_n^{(j)}| \leq p_n^\beta$$

with constants $C_1 > 0$ and $\alpha, \beta \geq 0$. Define

$$F(s) = \prod_{n=1}^{\infty} A_n(p_n^{-s})^{-1},$$

where $A_n(X)$ are polynomials in X given by

$$A_n(X) = \prod_{j=1}^{g(n)} \left(1 - a_n^{(j)} X^{f(j,n)}\right).$$

Then, $F(s)$ is absolutely convergent on the half plane $\sigma > \alpha + \beta + 1$. The class $\mathcal{M}$ consists of all such functions F which further satisfy the following conditions.

(a) $F(s)$ can be meromorphically continued to $\sigma > \sigma_0$, where $\alpha+\beta+1/2 \leq \sigma_0 < \alpha+\beta+1$, and all the poles in this region are contained in a compact subset.

(b) $|F(\sigma + it)| = O((|t| + 1)^C)$ for any $\sigma \geq \sigma_0$, with a constant $C > 0$.

(c) The mean-square on the vertical line $\{\Re(s) = \sigma_0\}$ is bounded, i.e.,

$$\int_{-T}^{T} |F(\sigma_0 + it)|^2 \, dt = O(T).$$

The class $\mathcal{M}$ contains familiar L-functions such as the Riemann zeta-function, Dirichlet L-functions, Dedekind zeta-functions etc. In [26], Matsumoto proved the following generalization of the Bohr-Jessen limit theorem.

Theorem 6.1 (Matsumoto). *Let $F \in \mathcal{M}$. For any $\sigma > \sigma_0$, the limit*

$$W_\sigma(R; F) := \lim_{T \to \infty} \frac{1}{2T} V_\sigma(T, R; F)$$

exists.

The limit theorem can also be proved for certain Dirichlet series where the coefficients are not multiplicative. In this regard, an important result is due to Laurinčikas [21], [22].

Theorem 6.2 (Laurinčikas). *Assume that $F(s)$ is a meromorphic function in the half plane $\sigma > \sigma_0$ and all the poles of $F(s)$ in this region are included in a compact set. Suppose $F(s)$ satisfies the estimate*

$$|F(\sigma + it)| \ll t^\delta$$

for $\sigma > \sigma_0$ with some $\delta > 0$ and that

$$\int_0^T |F(\sigma + it + i\tau)|^2 \, d\tau \ll T$$

for all $\sigma > \sigma_0$. Further, suppose that $F(s)$ has a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where the series is absolutely convergent for $\sigma > \sigma_0 + 1/2$ and $\sum_{n \leq x} |a_n|^2 \ll x^{2\sigma_0}$. Then, the probability measure, defined by

$$\frac{1}{T} meas\{\tau \in [0, T] : F(s + i\tau) \in A\}$$

for $A \in \mathcal{B}(\mathcal{H}(D_0))$, where $D_0 := \{s \in \mathbb{C} : \Re(s) > \sigma_0\}$, converges weakly to the distribution of random element $g \in \Omega$, where

$$\Omega \ni \omega \mapsto \sum_{n=1}^{\infty} \frac{a_n \omega_n}{n^s}$$

for $s \in D_0$ as $T \to \infty$.

There are several further generalizations and extensions of the limit theorem, mostly due to the school of Laurinčikas. For instance, Laurinčikas [22], [23] proved a limit theorem for Matsumoto zeta-functions in the space of meromorphic functions. Recently, the limit theorem has also been established for periodic Hurwitz zeta-functions by Rimkecičienė [33]. For an overview of various results, the reader may refer to [19], [20] and [29].

7. **Concluding remarks**

Inspired by the study of Bohr and further building on his ideas, Voronin [39] proved the celebrated universality theorem for $\zeta(s)$ in 1975. This theorem states that every non-vanishing analytic function is well approximated by vertical shifts of $\zeta(s)$ in the critical strip. This is a remarkable result, suggesting that $\zeta(s)$ mimics the behaviour of all analytic functions in the critical strip.

Theorem 7.1 (Voronin's Universality theorem)**.** *Let $0 < r < \frac{1}{4}$ and suppose that $g(s)$ is a non-vanishing continuous function on the disc $\{s : |s| \leq r\}$, which is analytic in its interior. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\, meas \left\{ |\tau| < T : \max_{|s|<r} \left| \zeta\left(s + \frac{3}{4} + i\tau \right) - g(s) \right| < \epsilon \right\} > 0.$$

The advantage that Voronin enjoyed over Bohr was the availability of sophisticated machinery in probability theory and Perchersky's re-arrangement theorem, which is a generalization of Riemann re-arrangement theorem. These tools played a crucial role in his proof of universality. Decades of investigation were stirred by Voronin's universality, resulting in several generalizations and alternate approaches. In a written communication with Ibragimov, Linnik conjectured that all Dirichlet series with suitable growth conditions must satisfy the universality property. This is known as the Linnik-Ibragimov conjecture and is widely open. The conjecture is formulated precisely by the author in [11, Chapter 5]. For a comprehensive account on the topic of universality of L-functions, we refer the interested readers to the excellent book of Steuding [36].

Another avatar of the limit theorem worth mentioning, which is not discussed in this article is its "discrete" version. Let $F \in \mathcal{M}$ be a Matsumoto zeta-function and γ be a real number such that $\exp(2\pi k/\gamma) \notin \mathbb{Q}$ for all positive integers k. Then, for any s in a suitable strip, the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : \, F(s + in\gamma) \in A\}$$

exists where A is a Borel set of the appropriate function space. This is called the *discrete limit theorem* and the above result is due to Kačinskaitė and Laurinčikas [18]. The techniques used to prove such theorems are closely related to the ones used in the continuous case.

There are several unanswered questions and future directions in the study of limit theorems. For instance, Theorem 6.2 suggests that the limit theorem is a "purely analytic property" of Dirichlet series. One would expect the same to be true for density theorems as well. So, it is desirable to have a proof of the density theorem without using convexity or Jessen-Wintner type inequality, completely relying on the analytic nature of the underlying function.

One more front for further investigation is the region where the limit theorem holds. For any "nice" function F such as in Theorem 6.2, the half-plane where one has the limit theorem is given by $\Re(s) > \sigma_m$, where σ_m is the abscissa of mean value convergence, i.e.,

$$\sigma_m := \inf \left\{ \sigma : \frac{1}{T} \int_{-T}^{T} |F(\sigma + it)|^2 \, dt \ll 1 \right\}.$$

It is unclear if this is the best region where the limit theorem is realized. One would like to believe that the limit theorem should hold in a larger vertical strip, but we do not know how to prove this yet.

8. **Acknowledgements**

I thank Prof. Ram Murty and Siddhi Pathak for carefully reading through an earlier version of the article and their valuable comments. I also thank the referees for helpful suggestions.

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