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On Ihara's conjectures for Euler–Kronecker constants

by

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Dedicated to the memory of Prof. A. Schinzel

1. Introduction. The Euler–Mascheroni constant denoted by γ is defined as

$$\gamma := \lim_{x\to\infty}\Big(\sum_{n\le x}\frac{1}{n} - \log x\Big).$$

This constant γ appears in many areas of mathematics. For instance, it is given by the constant term in the Laurent expansion of the Riemann zeta-function,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1). \tag{1}$$

Motivated by (1), Y. Ihara [15] introduced a generalization of γ to any number field K, using the Dedekind zeta-function $\zeta_K(s)$. The Dedekind zeta-function $\zeta_K(s)$ associated to a number field K is defined on the half-plane $\Re(s) > 1$ as

$$\zeta_K(s) := \sum_{\mathfrak{a}\subset\mathcal{O}_K}\frac{1}{N\mathfrak{a}^s},$$

where $\mathfrak{a}$ runs over all non-zero integral ideals of the ring of integers $\mathcal{O}_K$. The function $\zeta_K(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$. Analogous to the Riemann hypothesis for $\zeta(s)$, the generalized Riemann hypothesis (GRH) asserts that all zeros of $\zeta_K(s)$ in the strip $0 < \Re(s) < 1$ must satisfy $\Re(s) = 1/2$.

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If the Laurent expansion of $\zeta_K(s)$ near $s=1$ is written in the form

$$\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1),$$

then the Euler–Kronecker constant associated to K is defined as

$$\gamma_K := \frac{c_0}{c_{-1}}.$$

One could also view γ_K as the constant term in the Laurent expansion of the logarithmic derivative of $\zeta_K(s)$ at $s=1$, i.e.,

$$-\frac{\zeta_K'}{\zeta_K}(s) = \frac{1}{s-1} - \gamma_K + O(s-1). \tag{2}$$

We will show:

THEOREM 1.1. *Let K be an algebraic number field and write*

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_K(n)}{n^s}.$$

Setting

$$\Delta_K(N) := \sum_{n\le N} \Lambda_K(n) - N,$$

we have, for any $N \ge 1$,

$$\gamma_K = \Big[\log N - \sum_{n\le N-1} \frac{\Lambda_K(n)}{n}\Big] + \frac{\Delta_K(N-1)}{N} + \frac{1}{N} - \int_N^{\infty} \frac{\Delta_K(x)}{x^2}\,dx. \tag{3}$$

In particular, choosing $N=1$, we have

$$\int_1^{\infty} \frac{\Delta_K(x)}{x^2}\,dx = 1 - \gamma_K. \tag{4}$$

Formula (4) gives us a connecting link between the error term in the prime ideal theorem and γ_K. But there are other links we will establish. In [15], Ihara proved the following bounds for γ_K:

$$\begin{aligned} \gamma_K &\le 2\log\log\sqrt{|d_K|} \quad &&\text{(under GRH)},\\ \gamma_K &\ge -\log\sqrt{|d_K|} &&\text{(unconditionally)}, \end{aligned} \tag{5}$$

where d_K denotes the discriminant of K over $\mathbb{Q}$. Ihara used Weil's explicit formula to prove (5). We will give simple proofs of (5) and derive new unconditional upper bounds for γ_K.

In [4], the first author proved some unconditional bounds for γ_K where K is an almost normal field or has solvable Galois closure. These bounds depended in an essential way on the earlier work of V. Kumar Murty [27, 28]. In this paper, we obtain unconditional upper bounds for an arbitrary number field K. In particular, our bounds improve those in [4]. The upper bounds are closely related to the size of the hypothetical Siegel's zero of $\zeta_K(s)$. Recall that for $K \neq \mathbb{Q}$, this zero β_0, if it exists, is real, simple and the only zero in the region

$$|\Re(s)| > 1 - \frac{1}{4\log|d_K|}, \qquad |\Im(s)| < \frac{1}{4\log|d_K|}.$$

If K does not contain a quadratic field, we know by the work of Stark [34] that the Siegel zero does not exist. In such a case, our unconditional upper bound is $\gamma_K = O(\log|d_K|)$. More precisely, we prove:

THEOREM 1.2. *For any number field K, we have*

$$\gamma_K = O(\log|d_K|)$$

if $\zeta_K(s)$ has no Siegel zero. If it has a Siegel zero β_0, the bound is

$$\gamma_K = \frac{1}{2\beta_0(1-\beta_0)} + O(\log|d_K|).$$

Using known results about the location of Siegel's zero, we obtain the effective estimate $\gamma_K = O(|d_K|^{1/n})$ with $n = [K : \mathbb{Q}]$ and the ineffective estimate $\gamma_K = O(|d_K|^{\epsilon/n})$ for any $\epsilon > 0$. Here the implied constants are absolute.

If for an algebraic number field K, we let $\mathscr{P}_K$ be the smallest norm of a prime ideal in K, then (see Theorem 4.1 below)

$$\gamma_K = \log \mathscr{P}_K + \frac{1}{\mathscr{P}_K} - \int_{\mathscr{P}_K}^{\infty} \frac{\Delta_K(x)}{x^2}\,dx. \tag{6}$$

Formula (3) allows us to focus on the interval where the bulk of the contribution to γ_K arises. Indeed, if we assume GRH (see (17) below), then the optimal choice of N is

$$N_0 = (\log|d_K|)^2(\log\log|d_K|)^2,$$

which leads to

$$\gamma_K = \log N_0 - \sum_{n \le N_0} \frac{\Lambda_K(n)}{n} + O(1), \tag{7}$$

showing that the main contribution comes from primes of "small" norm. This formula highlights the intimate relationship between $\mathscr{P}_K$ and γ_K.

If q is prime and $K = \mathbb{Q}(\zeta_q)$ is the qth cyclotomic field, then $\mathscr{P}_K = q$. In this case, we are led to study error terms in the prime number theorem for arithmetic progressions. Let us recall several results in this context.

Let $\pi(x)$ denote the number of primes $\le x$. The classical prime number theorem, which was proved independently by Hadamard [12] and de la Vallée

Poussin [36], states that as $x \to \infty$,

$$\pi(x) \sim \mathrm{Li}(x), \quad \text{where} \quad \mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Equivalently, if

$$\psi(x) := \sum_{\substack{p^n \leq x \\ p \text{ prime}}} \log p,$$

then as $x \to \infty$, we have $\psi(x) \sim x$. For $(a, q) = 1$, let

$$\psi(x; q, a) := \sum_{\substack{p^m \leq x, \\ p^m \equiv a \bmod q}} \log p.$$

Then, the prime number theorem for arithmetic progressions asserts that, as $x \to \infty$,

$$\psi(x; q, a) \sim \frac{x}{\phi(q)}.$$

Under the assumption of the generalized Riemann hypothesis, one can show that

$$\left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| = O(\sqrt{x}\,(\log qx)^2).$$

Unconditionally, the Siegel–Walfisz theorem states that

$$\left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| = O(xe^{-c_N\sqrt{\log x}}), \tag{8}$$

uniformly for $q \leq (\log x)^N$ and a constant $c_N > 0$ dependent on N. The famous Bombieri–Vinogradov theorem (see [2, 37]) establishes that the GRH-error term holds on average. More precisely, if

$$\sqrt{x}(\log x)^{-A} \leq Q \leq \sqrt{x},$$

then

$$\sum_{q \leq Q} \max_{y \leq x} \max_{\substack{1 \leq a \leq q \\ (a,q)=1}} \left| \psi(y; q, a) - \frac{y}{\phi(q)} \right| = O(\sqrt{x}\,Q(\log x)^5).$$

Extending the range of the Bombieri–Vinogradov theorem beyond $\sqrt{x}$ has been an important theme of research in number theory. For example, Fouvry (see [10, Lemma 6]) has shown that for every $A > 0$,

$$\sum_{m \leq Q} m \left(\psi(x, m_0 m, a) - \frac{\psi(x)}{\phi(m_0 m)} \right) = O_A(Qx(\log x)^{-A+1}),$$

uniformly for $x \geq 2$, $Q \leq x(\log x)^{-200A-200}$, where a and m_0 are coprime integers such that $m_0 \geq 1$ and $1 \leq |a| \leq (\log x)^A$. The reader will observe that there are no absolute values in the summand. In this context, there is the conjecture of Elliott and Halberstam [8]: for any $A > 0$ and $\epsilon > 0$,

$$\sum_{q \le x^{1-\epsilon}} \max_{y \le x} \max_{\substack{1 \le a \le q \\ (a,q)=1}} \left| \psi(y; q, a) - \frac{y}{\phi(q)} \right| \ll \frac{x}{\log^A x}. \tag{9}$$

We will discuss below how these results and conjectures are related to the study of Euler–Kronecker constants.

There are several conjectures on bounding the size of γ_K in terms of d_K, over specific families of number fields (see [16, 35]). In particular, for a cyclotomic field $K = \mathbb{Q}(\zeta_m)$, we write γ_m for γ_K and d_m for its discriminant. It is known (see [38, p. 11]) that

$$\frac{\log |d_m|}{\phi(m)} = \sum_{p \mid m} \frac{\log p}{p-1}.$$

Thus, our general bound for γ_K in this case leads to $\gamma_m = O(\phi(m) \log m)$. Of course, GRH implies that the upper bound should be $O(\log m)$.

Henceforth, the letters p and q will denote prime numbers. The link between γ_q and primes in arithmetic progressions is revealed by our formula (6), which in our case reduces to

$$\gamma_q = \log q - (q-1) \int_q^\infty \frac{E(x,q,1)}{x^2}\, dx + O(1), \tag{10}$$

where

$$E(x,q,a) := \psi(x,q,a) - \frac{x}{\phi(q)}.$$

Perhaps based on this kind of reasoning, or on numerical computations for $m \le 8000$, Ihara [16] made the following conjecture.

Conjecture 1 (Ihara). *For $K = \mathbb{Q}(\zeta_m)$:*

(a) $\gamma_m > 0$ *for all m.*

(b) *There exist positive constants $c_1, c_2 \le 2$ such that for any $\epsilon > 0$,*

$$(c_1 - \epsilon) \log m < \gamma_m < (c_2 + \epsilon) \log m$$

for sufficiently large m. If m is a prime, one can choose $c_1 = 1/2$ and $c_2 = 3/2$.

In 2014, K. Ford, F. Luca and P. Moree [9] showed that the prime k-tuple conjecture, as formulated by Hardy and Littlewood, implies $\gamma_q < 0$ infinitely often. In fact, they explicitly produced a prime, namely $q = 964477901$, for which

$$\gamma_q = -0.18237\ldots.$$

There are further negative values known for γ_q and three such values have been obtained in the recent works of Languasco [21] and Languasco and

Righi [22] exploiting a new efficient algorithm to compute $L'/L(1,\chi)$. However, it is still not known unconditionally if $\gamma_m < 0$ for infinitely many positive integers m.

It is also worth mentioning that there are similarities between Conjecture 1(b) and Kummer's conjecture for cyclotomic extensions. Suppose $h_1(q)$ denotes the ratio of the class numbers of $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$. Then Kummer proved that $h_1(q)$ is an integer and conjectured that

$$h_1(q)\left(\frac{4\pi^2}{q}\right)^{(q-1)/4}$$

tends to 1 as $q \to \infty$. This is famously known as Kummer's conjecture. It would appear that the quantity above has similar analytic properties to $\gamma_q/\log q$. We refer interested readers to the excellent survey article of P. Moree [24] for further details.

1.1. Conjecture 1(b) and primes in arithmetic progressions. The main theme of our paper is to connect the behaviour of the γ_q to error terms in the prime number theorem for arithmetic progressions. Recently, there have been developments towards establishing upper bounds for $|\gamma_q|$ as suggested by Conjecture 1(b). In this direction, Y. Ihara, V. Kumar Murty and M. Shimura [17] proved that under GRH, for any prime q,

$$|\gamma_q| = O((\log q)^2) \tag{11}$$

and unconditionally, for any $\epsilon > 0$,

$$|\gamma_q| = O_\epsilon(q^\epsilon). \tag{12}$$

The estimate (12) follows from our root discriminant bound mentioned above. But we will give another proof of the result (12) (see Corollary 4). The bound (11) was improved by A. I. Badzyan [1], who showed that under GRH,

$$|\gamma_m| = O(\log m \log\log m),$$

for any positive integer m. Let us note that using the explicit formula for the discriminant of $\mathbb{Q}(\zeta_m)$, we obtain via (5) the inequality $\gamma_m \leq 2\log m + O(\log\log m)$.

As for lower bounds, M. Mourtada and V. Kumar Murty [25] have shown that $\gamma_q > -11\log q$ for almost all primes q.

A natural question is whether the bounds in Conjecture 1(b) hold on average. In this context, V. Kumar Murty [26] proved that

$$\sum_{q\sim Q,\, q\,\mathrm{prime}} |\gamma_q| \ll \pi^*(Q)\log Q, \tag{13}$$

where $q \sim Q$ means $Q \leq q \leq 2Q$ and $\pi^*(Q)$ denotes the number of primes in this interval. E. Fouvry [10] generalized this to

$$\frac{1}{Q}\sum_{m\sim Q}\gamma_m = \log Q + O(\log\log Q),$$

where m runs over all positive integers in the interval and $Q \geq 3$. Both of these results are quite deep. Kumar Murty's result uses zero density estimates for Dirichlet L-functions, and Fouvry's theorem uses the latest results regarding primes in arithmetic progressions. We will give a simplified proof of (13) and also show:

THEOREM 1.3. *Assuming the Elliott–Halberstam conjecture, we have*

$$\sum_{q\sim Q,\, q\ prime} |\gamma_q - \log q| = o(Q). \tag{14}$$

Consequently,

$$\sum_{q\sim Q,\, q\ prime} \gamma_q = \sum_{q\sim Q,\, q\ prime} \log q + o(Q) = Q + o(Q),$$

and

$$\sum_{q\sim Q,\, q\ prime} |\gamma_q| = \sum_{q\sim Q,\, q\ prime} \log q + o(Q) = Q + o(Q).$$

Fouvry's methods do not lead to this result because, as he remarks, one needs a dense set of moduli to apply his method, and the set of primes is too sparse. Thus, assuming the Elliott–Halberstam conjecture, we see from our theorem that γ_q is "usually" $\log q$, and the number of primes $q \sim Q$ for which $|\gamma_q - \log q| > \varepsilon \log q$ is $o(\pi^*(Q)/\varepsilon)$. We note that a similar result under the Elliot–Halberstam conjecture was also recently obtained by L. Hong, K. Ono and S. Zhang [14], who showed that

$$\frac{1}{Q}\sum_{Q\leq m\leq 2Q} |\gamma_m - \log m| = o(\log Q).$$

Theorem 1.3 also implies that

$$\sum_{q\leq Q} \frac{|\gamma_q|}{q-1} = \log Q + o(\log Q), \tag{15}$$

and it may be possible to prove this weaker assertion unconditionally. To this end, formula (10) suggests we study

$$A_{q,x} := \int_2^x \frac{\psi(t;q,1) - t/(q-1)}{t^2}\, dt,$$

for a prime q and real number x. The term $A_{q,x}$ can be thought of as a weighted average of the error term in the prime number theorem for a given arithmetic progression. We prove the following average result for $A_{q,x}$.

THEOREM 1.4. *Fix $\delta > 0$. For $x, y > 2$ satisfying $\log x > y^\delta$ and $x > x_0(\delta)$, we have*

$$\sum_{\substack{q \le y \\ q \text{ prime}}} |A_{q,x}| = O(\log y),$$

where the implied constant only depends on δ.

The condition $\log x > y^\delta$ in Theorem 1.4 can be relaxed to $\log x > A(\log y)^2$ for a constant $A > 0$ if we assume that $L(s, \chi)$ has no Siegel zeros for all χ mod q and all $q \le y$.

REMARK. Naively using the GRH-error term, we have $A_{q,x} = O((\log q)^2)$ and therefore, we find that the sum of the errors over primes q up to y is $O(y \log y)$. Hence, the average-error in Theorem 1.4 is much smaller than expected.

As a consequence of the proof of Theorem 1.4, along with an application of explicit formula due to Gun, the second author and Rath, we get the following result.

THEOREM 1.5. *Let $K = \mathbb{Q}(\zeta_q)$ for a prime q and $\log x > q^\delta$. Then*

$$\sum_\rho \frac{1}{x^\rho \rho} = O\big((\log x)^{\frac{1}{2} + \frac{1}{\delta}} e^{-c\sqrt{\log x}}\big),$$

where ρ runs over all the non-trivial zeros of $\zeta_K(s)$ and c is an absolute constant. The implied constant in the O-term depends on δ, but is independent of q and x.

We conclude this introduction with a curious remark related to almost primes. As noted earlier, if we assume GRH, we have the formula

$$-\gamma_q = (q-1) \sum_{n \le N_0} \frac{\Lambda(n)}{n} - \log N_0 + O(1),$$

where $N_0 = Aq^2(\log q)^4$ with A a sufficiently large constant. Then

$$-\sum_{2<q<y} \frac{\gamma_q}{q-1} = \sum_{n \le Ay^2(\log y)^4} \frac{\Lambda(n)}{n} \omega_y^*(n-1) - 2\log y = o(\log y),$$

where $\omega_y^*(n)$ is the number of odd prime divisors of n in the interval

$$[\sqrt{n/A}\,(\log y)^{-2}, y].$$

If we assume the Elliott–Halberstam conjecture, the left hand side is asymptotic to $-\log y$, and we deduce

$$\sum_{n \le Ay^2(\log y)^4} \frac{\Lambda(n)}{n} \left[\omega_y^*(n-1) - \frac{1}{2}\right] = o(\log y).$$

This essentially means that there are infinitely many primes p such that $(p-1)/2$ is either prime or a product of two prime factors, a result known unconditionally via the sieve method. The notorious parity problem of sieve theory is therefore not resolved. Sadly, the theory of the Euler–Kronecker constants will not capture the fugitive twin prime problem! However, the calculation does lead to the interest in the study of $\omega_y^*(n)$, which we discuss at the end of the paper.

2. Prolegomena and proof of Theorem 1.1. It may be useful to derive a very simple limit formula for a general class of Dirichlet series that will help us understand the nature of the Euler–Kronecker constants. This is the goal of the following proposition.

PROPOSITION 2.1. *Let $L(s)$ be a Dirichlet series such that for some $m \geq 0$, we can write the Laurent series at $s=1$ as*

$$-\frac{L'(s)}{L(s)} = \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \frac{m}{s-1} + c + O(s-1).$$

Put

$$S(x) := \sum_{n \leq x} b_n, \qquad E(x) := S(x) - mx.$$

Assume that

$$\int_1^{\infty} \frac{|E(x)|}{x^2}\, dx < \infty.$$

Then

$$c = m + \int_1^{\infty} \frac{E(x)}{x^2}\, dx.$$

Moreover, for any $N \geq 1$, we have

$$c = m + \left[\sum_{n \leq N-1} \frac{b_n}{n} - m \log N - \frac{S(N-1)}{N} \right] + \int_N^{\infty} \frac{E(x)}{x^2}\, dx. \tag{16}$$

In particular,

$$c = m + \lim_{N \to \infty} \left[\sum_{n \leq N-1} \frac{b_n}{n} - m \log N - \frac{S(N-1)}{N} \right].$$

Proof. By partial summation, we see that

$$-\frac{L'(s)}{L(s)} = s \int_1^{\infty} \frac{S(x)}{x^{s+1}}\, dx = \frac{ms}{s-1} + s \int_1^{\infty} \frac{E(x)}{x^{s+1}}\, dx$$

$$= \frac{m}{s-1} + m + s \int_1^{\infty} \frac{E(x)}{x^{s+1}}\, dx.$$

By our hypothesis, the last integral is analytic for $\Re(s) \geq 1$ and therefore

$$c = m + \int_1^\infty \frac{E(x)}{x^2}\,dx.$$

On the other hand,

$$\int_1^N \frac{E(x)}{x^2}\,dx = \int_1^N \frac{S(x) - mx}{x^2}\,dx = \int_1^N \frac{S(x)}{x^2}\,dx - m\log N,$$

and the integral on the right hand side is equal to

$$\begin{aligned}\sum_{j=1}^{N-1} \int_j^{j+1} \Big(\sum_{n\leq j} b_n\Big)\frac{dx}{x^2} &= \sum_{n\leq N-1} b_n \sum_{n\leq j\leq N-1} \int_j^{j+1} \frac{dx}{x^2}\\ &= \sum_{n\leq N-1} b_n \sum_{n\leq j\leq N-1} \Big(\frac{1}{j} - \frac{1}{j+1}\Big).\end{aligned}$$

The innermost sum on the right hand side telescopes and we get

$$\int_1^N \frac{S(x)}{x^2}\,dx = \sum_{n\leq N-1} b_n\Big(\frac{1}{n} - \frac{1}{N}\Big).$$

As

$$c = m + \int_1^N \frac{E(x)}{x^2}\,dx + \int_N^\infty \frac{E(x)}{x^2},$$

this proves (16). Taking limits as $N \to \infty$ in (16) gives the final assertion. ∎

We apply this to derive the limit formula for the Euler–Kronecker constant encoded in Theorem 1.1.

Proof of Theorem 1.1. In the case of the Dedekind zeta-function, $m = 1$, and the prime ideal theorem with error term certifies that the hypotheses of the theorem are satisfied which gives the result via (16). ∎

3. Ihara's theorems revisited. The prime ideal theorem assuming GRH is (see [33, Theorem 4])

$$|\Delta_K(x)| \ll \sqrt{x}\,(\log|d_K| + [K:\mathbb{Q}]\log x)\log x. \tag{17}$$

We will use this fact along with the formula (3) to give simpler proofs of Ihara's theorems (5). Ihara used Weil's explicit formula to prove his theorems. Our treatment shows that there is no need for such heavy artillery, and our elementary approach suggested by the previous section leads to the results directly.

THEOREM 3.1. *Assuming GRH, we have*

$$\gamma_K \le [2\log\log|d_K|]\Big(1+O\Big(\frac{\log\log\log|d_K|}{\log\log|d_K|}\Big)\Big).$$

Proof. Indeed, given the positivity of $\Lambda_K(n)$ for all n, from (3) we have the inequality

$$\gamma_K \le \log N + \frac{\Delta_K(N-1)}{N} + \frac{1}{N} - \int_N^\infty \frac{\Delta_K(x)}{x^2}\,dx,$$

for any value of $N \ge 1$. Inserting the bound (17) in the above inequality and estimating the integral via basic calculus gives

$$\gamma_K \le \log N + O\Big(\frac{\log|d_K|}{\sqrt{N}} + [K:\mathbb{Q}]\frac{\log N}{\sqrt{N}}\Big).$$

Choosing

$$N = (\log|d_K|)^2(\log\log|d_K|)^4$$

and using a simple corollary of the celebrated Minkowski bound that $[K:\mathbb{Q}]/(\log|d_K|)$ is bounded for all number fields $K \ne \mathbb{Q}$ (see for example [20, Theorem 5, p. 121] or [29, Exercise 6.5.21]), we deduce the desired result. ∎

4. Euler–Kronecker constants and the least prime ideal norm. For any algebraic number field K, let $\mathscr{P}_K$ be the smallest norm of a prime ideal in K. Using (17), we deduce that

$$\mathscr{P}_K \ll (\log|d_K|)^2.$$

The function $\mathscr{P}_K$ is similar (but not necessarily equal) to the least prime that splits completely in K, which we will discuss in the next section. The connecting link between γ_K and $\mathscr{P}_K$ is provided by the following theorem.

THEOREM 4.1. *For any algebraic number field K, we have*

$$\gamma_K = \log\mathscr{P}_K + \frac{1}{\mathscr{P}_K} - \int_{\mathscr{P}_K}^\infty \frac{\Delta_K(x)}{x^2}\,dx. \tag{18}$$

Assuming GRH, we have

$$\gamma_K = \log\mathscr{P}_K + O\Big(\frac{\log|d_K|}{\sqrt{\mathscr{P}_K}}\Big).$$

Proof. We apply (3) and observe that

$$\sum_{N(\mathfrak{a})\le\mathscr{P}_K-1} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})} = \sum_{n\le\mathscr{P}_K-1} \frac{\Lambda_K(n)}{n} = 0,$$

where $\Lambda(\mathfrak{a})$ denotes the usual von Mangoldt function defined on the ideals of the ring of integers of K. This gives (18). Injecting the error term obtained assuming GRH in the prime ideal theorem into (18), the result is now immediate. ∎

Based on similarities of $\mathscr{P}_K$ with the least prime that splits completely in K, it is reasonable to conjecture that $\mathscr{P}_K \ll (\log|d_K|)^{1+\epsilon}$ for any $\epsilon > 0$. In the case of the cyclotomic field, this would be tantamount to saying that the least prime p congruent to 1 (mod m) is $O(m^{1+\epsilon})$, which is related to the problem of Linnik's constant. We see that the error term in the theorem dominates if in fact

$$\mathscr{P}_K \ll \left(\frac{\log|d_K|}{\log\log|d_K|}\right)^2,$$

which is the case with the counterexamples to Ihara's conjecture provided in [9].

5. The least prime of degree one. We can split the sum

$$\sum_{N(\mathfrak{a})\le x} \Lambda(\mathfrak{a})$$

into two parts, according as $N(\mathfrak{a})$ is prime or not. In the second sum, the norm is a power of a prime p^f with $f \ge 2$. As there are at most $[K:\mathbb{Q}]$ prime ideals above a given prime, we see that the second sum is $O([K:\mathbb{Q}]\sqrt{x}\log x)$. Thus,

$$\sum_{N(\mathfrak{a})\le x} \Lambda(\mathfrak{a}) = \sum_{p\le x} \Lambda_K(p) + O([K:\mathbb{Q}]\sqrt{x}\log x).$$

Comparing this with (17), we deduce

$$\sum_{N(\mathfrak{a})\le x} \Lambda(\mathfrak{a}) = \sum_{p\le x} \Lambda_K(p) + O(\Delta_K(x)).$$

Therefore, applying Proposition 2.1 to the series

$$\sum_p \frac{\Lambda_K(p)}{p^s}$$

we deduce from (3):

PROPOSITION 5.1. *Assuming GRH for $\zeta_K(s)$, we have*

$$\gamma_K = \left[\log N - \sum_{p\le N-1} \frac{\Lambda_K(p)}{p}\right] + O\left(\frac{\log|d_K|}{\sqrt{N}} + [K:\mathbb{Q}]\frac{\log N}{\sqrt{N}}\right). \tag{19}$$

If we let P_K be the smallest prime of degree one in K, then the method of the preceding section now leads to:

COROLLARY 1. *Assuming GRH, we have*

$$\gamma_K = \log P_K + O\left(\frac{\log|d_K|}{\sqrt{P_K}}\right).$$

Thus, from these theorems, we see the intimate connection between γ_K and P_K, modulo GRH. One can also write down unconditional results using the known (unconditional) error term in the prime ideal theorem. We will not do so since we move towards another perspective that reveals a link between γ_K and the hypothetical Siegel zero.

6. An application of Stark's lemma. The following simple lemma due to Stark [34] is derived from the Hadamard factorization theorem applied to the Dedekind zeta-function. We will give a streamlined proof.

LEMMA 6.1 (Stark, 1974). *Let K be an algebraic number field of degree $n = r_1 + 2r_2$, where K has r_1 real conjugate fields and $2r_2$ complex conjugate fields. Recall that*

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)}$$

is the digamma function. Then, for any $s \in \mathbb{C}$,

$$(20)\qquad -\frac{\zeta_K'(s)}{\zeta_K(s)} - \frac{1}{s-1} + \sum_\rho \frac{1}{s-\rho} = \frac{1}{2}\log|d_K| + \left(\frac{1}{s} - \frac{n}{2}\log\pi\right) + \frac{r_1}{2}\psi\left(\frac{s}{2}\right) + r_2(\psi(s) - \log 2),$$

where the summation is over the non-trivial zeros of $\zeta_K(s)$. In particular, for $\sigma > 1$,

$$(21)\qquad -\frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)} - \frac{1}{\sigma-1} < \frac{1}{\sigma} + \frac{1}{2}\log\left(\frac{|d_K|}{2^{2r_2}\pi^n}\right) + \frac{r_1}{2}\psi\left(\frac{\sigma}{2}\right) + r_2\psi(\sigma).$$

If $K \neq \mathbb{Q}$, then $\zeta_K(s)$ has at most one zero in the region S given by

$$\Re(s) > 1 - \frac{1}{4\log|d_K|}, \qquad |\Im(s)| \le \frac{1}{4\log|d_K|}.$$

If such a zero exists, it is real and simple and called a Siegel zero.

Proof. Formula (20) is [34, (9)]. As usual, we should group ρ and $\overline{\rho}$ in the summation to ensure absolute convergence. For $\sigma > 1$, we have

$$\frac{1}{\sigma-\rho} + \frac{1}{\sigma-\overline{\rho}} > 0,$$

and this gives (21). Finally, if there are two non-real zeros in the region S, we rewrite (20) as

$$(22)\qquad \sum_\rho \frac{1}{s-\rho} = \frac{1}{s-1} + \frac{1}{2}\log|d_K| + \left(\frac{1}{s} - \frac{n}{2}\log\pi\right) + \frac{r_1}{2}\psi\left(\frac{s}{2}\right) + r_2(\psi(s) - \log 2) + \frac{\zeta_K'(s)}{\zeta_K(s)}.$$

As $\psi(\sigma)$ is monotonically increasing for $0 < \sigma \leq 2$ and

$$\psi(1) = -\gamma, \qquad \psi(2) = -\gamma + 1 < \log 2,$$

we see that all the terms on the right hand side of (22) after the term $\frac{1}{2}\log|d_K|$ are negative. Therefore, for $1 < \sigma < 2$, we have

$$\sum_{\rho\in S} \frac{1}{\sigma-\rho} < \frac{1}{\sigma-1} + \frac{1}{2}\log|d_K|.$$

If $\rho = \beta + i\eta$ is in S with $\eta \neq 0$, then $\overline{\rho}$ is also in S and pairing them together gives the inequality

$$\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\eta^2} < \frac{1}{\sigma-1} + \frac{1}{2}\log|d_K|.$$

As $K \neq \mathbb{Q}$, $|d_K| \geq 3$, and for $\sigma = 1 + \frac{1}{\log|d_K|} < 2$ we have

$$\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\eta^2} < \frac{3}{2}\log|d_K|. \tag{23}$$

Writing $A = \frac{1}{2}\log|d_K|$ and $\beta = 1-\epsilon$, we see that $\sigma - \beta = \frac{1}{2A} + \epsilon$ and the above inequality is equivalent to

$$2\left(\frac{1}{2A}+\epsilon\right) < 3A\left[\left(\frac{1}{2A}+\epsilon\right)^2 + \eta^2\right].$$

As $|\epsilon| < \frac{1}{8A}$ and $|\eta| < \frac{1}{8A}$, we deduce that $32A < 28A$, a contradiction. A similar analysis shows we cannot have two real zeros or a real zero which is not simple in the region S. ∎

The fecundity of (22) is amazing. We record two results needed later. The first is a fundamental inequality, and the second is an effective bound on the number $n(t)$ of zeros ρ of $\zeta_K(s)$ with $t < |\Im(\rho)| < t+1$. The method is standard (see for example [30, p. 246]). For future reference, we record these in the following lemmas.

Lemma 6.2. *For real $s > 1$,*

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} < -\frac{\zeta_K'(s)}{\zeta_K(s)} + \sum_{\rho} \frac{1}{s-\rho} < \frac{1}{s-1} + \frac{1}{2}\log|d_K|. \tag{24}$$

Lemma 6.3. *We have*

$$n(t) \ll \log|d_K| + [K:\mathbb{Q}]\log(|t|+1),$$

where the implied constant is absolute.

Theorem 6.4. *We have*

$$\sum_{n\leq x} \frac{\Lambda_K(n)}{n} \leq e\left(\log x + \frac{1}{2}\log|d_K|\right). \tag{25}$$

Proof. For any $\epsilon > 0$, by (24) we have

$$\sum_{n\le x}\frac{\Lambda_K(n)}{n} \le \sum_{n=1}^{\infty}\frac{\Lambda_K(n)}{n}\left(\frac{x}{n}\right)^{\epsilon} < x^{\epsilon}\left(\epsilon^{-1}+\frac{1}{2}\log|d_K|\right).$$

Choosing $\epsilon = 1/\log x$ gives the result. ∎

We will improve Ihara's unconditional lower bound for γ_K using the above lemma.

THEOREM 6.5. *For any algebraic number field K, we have*

$$\sum_{\rho}\frac{1}{\rho} = \gamma_K + \frac{1}{2}\log|d_K| - \frac{\gamma n}{2} - (r_1+r_2)\log 2 - \frac{n}{2}\log\pi + 1, \tag{26}$$

$$\gamma_K \ge -\frac{1}{2}\log|d_K| + \frac{\gamma n}{2} + (r_1+r_2)\log 2 + \frac{n}{2}\log\pi - 1. \tag{27}$$

Proof. Recall that from (2), we have

$$-\gamma_K = \lim_{s\to 1^+}\left[-\frac{\zeta_K'(s)}{\zeta_K(s)} - \frac{1}{\sigma-1}\right].$$

Taking the limit $s \to 1$ in (20) and noting the well-known values for the digamma function,

$$\psi(1) = -\gamma, \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2,$$

along with the observation that

$$\sum_{\rho}\frac{1}{1-\rho} = \sum_{\rho}\frac{1}{\rho},$$

gives the first assertion. For the second, we take the limit $\sigma \to 1^+$ in (21), and deduce

$$-\gamma_K \le 1 + \frac{1}{2}\log\left(\frac{|d_K|}{2^{2r_2}\pi^n}\right) + \frac{r_1}{2}\psi\left(\frac{1}{2}\right) + r_2\psi(1).$$

Again inserting the special values of the digamma function, we deduce

$$\gamma_K \ge -\frac{1}{2}\log|d_K| + \frac{\gamma n}{2} + (r_1+r_2)\log 2 + \frac{n}{2}\log\pi - 1,$$

as claimed. This improves Ihara's bound for $n \ge 2$ as $\log\pi > 1$. ∎

We make some editorial remarks. The positivity of (26) is step one in Li's criterion for the generalized Riemann hypothesis. Formula (26) also appears in earlier literature (see [13, Theorem B]) and it allows us to derive an unconditional upper bound for γ_K in terms of Siegel's zero, which does not seem to have been noted anywhere though both (26) and (27) are nascent in Ihara's paper [15].

Proof of Theorem 1.2. By pairing ρ and $1-\rho$ on the left hand side of (26) and taking real parts, we have

$$\frac{1}{2}\sum_{\rho}\Re\left(\frac{1}{\rho}+\frac{1}{1-\rho}\right)=\gamma_K+\frac{1}{2}\log|d_K|-\frac{\gamma n}{2}-(r_1+r_2)\log 2-\frac{n}{2}\log\pi+1.$$

The sum on the left hand side is absolutely convergent and splits into two parts according as $|\Im(\rho)|\le 1$ and $|\Im(\rho)|>1$. Using Lemma 6.3, the second sum is easily seen to be $\log|d_K|$. Again by Lemma 6.3, the number of terms in the first sum is $O(\log|d_K|)$. We will estimate

$$\sum_{\rho\notin S\colon|\Im(\rho)|<1}\Re\left(\frac{1}{1-\rho}\right).$$

By the cosine law,

$$|1+\delta-\rho|^2\le|1-\rho|^2+\delta^2+2|1-\rho|\delta.$$

Choosing $\delta=1/\log|d_K|$, and noting that for $\rho\notin S$, $4|1-\rho|\ge\delta$, we deduce

$$|1+\delta-\rho|^2\le 25|1-\rho|^2.$$

Consequently,

$$\Re\left(\frac{1}{1-\rho}\right)=\frac{1-\Re(\rho)}{|1-\rho|^2}\le 25\frac{1+\delta-\Re(\rho)}{|1+\delta-\rho|^2}=25\Re\left(\frac{1}{1+\delta-\rho}\right),$$

and (24) now implies

$$\sum_{\rho\notin S\colon|\Im(\rho)|<1}\Re\left(\frac{1}{1+\delta-\rho}\right)\le\frac{1}{\delta}+\log|d_K|.$$

Combining everything gives the result if there is no Siegel zero. If there is a Siegel zero β_0, the contribution from it must be taken into account in our sum over the zeros, and the result is now clear. ∎

Corollary 2. *If $K/\mathbb{Q}$ is normal and contains no quadratic subfield, then $\gamma_K=O(\log|d_K|)$.*

Proof. By [34, Lemma 8], the existence of β_0 implies that K contains a quadratic field F such that $\zeta_F(\beta_0)=0$. Hence the corollary. ∎

There are effective and ineffective bounds for β_0. We will not discuss here the ramifications of all the variegated results that one can obtain. Rather, we single out one. As noted in the proof of the previous corollary, β_0 arises from a quadratic subfield F of discriminant d_F, and Siegel's bound for β_0 gives the ineffective

$$1-\beta_0\gg|d_F|^{-\epsilon}.$$

As $1/2 < \beta_0 < 1$, this means $\beta_0(1-\beta_0) \gg |d_F|^{-\epsilon}$. As $|d_F|^{n/2}$ divides d_K (see, for example, [29, Exercise 5.6.25]), we deduce that

$$\beta_0(1-\beta_0) \gg |d_K|^{-2\epsilon/n}.$$

Injecting this into Theorem 1.2, we obtain:

COROLLARY 3. *For any number field K and for any $\epsilon > 0$, we have*

$$\gamma_K = O(|d_K|^{2\epsilon/n}),$$

where the implied constant depends only on ϵ.

As Siegel's theorem is effective if we take $\epsilon = 1/2$, we get the effective estimate $\gamma_K = O(|d_K|^{1/n})$. It is worth noting that in the case of the cyclotomic field, we deduce from the above corollary the result of Ihara, V. Kumar Murty and M. Shimura [17] cited in the introduction.

7. Explicit formula and some identities. The previous sections underscore the need for error terms to estimate the size of γ_K, and this motivates our discussion of the explicit formula method of the next and later sections. One of the main ingredients to prove Theorem 1.5 is an explicit formula by S. Gun, the second author and P. Rath [11]. As usual, $\Lambda(n)$ denotes the von Mangoldt function given by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < x < 1$ and $1/x$ not a prime power, we have the following explicit formula (see Ingham [18, p. 81]):

$$\sum_{n \le 1/x} \frac{\Lambda(n)}{n} = -\log x - \gamma + \sum_{\rho} \frac{x^\rho}{\rho} + \frac{1}{2}\log\frac{1+x}{1-x} - x,$$

where γ is the Euler–Mascheroni constant and ρ runs over all the non-trivial zeros of $\zeta(s)$. The generalization of the above explicit formula to the Selberg class $\mathbb{S}$ was established in [11].

Let us recall that the Selberg class $\mathbb{S}$, introduced by A. Selberg [32], consists of meromorphic functions $F(s)$ with the following properties:

(1) (*Dirichlet series*) F can be expressed as a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

which is absolutely convergent in the region $\Re(s) > 1$. We also normalize the leading coefficient as $a_F(1) = 1$.

(2) (*Analytic continuation*) There exists a non-negative integer m_F such that $(s-1)^{m_F} F(s)$ is an entire function of finite order.

(3) (*Functional equation*) There exist real numbers $Q > 0$ and $\lambda_i > 0$, and complex numbers μ_i and w, with $\Re(\mu_i) \geq 0$ and $|w| = 1$, such that

$$\Phi(s) := Q^s \prod_i \Gamma(\lambda_i s + \mu_i) F(s) \tag{28}$$

satisfies the functional equation

$$\Phi(s) = w\bar{\Phi}(1 - \bar{s}).$$

(4) (*Euler product*) There is an Euler product of the form

$$F(s) = \prod_{p \text{ prime}} F_p(s), \tag{29}$$

where

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}$$

with $b_F(p^k) = O(p^{k\theta})$ for some $\theta < 1/2$.

(5) (*Ramanujan hypothesis*) For any $\epsilon > 0$,

$$|a_F(n)| = O_\epsilon(n^\epsilon). \tag{30}$$

The Selberg class has been extensively studied, and interested readers may refer to survey articles [5, 31, 19] to get an account of the recent developments.

For $\Re(s) > 1$ and $F \in \mathbb{S}$, write

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s}.$$

Since F satisfies an Euler product, the coefficients $b_F(n)$ are only supported on prime powers. Thus, the logarithmic derivative of $F(s)$ for $\Re(s) > 1$ is given by

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s},$$

where $\Lambda_F(n)$ is given by $\Lambda_F(n) = b_F(n) \log n$. This function $\Lambda_F(n)$ is considered the generalization of the von Mangoldt function $\Lambda(n)$. In analogy with Ihara's definition of the Euler–Kronecker constants of a number field, we define, for $F \in \mathbb{S}$, the Euler–Kronecker constant γ_F via the Laurent expansion

$$-\frac{F'(s)}{F(s)} = \frac{m}{s-1} - \gamma_F + O(s-1).$$

We now state the explicit formula for elements in $\mathbb{S}$, as shown in [11].

Theorem 7.1 (GMR explicit formula). *Let $F \in \mathbb{S}$. For any $x \in (0,1)$ such that $1/x$ is not a prime power, we have*

$$\sum_{n\le 1/x} \frac{\Lambda_F(n)}{n} = -m_F \log x - \gamma_F + \sum_\rho \frac{x^\rho}{\rho} + \sum_{j=1}^{r}\biggl(\lambda_j x^{1+\mu_j/\lambda_j} f_{\lambda_j+\mu_j}(x^{1/\lambda_j}) + \frac{\lambda_j x^{1+\mu_j/\lambda_j}}{\lambda_j+\mu_j}\biggr),$$

where ρ runs over all the non-trivial zeros of F and

$$f_u(z) := \sum_{n=1}^{\infty} \frac{z^n}{n+u}.$$

For a number field $K/\mathbb{Q}$, the Dedekind zeta-function $\zeta_K(s)$ is an element in the Selberg class. Suppose r_1 and $2r_2$ denote the number of real and complex embeddings of K respectively. The function

$$\xi_K(s) := |d_K|^{s/2}\, \Gamma_{\mathbb{R}}(s)^{r_1}\, \Gamma_{\mathbb{C}}(s)^{r_2}\, \zeta_K(s),$$

satisfies the functional equation $\xi_K(s) = \xi_K(1-s)$. Here,

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\,\Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\,\Gamma(s).$$

We review some basic facts about cyclotomic fields that will be used below. First, the only prime that ramifies in $\mathbb{Q}(\zeta_q)$ is q and it ramifies totally. That is, $(q) = \mathfrak{q}^{q-1}$ with $N\mathfrak{q} = q$. If p is a prime coprime to q, then p factors as a product of g distinct prime ideals $\mathfrak{p}_i$ each with norm p^{f_p} and f_p is the order of $p \pmod q$. Thus, $q-1 = gf_p$.

Next, in the case of the cyclotomic field $\mathbb{Q}(\zeta_q)$, $r_1 = 0, r_2 = (q-1)/2$ and $\lambda_j = 1$, $\mu_j = 0$. In this setting, the last term in Theorem 7.1 simplifies considerably. With x replaced by $1/x$, we have

$$\sum_{n\le x} \frac{\Lambda_q(n)}{n} = \log x - \gamma_q + \sum_\rho \frac{1}{x^\rho \rho} - \Bigl(\frac{q-1}{2}\Bigr)\log\Bigl(1-\frac{1}{x}\Bigr). \tag{31}$$

Lemma 7.2. *For a cyclotomic field $K = \mathbb{Q}(\zeta_q)$, where q is a prime and $x > 1$ not a prime power, we have*

$$\gamma_q = -(q-1)\sum_{\substack{n\equiv 1 \bmod q\\ n\le x}} \frac{\Lambda(n)}{n} + \log x + \sum_\rho \frac{1}{x^\rho\,\rho} - \frac{q-1}{2}\log\Bigl(1-\frac{1}{x}\Bigr) - \frac{\log q}{q-1} + O\Bigl(\frac{\log q}{q^2}\Bigr),$$

where the implied constants are independent of q and x.

Proof. Let $K = \mathbb{Q}(\zeta_q)$ and $p\mathcal{O}_K = \mathfrak{p}_1\cdots\mathfrak{p}_g$ be the decomposition of the ideal $p\mathcal{O}_K$. Since $K/\mathbb{Q}$ is a Galois extension with degree $q-1$, we have

$q-1 = gf_p$, where $p^{f_p} = |\mathcal{O}_K/\mathfrak{p}_j| = N\mathfrak{p}_j$ for all $1 \le j \le g$. Thus, for $\Re(s) > 1$,

$$\zeta_K(s) = \left(1 - \frac{1}{q^s}\right)^{-1} \prod_{\substack{p \ne q \\ p \text{ prime}}} \left(1 - \frac{1}{p^{f_p s}}\right)^{-\frac{q-1}{f_p}}.$$

Taking log, we obtain

$$\log \zeta_K(s) = -\log\left(1 - \frac{1}{q^s}\right) - \sum_{\substack{p \ne q \\ p \text{ prime}}} \frac{q-1}{f_p} \log\left(1 - \frac{1}{p^{f_p s}}\right).$$

Differentiating both sides

$$\frac{\zeta_K'}{\zeta_K}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda_q(n)}{n^s} = -\sum_{l \ge 1} \frac{\log q}{q^{ls}} - \sum_{\substack{p \ne q \\ p \text{ prime}}} \sum_{l \ge 1} \frac{(q-1)\log p}{p^{l f_p s}}.$$

Hence,

$$\Lambda_q(n) = \begin{cases} (q-1)\log p & \text{if } n = p^{kf_p} \text{ for some } k \in \mathbb{N},\ p \ne q, \\ \log q & \text{if } n = q^l \text{ for some } l \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Thus,

$$\sum_{n \le x} \frac{\Lambda_q(n)}{n} - \sum_{\substack{n \equiv 1 \bmod q \\ n \le x}} \frac{(q-1)\Lambda(n)}{n} = \sum_{q^l \le x} \frac{\log q}{q^l} = \frac{\log q}{q-1} + O\left(\frac{\log q}{q^2}\right), \tag{33}$$

noting the remark about f_p being the order of $p \pmod q$, underlined earlier. Using the GMR explicit formula (31), for any $x > 1$, not a prime power,

$$\gamma_q = -\sum_{n \le x} \frac{\Lambda_q(n)}{n} + \log x + \sum_{\rho} \frac{1}{x^\rho \rho} - \frac{q-1}{2}\left(\log\left(1 - \frac{1}{x}\right)\right).$$

Comparing with (33), we have the lemma. ∎

The function $Z(x) = \sum_\rho 1/(x^\rho \rho)$, which appears in the above lemma, is rather mysterious. Note that $\zeta_K(\rho) = 0$ if and only if $\zeta_K(\overline{\rho}) = 0$. Hence, $Z(x)$ is a real-valued function. From the GMR explicit formula, we see that the series $Z(x)$ converges for $x > 1$. Denoting the sum of the reciprocals of the real zeros in the critical strip by R, we note that

$$Z(1) - R = \sum_\rho \frac{1}{\rho} = \frac{1}{2} \sum_\rho \left(\frac{1}{\rho} + \frac{1}{\overline{\rho}}\right) = \frac{1}{2} \sum_\rho \frac{\rho + \overline{\rho}}{|\rho|^2}, \tag{34}$$

where the summations are over the non-real zeros. Since the number of zeros of $\zeta_K(s)$ in the critical strip with $|\Im(s)| < T$ is $\ll qT \log qT$, using partial summation, one can conclude that $Z(1)$ converges and is $O(q \log q)$. Therefore, it is of interest to understand whether $Z(x)$ is bounded as $x \to \infty$, as we vary over the number fields K. For cyclotomic fields, this is answered in Theorem 1.5.

It would appear that understanding $Z(x)$ for smaller values of x holds the key to unravelling the nature of γ_q. The study of $Z(x)$ associated to $\zeta_K(s)$ for any number field K is of independent interest. For instance, we know that $Z(1) > 0$ from (34). However, for $x > 1$, the sign of $Z(x)$ is unclear.

As we are interested in the real part of $Z(x)$, we can estimate it using GRH. Indeed, pairing up ρ and $\overline{\rho}$, we get

$$\Re(Z(x)) \ll x^{-1/2} \sum_{\rho} \frac{1}{|\rho|^2} \ll \frac{\log |d_K|}{\sqrt{x}},$$

an estimate consonant with our earlier study in previous sections.

8. Averaging γ_q and proof of Theorem 1.3. Let

$$\psi_q(x) = \sum_{n \le x} \Lambda_q(n).$$

Then $\psi_q(x) = 0$ for $x < q$, and for $x \ge q$, from (32) and formula (4) of Theorem 1.1 we have

$$\psi_q(x) = \left[\frac{\log x}{\log q}\right] + (q-1)\psi(x, q, 1).$$

Writing $\Delta_q(x) = \psi_q(x) - x$ and using formula (18) we have

$$\gamma_q = \log q - \int_q^\infty \frac{\Delta_q(x)}{x^2}\, dx.$$

For $x \ge q$, we have

$$\Delta_q(x) = (q-1)E(x, q, 1) + \left[\frac{\log x}{\log q}\right],$$

where

$$E(x, q, 1) = \psi(x, q, 1) - \frac{x}{q-1}.$$

Therefore,

$$\gamma_q = \log q + (q-1)\int_q^\infty \frac{E(x, q, 1)\, dx}{x^2} + O\left(\frac{1}{q}\right), \tag{35}$$

making it patently clear that the sign behaviour of γ_q is intimately connected with oscillations of the error term in the prime number theorem for arithmetic progressions. We will average this formula over primes q to deduce both (13) and Theorem 1.3. To this end, we need two lemmas. The first is a standard sieve result, and the second is a variation of a lemma of Fouvry [10].

LEMMA 8.1. *For a fixed positive integer m and $x > m$, the number of solutions of the equation*

$$p - 1 = mq,$$

with p and q prime $\le x$ is

$$\ll \frac{x}{\phi(m)\log^2(x/m)}.$$

Proof. This is a classical application of Brun's sieve and can be found in many places such as, for example, [3, Exercise 13, p. 110]. ■

LEMMA 8.2. *We have*

$$\sum_{q\sim Q,\, q\,\text{prime}} \psi(2x,q,1) - \psi(x,q,1) \ll \frac{x}{\log Q} + \frac{x}{\log x} + \sqrt{x}\log^2 x.$$

Proof. The sum in question is clearly bounded by

$$(\log x)\#\{(p,r,q) : p^r - 1 = qm,\ p^r \sim x,\ q \sim Q\}.$$

As in [10], we observe that the contribution from $r \ge 2$ is at most

$$\sum_{p^r\sim x,\, r\ge 2} d(p^r - 1) \ll x^{1/2}\log x,$$

where $d(n)$ denotes the divisor function. When $r = 1$, $Q \le \sqrt{x}$, we can apply the Brun–Titchmarsh theorem to bound the size of the set by

$$\sum_{q\sim Q,\, q\,\text{prime}} (\pi(2x,q,1) - \pi(x,q,1)) \ll \sum_{q\sim Q,\, q\,\text{prime}} \frac{x}{q\log x} \ll \frac{\pi(x)}{\log Q},$$

using the Chebyshev estimate

$$\sum_{q\sim Q,\, q\,\text{prime}} \frac{1}{q} = O\left(\frac{1}{\log Q}\right).$$

When $Q > \sqrt{x}$, we have $m \le \sqrt{x}$ and so we can reverse the roles of q and m. Applying Lemma 8.1, we see that the set is bounded by

$$\sum_{m\sim x/Q} \frac{x}{\phi(m)\log^2 x} \ll \frac{x}{\log^2 x},$$

using the well-known formula due to Landau,

$$\sum_{m\le x} \frac{1}{\phi(m)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\log x + c + O\left(\frac{\log x}{x}\right).$$

This completes the proof. ■

Proof of Theorem 1.3. Summing (35) over primes q, we have

$$\sum_{q\sim Q} |\gamma_q - \log q| \ll Q \int_Q^\infty \Big(\sum_{q\sim Q} |E(x,q,1)| \Big) \frac{dx}{x^2}.$$

We split the integral into two parts according as $x < Q^{1+\epsilon}$ and $x \geq Q^{1+\epsilon}$. In the second part, we apply the Elliott–Halberstam conjecture and deduce the total contribution to the right hand side of the above inequality is

$$\ll \frac{Q}{\log^A Q}$$

for any $A > 0$. The first part is

$$\ll Q \int_Q^{Q^{1+\epsilon}} \sum_{q\sim Q} \Big(\psi(x,q,1) + \frac{x}{q} \Big) \frac{dx}{x^2} \ll Q \int_Q^{Q^{1+\epsilon}} \Big(\sum_{q\sim Q} \psi(x,q,1) \Big) \frac{dx}{x^2} + \epsilon Q.$$

By Lemma 8.2, we have

$$Q \int_Q^{Q^{1+\epsilon}} \Big(\sum_{q\sim Q} \psi(x,q,1) \Big) \frac{dx}{x^2}$$
$$\ll Q \int_Q^{Q^{1+\epsilon}} \Big(\frac{x}{\log Q} + \frac{x}{\log x} + \sqrt{x}\, \log^2 x \Big) \frac{dx}{x^2} \ll \epsilon Q.$$

This completes the proof. ∎

We remark that if instead of breaking the integral at $Q^{1+\epsilon}$, we broke it at $Q^{2+\epsilon}$, then we could have used the Bombieri–Vinogradov theorem instead of the Elliott–Halberstam conjecture. This would then give an unconditional estimate and a new proof of (13).

9. Proof of Theorems 1.4 and 1.5. Towards the proof of Theorem 1.4, we first prove the following lemma, which is a more precise version of [9, Proposition 3].

LEMMA 9.1. *Let $\delta > 0$ be fixed. There is an $x_0(\delta)$ such that for any $x > x_0(\delta)$ and any prime q satisfying $\log x > q^\delta$,*

(36)
$$\gamma_q = -(q-1) \sum_{\substack{n\equiv 1 \bmod q \\ n\leq x}} \frac{\Lambda(n)}{n} + \log x - \frac{\log q}{q-1} + O((\log x)^{\frac{1}{2}+\frac{1}{\delta}} \exp(-c\sqrt{\log x}))$$

for some constant $c > 0$ independent of q and x.

Proof. For $K = \mathbb{Q}(\zeta_q)$,

$$\zeta_K(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s, \chi),$$

where the product is over all irreducible characters χ mod q and χ_0 is the principal character. Taking the logarithmic derivative on both sides, we deduce

$$\gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'}{L}(1, \chi). \tag{37}$$

For $\chi \neq \chi_0$,

$$\frac{L'}{L}(1, \chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n}.$$

Using similar methods as in the proof of prime number theorem, it is known (see [23, Chapter 11, Cor. 11.18]) that

$$B(x) := \sum_{n \leq x} \chi(n)\Lambda(n) \ll x \exp(-c\sqrt{\log x})$$

for $x > x_0(\delta)$ and $\log x > q^\delta$. By partial summation,

$$\sum_{n > x} \frac{\chi(n)\Lambda(n)}{n} = \frac{1}{x} B(x) + \int_x^\infty \frac{B(u)}{u^2}\, du = O(\sqrt{\log x}\, \exp(-c\sqrt{\log x})).$$

Therefore,

$$\frac{L'}{L}(1, \chi) = -\sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n} + O(\sqrt{\log x}\, \exp(-c\sqrt{\log x})).$$

From (37),

$$\begin{aligned} \gamma_q &= \gamma - \sum_{\substack{n \leq x \\ \chi \neq \chi_0}} \frac{\chi(n)\Lambda(n)}{n} + O(q\sqrt{\log x}\, \exp(-c\sqrt{\log x})) \\ &= \gamma - \sum_{\substack{n \leq x \\ \chi \neq \chi_0}} \frac{\chi(n)\Lambda(n)}{n} + O((\log x)^{\frac{1}{2}+\frac{1}{\delta}}\, \exp(-c\sqrt{\log x})). \end{aligned} \tag{38}$$

It is well-known (see [23, Exercise 4, Chapter 6.2]) that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x - \gamma + O(\exp(-c\sqrt{\log x})).$$

Using this and the fact that

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} q - 1 & \text{if } n \equiv 1 \bmod q, \\ 0 & \text{otherwise,} \end{cases}$$

we can write (38) as

$$\begin{aligned}\gamma_q &= \log x - \sum_{n\le x}\frac{\Lambda(n)}{n} - \sum_{\substack{n\le x\\ \chi\ne\chi_0}}\frac{\chi(n)\Lambda(n)}{n} + O((\log x)^{\frac12+\frac1\delta}\exp(-c\sqrt{\log x}))\\ &= -\frac{\log q}{q-1} + \log x - (q-1)\sum_{\substack{n\equiv 1 \bmod q\\ n\le x}}\frac{\Lambda(n)}{n}\\ &\quad + O\Big(\frac{\log x}{x}\Big) + O((\log x)^{\frac12+\frac1\delta}\exp(-c\sqrt{\log x})),\end{aligned}$$

where in the last step we have separated $q\,|\,n$ and q coprime to n in the first sum of the penultimate step and used

$$\sum_{n\le x,\, q|n}\frac{\Lambda(n)}{n} = \sum_{j=1}^{\infty}\frac{\log q}{q^j} - \sum_{q^j>x}\frac{\log q}{q^j} = \frac{\log q}{q-1} + O\Big(\frac{\log x}{x}\Big).$$

Noting that

$$O\Big(\frac{\log x}{x}\Big) = O((\log x)^{\frac12+\frac1\delta}\exp(-c\sqrt{\log x}))$$

completes the proof. ∎

Corollary 4. *For any $\epsilon>0$, we have $\gamma_q = O(q^\epsilon)$.*

Proof. We put $\log x = q^\epsilon$ in (36). The error term is clearly $O(q^\epsilon)$. The Brun–Titchmarsh inequality (see for example [9, Proposition 6]) shows that for $x\ge 10q$,

$$\sum_{\substack{n\equiv 1 \bmod q\\ n\le x}}\frac{\Lambda(n)}{n} \ll \frac{\log x + (\log q)(\log\log(x/q))}{q-1}.$$

By formula (36), the result follows. ∎

Proof of Theorem 1.4. Using partial summation and the Siegel–Walfisz theorem (noting that q is in the range of applicability), we get the identity

$$\sum_{\substack{n\equiv 1 \bmod q\\ n\le x}}\frac{\Lambda(n)}{n} = \int_2^x\frac{\psi(t;q,1)}{t^2}\,dt + O\Big(\frac1q\Big).$$

Moreover,

$$\int_2^x\frac{\psi(t;q,1)}{t^2}\,dt = \int_2^x\frac{\psi(t;q,1)-\big(\frac{t}{q-1}\big)}{t^2}\,dt + \int_2^x\frac{\big(\frac{t}{q-1}\big)}{t^2}\,dt = A_{q,x} + \frac{\log x}{q-1} - \frac{\log 2}{q-1}.$$

Therefore,

$$A_{q,x} = \sum_{\substack{n\equiv 1 \bmod q\\ n\le x}} \frac{\Lambda(n)}{n} - \frac{\log x}{q-1} + O\Big(\frac{1}{q}\Big). \tag{39}$$

From Lemma 9.1, for $\log x > q^{\delta}$,

$$\begin{aligned} A_{q,x} &= -\frac{\gamma_q}{q-1} + O\Big(\frac{\log q}{q^2}\Big) + O\Big(\frac{1}{q-1}(\log x)^{\frac{1}{2}+\frac{1}{\delta}}\exp(-c\sqrt{\log x})\Big) \\ &= -\frac{\gamma_q}{q-1} + O\Big(\frac{1}{q}\Big). \end{aligned} \tag{40}$$

We now invoke V. Kumar Murty's result [26], as stated in (13), given by

$$\frac{1}{\pi(y)}\sum_{q\le y}|\gamma_q| = O(\log y).$$

By partial summation, we obtain

$$\sum_{q\le y}\frac{|\gamma_q|}{q-1} = O(\log y).$$

Using this in (40), we conclude that $\sum_{q\le y}|A_{q,x}| \ll \log y$. This proves Theorem 1.4. ∎

Proof of Theorem 1.5. Combining Lemmas 7.2 and 9.1, for $\log x > q^{\delta}$ and $x > x_0(\delta)$, the result now follows immediately. ∎

10. Concluding remarks. Two formulas of independent interest emerge from our study. One is (see (40))

$$\begin{aligned} A_{q,x} &= -\frac{\gamma_q}{q-1} + O\Big(\frac{\log q}{q^2}\Big) + O\Big(\frac{1}{q-1}(\log x)^{\frac{1}{2}+\frac{1}{\delta}}\exp(-c\sqrt{\log x})\Big) \\ &= -\frac{\gamma_q}{q-1} + O\Big(\frac{1}{q}\Big), \end{aligned}$$

and the other is (see (39))

$$A_{q,x} = \sum_{\substack{n\equiv 1 \bmod q\\ n\le x}} \frac{\Lambda(n)}{n} - \frac{\log x}{q-1} + O\Big(\frac{1}{q}\Big),$$

both valid for $\log x > q^{\delta}$. Thus,

$$-\sum_{q\le y}\frac{\gamma_q}{q-1} = \sum_{n\le x}\frac{\Lambda(n)}{n}\Big[\omega_y(n-1) - \sum_{q\le y}\frac{1}{q-1}\Big] + O(\log\log y).$$

Here, $\omega_y(n)$ denotes the number of distinct prime divisors of n not exceeding y. Heuristic reasoning would suggest that $\omega_y(n)$ has normal order $\log\log y$ and the Erdős–Kac theorem is inapplicable given our range of y compared to x. This naturally leads to the question of studying small prime divisors of $p+a$, where p runs over prime numbers and a is a fixed integer. This motivated us to study the distribution of $\omega_y(p+a)$, where p runs over primes, and in [7], we establish a "localized" Erdős–Kac theorem for $\omega_y(p+a)$, which relies on a general method initiated in [6].

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Abstract (will appear on the journal's web site only)

As a natural generalization of the Euler–Mascheroni constant γ, Y. Ihara introduced the Euler–Kronecker constant γ_K attached to any number field K. He obtained bounds on γ_K conditional upon the generalized Riemann hypothesis. In this paper, we establish unconditional bounds on γ_K in terms of the Siegel zero of $\zeta_K(s)$. We also produce an alternative proof of Ihara's theorem without invoking the explicit formula. Furthermore, using known upper bounds on $\gamma_{\mathbb{Q}(\zeta_q)}$, we obtain a bound on the error term in the prime number theorem, averaging over certain arithmetic progressions.