AN EFFECTIVE BOUND ON GENERALIZED DIOPHANTINE m-TUPLES

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Abstract. For $k \geq 2$ and a non-zero integer n, a generalized Diophantine m-tuple with property $D_k(n)$ is a set of m positive integers $S = \{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a k-th power for $1 \leq i < j \leq m$. Define $M_k(n) := \sup\{|S| : S \text{ has property } D_k(n)\}$. In a recent work, the second author, S. Kim and M. R. Murty proved that $M_k(n)$ is $O(\log n)$, for a fixed k, as we vary n. In this paper, we obtain effective upper bounds on $M_k(n)$. In particular, we show that for $k \geq 2$, $M_k(n) \leq 3\,\phi(k)\,\log n$, if n is sufficiently larger than k.

1. Introduction

Given a non-zero integer n, we say a set of natural numbers $S = \{a_1, a_2, \ldots, a_m\}$ is a Diophantine m-tuple with property $D(n)$ if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Diophantus first studied such sets of numbers and found the quadruple $\{1, 33, 68, 105\}$ with property $D(256)$. The first $D(1)$-quadruple $\{1, 3, 8, 120\}$ was discovered by Fermat, and this was later generalized by Euler, who found the following infinite family of quadruples with property $D(1)$, namely

$$\{a, b, a + b + 2r, 4r(r + a)(r + b)\},$$

where $ab + 1 = r^2$. In fact, it is known that any $D(1)$-triple can be extended to a Diophantine quadruple [1]. In 1969, using Baker's theory of linear forms in logarithm of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [2] proved that Fermat's example is the only extension of $\{1, 3, 8\}$ with property $D(1)$. In 2004, Dujella [10], using similar methods, proved that there are no $D(1)$-sextuples and there are only finitely many $D(1)$-quintuples, if any. The conjecture on the non-existence of $D(1)$-quintuples was finally settled in 2019 by He, Togbé, and Ziegler in [15].

However, in general, there are $D(n)$-quintuples for $n \neq 1$. For instance,

$$\{1, 33, 105, 320, 18240\} \quad \text{and} \quad \{5, 21, 64, 285, 6720\}$$

are Diophantine quintuples satisfying property $D(256)$. Also, we note that there are known examples of $D(n)$-sextuples, but no $D(n)$-septuple is known. So, it is natural to study the size of the largest tuple with property $D(n)$. Define

$$M_n := \sup\{|S| : S \text{ satisfies property } D(n)\}.$$

In 2004, Dujella [9] showed that

$$M_n \leq C \log |n|,$$

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where C is an absolute constant. He also showed that for $n > 10^{100}$, one can choose $C = 8.37$. This constant was improved by Becker and Murty [3], who showed that for any n,

$$M_n \leq 2.6071 \log |n| + \mathcal{O}\left(\frac{\log |n|}{(\log \log |n|)^2} \right). \tag{1}$$

Our goal is to study this problem when squares are replaced by higher powers.

Definition 1 (generalized Diophantine m-tuples)**.** *Fix a natural number $k \geq 2$. A set of natural numbers $S = \{a_1, a_2, \ldots, a_m\}$ is said to satisfy property $D_k(n)$ if $a_i a_j + n$ is a k-th power for all $1 \leq i < j \leq m$.*

We analogously define the following quantity for each non-zero integer n,

$$M_k(n) := \sup\{|S| : S \text{ satisfies property } D_k(n)\}.$$

For $k \geq 3$ and $m \geq 3$, we can apply the celebrated theorem of Faltings [12] to deduce that a superelliptic curve of the form

$$y^k = f(x) = (a_1 x + n)(a_2 x + n)(a_3 x + n)(a_4 x + n) \ \cdots \ (a_m x + n)$$

has only finitely many rational points and a fortiori, finitely many integral points. Therefore, a set S satisfying property $D_k(n)$ must be finite. When $n = 1$, Bugeaud and Dujella [6] showed that

$$M_3(1) \leq 7, \quad M_4(1) \leq 5, \quad M_k(1) \leq 4 \text{ for } 5 \leq k \leq 176, \text{ and } M_k(1) \leq 3 \text{ for } k \geq 177.$$

In other words, the size of $D_k(1)$-tuples is bounded by 3 for large enough k. In the general case, for any $n \neq 0$ and $k \geq 3$, Bérczes, Dujella, Hajdu and Luca [5] obtained upper bounds for $M_k(n)$. In particular, they showed that for $k \geq 5$

$$M_k(n) \leq 2|n|^5 + 3.$$

In [8], the second author, S. Kim and M. R. Murty improved the above bounds on $M_k(n)$ for large n and a fixed k. Define

$$M_k(n; L) := \sup\{|S \cap [|n|^L, \infty)| : S \text{ satisfies property } D_k(n)\}$$

They proved that for $k \geq 3$, as $n \to \infty$,

$$M_k(n, L) \ll_{k,L} 1, \text{ for } L \geq 3 \quad \text{and} \quad M_k(n) \ll_k \log n. \tag{2}$$

The purpose of this paper is to explicitly obtain the implied constants in (2). In [8], the bounds for $M_k(n)$ were proven under the further assumption that $n > 0$. However, this is not necessary, as was remarked in [8], but an argument was not provided. We first prove the bounds (2) for all non-zero integers n.

Theorem 1.1. *Let $k \geq 3$ be an integer. Then the following holds as $|n| \to \infty$:*

(1) *For $L \geq 3$,*
$$M_k(n, L) \ll 1,$$
where the implied constant depends on k and L, but is independent of n.

(2) *Moreover,*
$$M_k(n) \ll \log |n|$$
where the implied constant depends on k.

We now state our main theorem, which is the effective version of Theorem 1.1.

Theorem 1.2. *Let $k \geq 3$ be a positive integer. Then the following holds.*

(a) *For $L \geq 3$,*

$$M_k(n, L) \leq 2^{28} \log(2k) \log(2 \log(2k)) + 14. \tag{3}$$

(b) *Suppose n and k vary such that as $|n| \to \infty$ and*

$$k = o(\log \log |n|).$$

Then

$$M_k(n) \leq 3\, \phi(k)\, \log |n| + \mathcal{O}\left(\frac{(\phi(k))^2 \log |n|}{\log \log |n|} \right),$$

where $\phi(n)$ denotes the Euler totient function.

Remarks.

i. It is possible to replace 14 on the right hand side of (3) with a smaller positive integer for large values of k.

ii. For a fixed $k > 2$, from Theorem 1.2(b), we obtain that as $|n| \to \infty$

$$M_k(n) \leq 3\, \phi(k)\, \log |n| + \mathcal{O}\left(\frac{\log |n|}{\log \log |n|} \right).$$

This upper bound is very close to the case $k = 2$, where the best known upper bound due to Becker and Murty is given by (1).

2. **Preliminaries**

In this section, we recall and develop the necessary tools to prove our main theorems.

2.1. **Gallagher's larger sieve.** In 1971, Gallagher [13] discovered an elementary sieve inequality which he called the larger sieve. We refer the reader to [7] for the general discussion but record the result in a form applicable to our context.

Theorem 2.1. *Let N be a natural number and $\mathcal{S}$ a subset of $\{1, 2, \ldots, N\}$. Let $\mathcal{P}$ be a set of primes. For each prime $p \in \mathcal{P}$, let $\mathcal{S}_p = \mathcal{S} \pmod p$. For any $1 < Q \leq N$, we have*

$$|\mathcal{S}| \leq \frac{\sum\limits_{p \leq Q, p \in \mathcal{P}} \log p - \log N}{\sum\limits_{p \leq Q, p \in \mathcal{P}} \frac{\log p}{|\mathcal{S}_p|} - \log N}, \tag{4}$$

where the summations are over primes $p \leq Q, p \in \mathcal{P}$ and the inequality holds provided the denominator is positive.

2.2. **A quantitative Roth's theorem.** Quantitative results related to counting exceptions in Roth's celebrated theorem on Diophantine approximations were established by a variety of authors. We will use the following result due to Evertse [11]. For an algebraic number ξ of degree r, we define the (absolute) height by

$$H(\xi) := \left(a \prod_{i=1}^{r} \max(1, |\xi^{(i)}|) \right)^{1/r},$$

where $\xi^{(i)}$ for $1 \leq i \leq r$ are the conjugates (over $\mathbb{Q}$) and a is the positive integer such that

$$a \prod_{i=1}^{r} (x - \xi^{(i)})$$

has rational integer coefficients with gcd 1.

Theorem 2.2. *Let α be a real algebraic number of degree r over $\mathbb{Q}$, and $0 < \kappa \leq 1$. The number of rational numbers p/q satisfying* $\max(|p|, |q|) \geq \max(H(\alpha), 2)$,

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{\max(|p|, |q|)^{2+\kappa}}$$

is at most

$$2^{25} \kappa^{-3} \log(2r) \log(\kappa^{-1} \log(2r)).$$

2.3. **Vinogradov's theorem.** The following bound on character sums was proved by Vinogradov (see [16]).

Lemma 2.3. *Let $\chi \pmod q$ be a non-trivial Dirichlet character and n be an integer such that $(n, q) = 1$. If $\mathcal{A} \subseteq (\mathbb{Z}/q\mathbb{Z})^*$ and $\mathcal{B} \subseteq (\mathbb{Z}/q\mathbb{Z})^* \cup \{0\}$, then*

$$\sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \chi(ab + n) \leq \sqrt{q |\mathcal{A}| |\mathcal{B}|}.$$

The original method of Vinogradov does not produce the bound above and instead gives the right hand side as $\sqrt{2q|\mathcal{A}||\mathcal{B}|}$. However, the above bound holds and a short proof of this can be found in [3, Proposition 2.5].

2.4. **Prime bounds in arithmetic progression.** Let Q, k, a are positive integers with $(a, k) = 1$. Denote by $\theta(Q; k, a)$ the sum of the logarithms of the primes $p \equiv a \pmod k$ with $p \leq Q$, i.e.,

$$\theta(Q; k, a) := \sum_{\substack{p \equiv a \bmod k \\ p \text{ prime} \leq Q}} \log p.$$

We will need the following bound on $\theta(Q; k, a)$ obtained by Bennet, Martin, O'Bryant, Rechnitzer in [4, Theorem 1.2]:

Theorem 2.4. *For $k \geq 3$ and $(a, k) = 1$,*

$$\left| \theta(Q; k, a) - \frac{Q}{\phi(k)} \right| < \frac{1}{160} \frac{Q}{\log Q} \tag{5}$$

for all $Q \geq Q_0(k)$ where

$$Q_0(k) = \begin{cases} 8 \cdot 10^9 & \text{if } 3 \leq k \leq 10^5 \\ \exp(0.03\sqrt{k} \log^3 k) & \text{if } k > 10^5. \end{cases}$$

2.5. **Gap principle.** The next two lemmas are variations of a gap principle of Gyarmati [14]. The following lemma was proved in [8].

Lemma 2.5 ([8], Lemma 2.4)**.** *Let $k \geq 2$. Suppose that a, b, c, d are positive integers such that $a < b$ and $c < d$. Suppose further that*

$$ac + n, \quad bc + n, \quad ad + n, \quad bd + n$$

are perfect k-th powers. Then,

$$bd \geq k^k n^{-k} (ac)^{k-1}.$$

An immediate Corollary of this lemma shows that "large" elements of any set with property $D_k(n)$ have "super-exponential growth."

Corollary 2.1 ([8] Corollary 2)**.** *Let $k \geq 3$ and $m \geq 5$. Suppose that $n^3 \leq a_1 < a_2 < \ldots < a_m$ and the set $\{a_1, a_2, ..., a_m\}$ has property $D_k(n)$. Then $a_{2+3j} \geq a_2^{(k-1)^j}$ provided $1 \leq j \leq (m-2)/3$.*

A modification of the proof of the above Lemma 2.5 yields a gap principle for negative values of n.

Lemma 2.6. *For $n > 0$ and natural numbers a, b, c, d such that $n^3 \leq a < b < c < d$, we have*

$$(ac - n)(bd - n) \geq \frac{abcd}{2}.$$

Proof. Since

$$(ac - n)(bd - n) = abcd - n(ac + bd) + n^2,$$

it is enough to prove that

$$\frac{abcd}{2} \geq n(ac + bd) - n^2.$$

Also, since $a \geq n^3$ and $c > n^3$, we get for all cases other than $n = 1, a = 1, b = 2, c = 3$,

$$\begin{aligned} abcd &\geq 4nbd \\ &\geq 2nbd + 2nac \\ &\geq 2nbd + 2nac - 2n^2 \end{aligned}$$

where the first inequality is obvious as $a \geq n^3$ and $c \geq n^3 + 2$. Thus, we get the desired result. For the case $n = 1, a = 1, b = 2, c = 3$, clearly

$$2n(ac + bd) - 2n^2 = 4 + 4d < 6d = abcd$$

as $d > c$. □

We are now ready to prove the following analog of Lemma 2.5.

Lemma 2.7. *Let $n > 0$ and $k \geq 2$. Suppose that a, b, c, d are positive integers such that $n^3 \leq a < b < c < d$. Suppose further that $ac - n, bc - n, ad - n, bd - n$ are perfect k-th powers. Then,*

$$bd \geq k^k 2^{-k} n^{-k} (ac)^{k-1}.$$

Proof. Since $(b - a)(d - c) > 0$, we have

$$bd + ac > ad + bc,$$

and it is easily seen that

$$(ad - n)(bc - n) > (ac - n)(bd - n).$$

As $(ac-n)(bd-n)$ and $(ad-n)(bc-n)$ are both perfect k-th powers, we have

$$\begin{aligned}(ad-n)(bc-n) &\geq [((ac-n)(bd-n))^{1/k}+1]^k \\ &\geq (ac-n)(bd-n)+k((ac-n)(bd-n))^{k-1/k} \\ &\geq (ac-n)(bd-n)+k\left(\frac{abcd}{2}\right)^{k-1/k}\end{aligned}$$

where the last inequality follows from Lemma 2.6. Thus,

$$-n(ad+bc) \geq -n(ac+bd)+k\left(\frac{abcd}{2}\right)^{k-1/k}.$$

Since $bd > ad+bc-ac > 0$, we have $bd+ac-ad-bc < bd$ and hence we obtain

$$nbd > k\left(\frac{abcd}{2}\right)^{k-1/k}.$$

Therefore,

$$bd \geq k^k 2^{1-k} n^{-k}(ac)^{k-1} \geq k^k 2^{-k} n^{-k}(ac)^{k-1},$$

which proves the lemma. □

This enables us to prove super-exponential growth for large elements of a set with $D_k(n)$, when $n<0$.

Corollary 2.2. *Let $k \geq 3$. If $n^3 \leq a < b < c < d < e$ are natural numbers such that the set $\{a,b,c,d,e\}$ has property $D_k(-n)$, then $e \geq b^{k-1}$.*

Proof. From Lemma 2.7,

$$ce \geq k^k 2^{-k} n^{-k}(bd)^{k-1} \geq k^k 2^{-k} n^{-k}(bc)^{k-1}.$$

Therefore,

$$e \geq b^{k-1}c^{k-2}n^{-k}2^{-k} \geq b^{k-1}n^{2k-6}2^{-k} \geq b^{k-1}.$$

□

Using induction on the previous corollary, we deduce

Corollary 2.3. *Let $k \geq 3$ and $m \geq 5$. Suppose that $n^3 \leq a_1 < a_2 < \ldots < a_m$ and the set $\{a_1, a_2, \ldots, a_m\}$ has property $D_k(-n)$. Then $a_{2+3j} \geq a_2^{(k-1)^j}$ provided $1 \leq j \leq (m-2)/3$.*

3. Proof of the main theorems

3.1. **Proof of Theorem 1.1.** We first prove Theorem 1.1 as the proof follows a similar method as in [8].

Let n be a positive integer and $m = M_k(-n)$ and $S = \{a_1, a_2, a_3, \cdots, a_m\}$ be a generalized m-tuple with the property $D_k(-n)$. Suppose $n^L < a_1 < a_2 < \cdots < a_m$ for some $L \geq 3$. Consider the system of equations

$$\begin{aligned}a_1x-n &= u^k \\ a_2x-n &= v^k. \end{aligned} \tag{6}$$

Clearly, $x = a_i$ for $i \geq 3$ are solutions to this system. Also,

$$\left| a_2 u^k - a_1 v^k \right| = n(a_2 - a_1).$$

Let $\alpha := (a_1/a_2)^{1/k}$ and $\zeta_k := e^{2\pi i/k}$. Then, we have the following two lemmas analogous to the ones proved in [8].

Lemma 3.1. *Let $k \geq 3$ be odd. Suppose u, v satisfy the system of equations* (6). *Let*

$$c(k) := \prod_{j=1}^{(k-1)/2} \left(\sin \frac{2\pi j}{k} \right)^2 .$$

Then, for $n > 2^{1/(L-1)} c(k)^{-1/(L-1)}$,

$$\left| \frac{u}{v} - \alpha \right| \leq \frac{a_2}{2v^k}.$$

We omit the proof of this lemma here as it is identical to the proof of [8, Lemma 3.1].

Lemma 3.2. *Let (u_i, v_i) denote distinct pairs that satisfy the system of equations* (6) *with $v_{i+1} > v_i$. For $n > 2^{1/(L-1)} c(k)^{-1/(L-1)}$ and $i \geq 14$,*

$$\left| \frac{u}{v} - \alpha \right| < \frac{1}{v_i^{k-1/2}},$$

and $v_i > a_2^4$.

Proof. From Lemma 3.1, we have $\left| \frac{u_i}{v_i} - \alpha \right| < \frac{a_2}{2v_i^k}$. Thus, we need to show $a_2 < 2v_i^{1/2}$ for $i > 14$. Since $v_i^k = a_2 a_i - n$, we have $v_i \geq a_i^{1/k}$. By Corollary 2.3,

$$a_{2+3j} \geq a_2^{(k-1)^j}$$

so that $v_{2+3j} \geq a_2^{(k-1)^j}$. We choose a positive integer j_0 such that $(k-1)^{j_0} > 4k$. Since $k \geq 3$, $j_0 = 4$ satisfies the condition. As $2 + 3j_0 = 14$, we have $v_i \geq v_{14} > a_2^4$ for all $i \geq 14$. This completes the proof. □

For larger values of k, the number 14 in the above Lemma can be improved to $2 + 3j_0$, where j_0 satisfies the condition $(k-1)^{j_0} > 4k$.

Now, assume that $(u_1, v_1), (u_2, v_2), \cdots, (u_m, v_m)$ satisfy the system of equations (6) with

$$v_i > \max(a_2^{1/k}, 2) \geq \max(H(\alpha), 2).$$

By Lemma 3.2, for $14 \leq i \leq m$,

$$\left| \frac{u}{v} - \alpha \right| < \frac{1}{v_i^{k-1/2}} \leq \frac{1}{v_i^{2.5}},$$

as $k \geq 3$. Since $\alpha = (a_1/a_2)^{1/k} < 1$ and $\max(u_i, v_i) = v_i$, from Theorem 2.2, the number of such i's is $\mathcal{O}(\log k \, \log\log k)$. This proves Theorem 1.1.

3.2. **Proof of Theorem 1.2.** Let $m = M_k(n)$ and $S = \{a_1, a_2, a_3, \dots a_m\}$ be a generalized m-tuple with the property $D_k(n)$. Suppose $|n|^L < a_1 < a_2 < \dots < a_m$ for some $L \geq 3$. We consider the system of equations

$$\begin{aligned} a_1 x + n &= u^k \\ a_2 x + n &= v^k. \end{aligned} \tag{7}$$

As before, $x = a_i$ for $i \geq 3$ are solutions to this system. The statements of Lemma 3.1 and 3.2 hold for all non-zero integers n. For $n > 0$ this was proved in [8].

Proof of Theorem 1.2(a). Let $(u_1, v_1), \cdots (u_m, v_m)$ satisfy the system of equations (7) with $v_i > \max(a_2^{1/k}, 2) \geq \max(H(\alpha), 2)$. By Lemma 3.2, we get for $14 \leq i \leq m$,

$$\left| \frac{u_i}{v_i} - \alpha \right| \leq \frac{1}{v_i^{k-1/2}} \leq \frac{1}{v_i^{2.5}},$$

as $k \geq 3$. Since $\alpha = (a_1/a_2)^{1/k} < 1$ and $\max(u_i, v_i) = v_i$, applying Theorem 2.2 with $\kappa = 0.5$, we get that the number of i's satisfying the above inequality is

$$2^{25}(0.5)^{-3}\log(2k)\log((0.5)^{-1}\log(2k))$$

which is

$$2^{28}\log(2k)\log(2\log(2k)).$$

So, the total number of solutions is at most

$$2^{28}\log(2k)\log(2\log(2k)) + 14$$

for $k \geq 3$. □

Proof of Theorem 1.2 (b). Let $S = \{a_1, a_2, \dots, a_m\}$ be a generalized Diophantine m-tuple with property $D_k(n)$ such that each $a_i \leq |n|^3$. We shall apply Gallagher's larger sieve with primes $p \leq Q$ satisfying $p \equiv 1 \bmod k$. Let $\mathcal{P}$ be the set of all primes $p \equiv 1 \bmod k$. For all such primes $p \in \mathcal{P}$, there exists a Dirichlet character $\chi(\operatorname{mod} p)$ of order k.

Denote by S_p the image of $S \pmod p$ for a given prime p. For $p \in \mathcal{P}$, applying Lemma 2.3 with $\mathcal{A} = \mathcal{B} = S_p$ and $\chi \bmod p$ a character of order k, we obtain

$$|S_p|\left(|S_p| - 1\right) \leq \sum_{a \in S_p - \{0\}} \sum_{b \in S_p} \chi(ab + n) + |S_p| \leq \sqrt{p}|S_p| + |S_p|.$$

Thus,

$$|S_p| \leq \sqrt{p} + 2.$$

Take $N = |n|^3$. Since $a_i \leq |n|^3$, applying Theorem 2.1, we obtain

$$|S| \leq \frac{\sum\limits_{p \in \mathcal{P},\, p \leq Q} \log p - \log N}{\sum\limits_{p \in \mathcal{P},\, p \leq Q} \frac{\log p}{|S_p|} - \log N}.$$

By Theorem 2.4,

$$\sum_{\substack{p \leq Q \\ p \equiv 1 \bmod k}} \log p = \frac{Q}{\phi(k)} + \mathcal{O}\left(\frac{Q}{\log Q}\right)$$

when $Q > Q_0(k)$. Using partial summation,

$$\sum_{\substack{p\leq Q\\ p\equiv 1 \bmod k}} \frac{\log p}{\sqrt{p}+2} = \frac{2\sqrt{Q}}{\phi(k)} + \mathcal{O}\left(\frac{\sqrt{Q}}{\log Q}\right).$$

Thus,

$$|S| \leq \frac{\frac{Q}{\phi(k)} + \mathcal{O}\left(\frac{Q}{\log Q}\right) - \log N}{\frac{2\sqrt{Q}}{\phi(k)} + \mathcal{O}\left(\frac{\sqrt{Q}}{\log Q}\right) - \log N}. \tag{8}$$

Choose $Q = (\phi(k)\log N)^2$. Note that the condition $Q > Q_0(k)$ is same as

$$\log N > \frac{\exp\left(0.015\sqrt{k}(\log k)^3\right)}{\phi(k)}. \tag{9}$$

Since $k = o(\log\log|n|)$, (9) holds for N large enough. Now, for both the numerator and the denominator, multiply by $\phi(k)$ and divide by $\log N$, to get

$$|S| \leq \frac{\phi(k)\log N - 1 + \mathcal{O}\left(\frac{Q}{\log N\log Q}\right)}{1 + \mathcal{O}\left(\frac{\sqrt{Q}}{\log N\log Q}\right)}. \tag{10}$$

Because $k = o(\log\log N)$, it is easy to see that

$$\frac{\sqrt{Q}}{\log N\log Q} = o(1).$$

Hence, the denominator in (10) is $1 + o(1)$ and we obtain

$$|S| \leq \phi(k)\log N + \mathcal{O}\left(\frac{(\phi(k))^2\log N}{\log\log N}\right).$$

Since $N = |n|^3$,

$$M_k(n) \leq 3\,\phi(k)\ \log|n| + \mathcal{O}\left(\frac{(\phi(k))^2\log|n|}{\log\log|n|}\right)$$

as required. □

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