TOPICS IN REPRESENTATION THEORY

OUTLINE OF SOLUTIONS TO THE MIDSEMESTER EXAMINATION

(1) Show that if an endomorphism of a Noetherian module is surjective then it is an isomorphism.
Solution. Let ϕ be an endomorphism of a Noetherian module. By the ascending chain condition the sequence

$$\ker\phi \subset \ker\phi^2 \subset \cdots$$

stabilises. Therefore, there exists a natural number N such that $\ker\phi^N = \ker\phi^{2N}$. Suppose $x \in \ker\phi^N$. Since ϕ is surjective, $x = \phi^N(y)$ for some y. Since $\phi^{2N}(y) = \phi^N(x) = 0$, $y \in \ker\phi^{2N}$. Therefore, $y \in \ker\phi^N$, and $x = \phi^N(y) = 0$.

(2) Let k be a field, and let R be the set of $n \times n$ triangular matrices of the from

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ * & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a \end{pmatrix}, \text{ for some } a \in k.$$

Show that, when thought of as a subring of $M_n(k)$, R is local.
Solution. The set of non-units consists of matrices of the above form where $a = 0$. This set is seen to be a two sided ideal.

(3) Show that the product of two commuting semisimple matrices with entries in a perfect field is semisimple.
Solution. Let A be a semisimple $n \times n$ matrix over a perfect field k. Since A is semisimple, there is an isomorphism of A-modules

$$k^n \cong \bigoplus_i (k[t]/p_i(t))^{m_i}$$

Since B commutes with A, B is an endomorphism of the above $k[t]$ modules. Since there are no morphisms between the different summands in the decomposition above, B is a direct sum of endomorphisms on the summands. A direct sum of semisimple matrices is semisimple. We may therefore assume, without loss of generality, that there is only one direct summand, i.e.,

$$k^n \cong (k[t]/p(t))^m$$

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for some irreducible $p(t) \in k[t]$ and $m \in \mathbf{N}$. For a perfect field, we know that this is isomorphic to $K[u]/u^m$, where $K = k[t]/p(t)$, and $t \equiv u \mod p(t)$. $K[u]/u^m$ is a cyclic $k[t]$-module, hence $B \in k[t]$. Say, $B = q(t)$. Then $AB = tq(t)$, which is a polynomial in the semisimple operator t, hence is also semisimple.

(4) Let $R = (\mathbf{Z}/2\mathbf{Z})[\mathbf{Z}/4\mathbf{Z}]$. List all the distinct isomorphism classes of simple R-modules. Describe the decomposition of ${}_RR$ into indecomposable R-modules.
Solution. Note that $a0 + b1 + c2 + d3 \in R$ generates a proper subspace of R if and only if

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = (a+b+c+d)^4 = 0$$

Therefore,

$$M = \{a0 + b1 + c2 + d3 | a + b + c + d = 0\}$$

is the unique maximal proper invariant subspace of R. It follows that R is local, R/M is the unique irreducible R-module and that ${}_RR$ is indecomposable.

(5) Let $\mathbf{F}_q$ denote the finite field with q elements. Compute the number of *semisimple* conjugacy classes in $GL_3(\mathbf{F}_q)$ (this is the same as the number of similarity classes in $M_3(\mathbf{F}_q)$ which consist of invertible semisimple matrices). **Solution.** Count the number of rational canonical forms that are semisimple. Suppose the invariant factors are $d_1|d_2|d_3$. The criterion for semisimplicity is that d_3 is square-free.
(a) $\deg(d_1) = 1$, then $d_i = (x - a)$ for all i. Get $q - 1$ classes.
(b) $\deg(d_1) = 0$ and $\deg(d_2) = 1$. Then $\deg(d_3) = 2$, $d_1 = (x - a)$, and $d_2 = (x-a)(x-b)$, where $n \neq a$. Get $(q-1)(q-2)$ classes.
(c) $\deg(d_1) = \deg(d_2) = 0$. Then d_3 is a square free polynomial of degree three. The number of such polynomials is q^3-2q^2+2q-1.
Therefore there are $q^2(q - 1)$ semisimple conjugacy classes.