TOPICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 17/11/2006

(1) Let p be a prime, $p > 3$. Let G be a cyclic group with p. Show that the $\mathbf{Z}/pZ[G]$ module where a generator of G acts by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not obtained from any $\mathbf{Q}[G]$-module (by the process $M \to M_P \to \overline{M}$ described in Wednesday's lecture).

(2) Show that $\mathbf{Q}[i]$ and $\mathbf{Z}/2\mathbf{Z}$ are splitting fields for the finite group $\mathbf{Z}/4\mathbf{Z}$.

(3) The ring $\mathbf{H}$ of quaternions is the $\mathbf{R}$-span in $M_2(\mathbf{C})$ of

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

Show that $H \otimes_{\mathbf{R}} \mathbf{C}$ is isomorphic to $M_2(\mathbf{C})$ as a $\mathbf{C}$-algebra.

(4) Show that the group $\mathbf{Z}/n\mathbf{Z}$ splits over any field of characteristic 0 which contains all the roots of the polynomial $x^n - 1$.

(5) Show that the group $\mathbf{Z}/n\mathbf{Z}$ splits over any field of characteristic p which contains all the roots of the polynomial $x^m - 1$, where m is such that $n = mp^a$ for some non-negative integer a, and $(m, p) = 1$. [Hint: $(x^n - 1) = (x^m - 1)^{p^a}$ in characteristic p.]

(6) The *exponent* of a group is the smallest positive integer n such that x^n is the identity element for every element x of the group[1]. What is the exponent of $GL_n(\mathbf{F}_q)$?

[1]A well known result (conjectured by Maschke and proved by Brauer) is that G splits over a field of characteristic zero if and only if that field contains all the roots of $x^n - 1$.