TOPICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 04/10/2006

(1) For any ring R show than an R-module is Noetherian if and only if every submodule of M is finitely generated.
(2) If R is a left Noetherian ring, then any finitely generated R-module is Noetherian.
(3) Consider the subring of $M_2(\mathbf{Q})$ given by
$$R = \left\{ \left(\begin{array}{cc} a & b \\ 0 & d \end{array} \right) \middle| \; a \in \mathbf{Z}, \; b, d \in \mathbf{Q} \right\}.$$
Show that R is right Noetherian, but not left Noetherian. Conclude that R is not isomorphic to R^{opp}.
(4) Let p be a prime number, e be any natural number, and let $R = \mathbf{Z}/p\mathbf{Z}[\mathbf{Z}/p^e\mathbf{Z}]$.
   (a) Determine all the simple R-modules up to isomorphism.
   (b) Determine all the indecomposable R-modules up to isomorphism.
(5) Let p be a prime number, n be a positive integer that is not divisible by p and let $R = \mathbf{Z}/p\mathbf{Z}[\mathbf{Z}/n\mathbf{Z}]$.
   (a) Determine all the simple R-modules up to isomorphism.
   (b) Determine all the indecomposable R-modules up to isomorphism.
(6) Let e and f be non-zero idempotents in a ring R. Construct an isomorphism between the abelian groups $\mathrm{Hom}_R(Re, Rf)$ and eRf. Show that the rings $\mathrm{End}_R(Re)$ and $(eRe)^{\mathrm{opp}}$ are isomorphic.
(7) Show that an $n \times n$ matrix A with entries in a field is semisimple if and only if the algebra $Z(A)$ of matrices which commute with A is a semisimple algebra.
(8) Let R be a unital ring satisfying the Noetherian and Artinian conditions. Show that $\frac{R}{\mathrm{Rad}R}$ is semisimple.
(9) Show that any finitely generated module over a semisimple ring is semisimple.
(10) Let R be a unital ring satisfying the Noetherian and Artinian conditions. If M is a finitely generated R-module on which $\mathrm{Rad}R$ acts trivially, show that M is semisimple.