TOPICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 15/09/2006

(1) Let R be any ring, and M be any R-module. Show that if $\mathrm{End}_R M$ is a local ring, then M is indecomposable.

(2) Let Q be the linear quiver with n-vertices:
$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n\ .$$
Show that if (π, V) is an indecomposable representation of Q such that $\pi(\alpha_j)$ is not surjective, then $V_i = 0$ for all $i \leq j$.

(3) Let Q be as in the previous exercise, and let $(\pi, V) = [j, i]$ denote the indecoposable representation of Q where $V_h \cong k$ for all $j \leq h \leq i$, and $\pi(\alpha_h) = \mathrm{id}$ for $j \leq h < i$. Compute $\mathrm{Hom}_Q([j, i], [j', i'])$.

(4) What are the irreducible representations of the quiver Q of the previous problem?

(5) What is the representation type of the quiver
$$\begin{tikzcd} \bullet \arrow[dr] & \bullet \arrow[d] & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd} \quad ?$$

(6) What is the representation type of the quiver
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \bullet \end{tikzcd} \ ?$$

(7) (Emmy Noether) A *left ideal* in a ring R is, by definition, a submodule of the the left R-module R. Show that a decomposition of a unital ring R as a direct sum
$$R = M_1 \oplus \cdots \oplus M_n$$
of left ideals is equivalent to a decomposition of the unit 1 as a sum of rothogonal idempotents, i.e.,
$$1 = e_1 + \cdots + e_n,$$
with
$$e_i^2 = e_i, \quad e_i e_j = 0 \text{ for } i \neq j, \text{ and } M_i = Re_i,$$
for all i and j.