TOPICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 08/09/2006

(1) A matrix $u \in M_n(k)$ is said to be *unipotent* if $u - I$ is nilpotent. Prove the multiplicative version of the Jordan decomposition:

> Every invertible matrix x with entries in a perfect field k can be written in a unique way as a product su of a semisimple matrix s and a unipotent matrix u with $su = us$. There exists a polynomial $q(t) \in k[t]$ such that $s = q(x)$.

(2) Suppose k is a field of characteristic $p > 0$, and n is a positive integer that is not divisible by p, or that k is of characteristic zero and n is an arbitrary positive integer. Show that if $A^n = I$, then A is semisimple.

(3) Let p be a prime number and let $q = p^e$ for some positive integer e. Show that a matrix $s \in GL_n(\mathbf{F}_q)$ is semisimple if and only if its order is not divisible by p. Show that a matrix $u \in GL_n(\mathbf{F}_q)$ is unipotent if and only if its order is a power of p.

(4) Use the previous problem to show that for any $x \in GL_n(\mathbf{F}_q)$, with multiplicative Jordan decomposition $x = su$ (Problem 1), s and u can be taken to be powers of x. [Hint: consider the finite cyclic group generated by x.]

(5) For any matrix $A \in M_n(k)$ for which $d_1(t), \ldots, d_s(t)$ are the invariant factors of $tI - A$, prove the following formula (due to Frobenius):

$$\dim_k Z(A) = \sum_{j=1}^{s} (2s - 2j + 1) \deg(d_s(t))$$

(6) Find a polynomial $q(t) \in \mathbf{R}[t]$ such that

$$q \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

[Hint: look at Problem 2 from the previous assignment.]

(7) Find the semisimple and nilpotent parts in the Jordan decomposition of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

[Hint: use the previous exercise.]