TOPICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 25/08/2006

(1) Let k be a field, and A and B be two $n \times n$ matrices with entries in k. Show that $\lambda I - A$ is *similar* to $\lambda I - B$ in $M_n(k[\lambda])$ if and only if $\lambda I - A$ is *equivalent* to $\lambda I - B$ in $M_n(k[\lambda])$.

(2) A vector $\mathbf{v} \in k^n$ is said to be *cyclic* for an $n \times n$ matrix A if the set

$$\mathbf{v},\ A\mathbf{v},\ A^2\mathbf{v}, \ldots$$

spans k^n. Show that a matrix is similar to the companion matrix of its characteristic polynomial if and only if it has a cyclic vector.

(3) Let K be an extension of the field k. Suppose A and B are two matrices in $M_n(k)$. Show that A and B are similar in $M_n(k)$ if and only if A and B are similar in $M_n(K)$.

(4) Let A be an $n \times n$ matrix with *real* entries such that $A^2 + I = 0$. Prove that n is even, and if $n = 2k$, then A is similar to a matrix of the block form

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I is the $k \times k$ identity matrix.

(5) Show that every non-zero vector in k^n is cyclic for a matrix A if and only if the characteristic polynomial of A is irreducible over k.

(6) Let A be an $n \times n$ matrix with *real* entries. Show that if the only subspaces of $\mathbf{R}^n$ invariant under A are $\{0\}$ and $\mathbf{R}^n$, then A is similar to a diagonal matrix in $M_n(\mathbf{C})$.

(7) Let $\mathbf{F}_q$ denote the finite field with q elements, where q is a power of a prime number. Calculate the number of similarity classes of matrices in $M_2(\mathbf{F}_q)$, and in $M_3(\mathbf{F}_q)$. What can you say about the number of similarity classes in $M_n(\mathbf{F}_q)$ for general n?

(8) Calculate the number of conjugacy classes in $GL_2(\mathbf{F}_q)$, and in $GL_3(\mathbf{F}_q)$. What can you say about the number of similarity classes in $GL_n(\mathbf{F}_q)$ for general n?