TOPICS IN REPRESENTATION THEORY

ASSIGNMENT DUE ON 18/08/2006

(1) Let $R = \mathbf{R}[\lambda]$, and suppose

$$M = R/(\lambda-1)^3 \oplus R/(\lambda^2+1)^2 \oplus R/(\lambda-1)(\lambda^2+1)^4 \oplus R/(\lambda+2)(\lambda^2+1)^2.$$

Determine the elementary divisors and invariant factors of M. Write down a matrix T such that M corresponds to T under the correspondence between torsion R-modules and similarity classes of matrices.

(2) Let R be a principal ideal domain, and fix an element $\lambda \in R$. Compute the invariant factors of the $n \times n$ matrix

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

(3) For any field k prove that two matrices A and B in $M_n(k)$ are *similar* if and only if the matrices $\lambda I - A$ and $\lambda I - B$ are *equivalent* in $M_n(k[\lambda])$.

(4) Recall that the elemantary row and column operations correspond to left and right multiplication respectively by certain 'elementary matrices':

Type 1. The matrix corresponding to the operation of adding a multiple of one row or column to another.

Type 2. The matrix corresponding to scaling a row or column by a unit.

Type 3. The matrix corresponding to interchanging two rows or columns.

Prove that if R is Euclidean then any invertible matrix in $M_n(R)$ is a product of elementary matrices. Show that any elementary matrix of Type 3 is a product of elementary matrices of Type 1 and Type 2 (begin with the case of 2×2 matrices). Conclude that any invertible matrix in $M_n(R)$ is a product of elementary matrices of Type 1 and Type 2.

(5) If k is a field, then any matrix in $M_n(k)$ of determinant 1 is a product of elementary matrices of Type 1.

(6) Let R be a principal ideal domain. Let $a_1, \ldots, a_n \in R$. Let d be the gcd of $a_1, \ldots, a_n$. Show that there exists an *invertible* matrix Q in $M_n(R)$ such that

$$(a_1, \ldots, a_n)Q = (d, 0, \ldots, 0).$$

(7) Suppose $a_{11}, \ldots, a_{1n}$ are relatively prime elements in a principal ideal domain. Show that there exists an *invertible* square matrix with first row $(a_{11}, \ldots, a_{1n})$.