Exercises
For mini-course as part of Amri's
Algebra course

October 2006

1. Prove that the following are all equal to the radical

   - The union of all quasi-regular right ideals.
   - The union of all quasi-regular left ideals.
   - $\{z \in A \mid xzy$ is quasi-regular for all x and $y\}$.

   That the radical is the union of all quasi-regular right ideals is just a restatement of the internal characterization of the radical in §1.5 of the notes. By the left analogue of this restatement, the left radical is the union of all quasi-regular left ideals. But, as observed in §1.5, the left radical equals the right radical.

2. Prove or disprove: the ring of finite rank linear transformations of a vector space (possibly infinite dimensional) is simple.

   Let t_0 and t be a finite rank linear transformations. We will prove that t can be written as a finite sum $\sum x_i t_0 y_i$ with x_i and y_i finite rank linear transformations. We may assume that the rank of t is 1, for, as is easily seen, t is a finite sum of rank 1 transformations.

   So let K is the codimension 1 subspace that is the kernel of t and let v be not in K. Since $t_0 \neq 0$, there exists v_0 such that $tv_0 \neq 0$. Let y be the linear transformation that is 0 on K and maps v to v_0; and let x be a linear transformation of finite rank such that $xtv_0 = w$. Then $xt_0y = t$, and we are done.

3. Let V be a finite dimensional vector space over a field k and $\mathrm{End}_k(V)$ the set of all k-linear maps from V to itself. Suppose that R is a subset of $\mathrm{End}_k(V)$ that is an additive subgroup and is closed under multiplication. Assume that given any two linearly independent elements v_1, v_2

of V and any two elements w_1, w_2 of V, there exists an element r of R such that $rv_1 = w_1$ and $rv_2 = w_2$. Show that R equals $\text{End}_k(V)$.

A proof from first principles should be easy to give, but here is a "high level" proof. Since R is 2-transitive, it follows from an observation in §3 of the notes that $\text{End}_R(V) = k$. But V is also irreducible for R (1-transitivity is enough for this), so, by the density theorem, R is dense for $\text{End}_R(V) = k$. So $R = \text{End}_k(V)$ by the finite dimensionality of V over k.

4. Prove or disprove: Let V be a finite dimensional vector space over a field k and $\text{End}_k(V)$ the set of all k-linear maps from V to itself. Suppose that A is a subring of $\text{End}_k(V)$ and that V is irreducible as an A-module. Then $\text{End}_R(V) = k$.

   The statement is not true as the following example shows. Let k be the field $\mathbb{R}$ of real numbers and let V be the Cartesian plane over $\mathbb{R}$. Let $G = SO(2)$ be the the circle group acting on V in the standard way. Then V is irreducible for G and so also for the group ring A over $\mathbb{R}$ of G. The group G being abelian, the commutator of A contains A in particular, and so is bigger than $k = \mathbb{R}$.

5. Show that there is a one-to-one inclusion reversing correspondence between Γ-subspaces W of a finite dimensional Γ-vector space V and the right ideals $\mathfrak{a}$ of $\text{End}_\Gamma V$: $W \mapsto W^\perp$ and $\mathfrak{a} \mapsto \mathfrak{a}^\perp$.

6. Let k be a field (take it to be the field $\mathbb{C}$ of complex numbers if you wish), x a variable over k, and $k(x)$ the field of rational functions[1]. Let $D : k(x) \to k(x)$ be the map which takes a rational function to its derivative with respect to x. Then D is k-linear and satisfies the Liebniz rule: $D(fg) = fD(g)+gD(f)$. We can think of a "polynomial" $a_m D^m + a_{m-1} D^{m-1} + \ldots + a_1 D + a_0$ in D with coefficients $a_m, a_{m-1}, \ldots, a_1, a_0$ in $k(x)$ as an operator on $k(x)$: elements of $k(x)$ act by multiplication and D^k acts as the k^{th}-derivative. Let R be the ring of all such polynomial operators on $k(x)$. It is associative and has an identity but is not commutative.

[1] A *rational function* is a quotient $p(x)/q(x)$ of polynomials $p(x)$ and $q(x)$ having coefficients in the field k with $q(x) \neq 0$.

(a) Show that R has no zero divisors: that is, if $rs = 0$ with r, s in R, then either $r = 0$ or $s = 0$.

(b) Show that every left ideal of R is generated by a single element.

(c) Show that every right ideal of R is generated by a single element.

(d) Show that every two-sided ideal of R has an element which generates it both as a left and as a right ideal.

(e) Let I be a two-sided ideal and a an element of R such that $Ra = I = aR$. Show that a commutes with every element of $k(x)$. Show further that a belongs to $k(x)$ so that either $I = 0$ or $I = R$. (Hint: Given b in $k(x)$ write $ba = ac$. Prove that c belongs to $k(x)$.)

(f) Show that R does not contain minimal non-zero one-sided (left or right) ideals.

(a) If $a_m D^m$ is the leading term of a polynomial and $b_n D^n$ the leading term of another, then $a_m b_n D^{m+n}$ is the leading term of the product the Liebniz rule not withstanding.

(b) Choose a polynomial of least degree in the ideal. We may take it to be monic by multiplying on the left by a suitable element of $k(x)$. Any polynomial of degree that is not less this polynomial can be divided (on the left) just like in the usual case. The remainder is a polynomial of lesser degree than the divider and belongs to the ideal. It must therefore be 0.

(c) Similar to item 6b. We can also divide "on the right" a polynomial of higher degree by a polynomial of lower degree.

(d) Let I be a 2-sided ideal. The two items above imply we can find a and b such that $I = Ra$ and $I = bR$. Write $a = bu$ and $b = va$ so that $a = vau$. Since au belongs to I, there exists u' such that $au = u'a$. We thus have $a = vu'a$. Cancelling a (see the first item) we get $1 = vu'$. Thus v is a unit. Substituting Rv for R in $I = Ra$, we get $I = Rva = Rb$.

(e) Continuing from where the hint left off, since the degrees on both sides of $ba = ac$ are equal, we conclude that c belongs to $k(x)$. Comparison of highest degree terms gives $b = c$. If a has degree $m > 1$, taking b in $k(x)$ and comparing the coefficient of D^{m-1} in

$ba = ab$ gives that $Db = 0$, a contradiction. Thus a belongs to $k(x)$.

(f) If we multiply a generator of a non-zero one-sided ideal by a polynomial of positive degree we get a generator of a strictly smaller non-zero ideal.