THE CARTAN MATRIX OF A CENTRALISER ALGEBRA

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ABSTRACT. We calculate the Cartan matrix of the algebra of the ring of matrices that commute with a given square matrix.

1. The problem

Let K be a perfect field and T be an $n \times n$-matrix with entries in K. Let A denote the K-algebra of all matrices B such that $AB = BA$. Let $P_1, \ldots, P_l$ be a complete set of representatives for the isomorphism classes of principal indecomposable A-modules. Each simple A-module is then of the form $D_i = P_i/\mathrm{Rad} P_i$ [CR62, Chapter VIII]. Given a finite dimensional A-module M and a simple A-module D, let $[M : D]$ denote the number of composition factors in a composition series for M that are isomorphic to D. The *Cartan matrix* of A is the $l \times l$ matrix $C = (c_{ij})$ defined by

$$c_{ij} = [P_i : D_j].$$

The goal of this article is to compute the matrix C. For this purpose T can always be replaced by a matrix similar to it.

2. Reduction to the primary case

We will use $T_1 \oplus \cdots \oplus T_r$ to denote the block diagonal matrix whose diagonal blocks are $T_1, \ldots, T_r$. Then T is similar to a matrix of the form

$$\bigoplus_p T_p,$$

where p ranges over a finite set of irreducible polynomials with coefficients in K and T_p is a matrix whose characteristic polynomial is a power of p. Moreover, A has a decomposition into two-sided ideals

$$A = \bigoplus A_p,$$

where A_p is the ring of matrices that commute with T_p.

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Moreover, for each p, there exists a unique partition

$$\lambda = (\underbrace{\lambda_1, \cdots, \lambda_1}_{m_1 \text{ terms}}, \underbrace{\lambda_2, \cdots, \lambda_2}_{m_2 \text{ terms}}, \underbrace{\cdots,}_{\cdots} \underbrace{\lambda_l, \cdots, \lambda_l}_{m_l \text{ terms}})$$

with $\lambda_1 < \lambda_2 < \cdots < \lambda_l$, $m_1, \ldots, m_l$ positive integers, which we will abbreviate as

$$\lambda = (\lambda_1^{m_1}, \ldots, \lambda_l^{m_l}),$$

such that

$$A_p \cong \mathrm{End}_{E[t]} E[t]/(t^{\lambda_1})^{\bigoplus m_1} \oplus \cdots \oplus E[t]/(t^{\lambda_l})^{\bigoplus m_l},$$

where E is the algebraic field extension $K[t]/p(t)$ of K. The ring A_p depends on T_p only through E and λ.

We shall use the notation

$$E^\lambda = E[t]/(t^{\lambda_1})^{\oplus m_1} \oplus \cdots \oplus E[t]/(t^{\lambda_l})^{\oplus m_l},$$

which is an $E[t]$-module, and

$$M_\lambda(E) = \mathrm{End}_{E[t]}(E^\lambda).$$

In particular $M_{1^m}(E)$ denotes the ring of $m \times m$ matrices over E.

3. A block matrix representation of $M_\lambda(E)$

An element $\mathbf{x} \in E^\lambda$ can be represented as a vector with entries $(\mathbf{x}_1, \ldots, \mathbf{x}_l)$, where $\mathbf{x}_j \in E[t](t^{\lambda_j})^{\oplus m_j}$. Accordingly, if such a vector is represented as a column, an element $a \in M_\lambda(E)$ can likewise be represented by a matrix $a = (a_{ij})$, where

$$a_{ij} \in \mathrm{Hom}_{E[t]}(E[t]/(t^{\lambda_j})^{\oplus m_i}, E[t]/(t^{\lambda_j})^{\oplus m_i}).$$

Given $a = (a_{ij})$ and $b = (b_{ij})$ in $M_\lambda(E)$, the composition ab has (i, j) entry

$$a_{i1}b1j + \cdots + a_{il}b_{lj}.$$

Each of the above summands $a_{ik}b_{kj}$ is obtained by composition:

$$E[t]/(t^{\lambda_j})^{\oplus m_j} \xrightarrow{b_{kj}} E[t]/(t^{\lambda_k})^{\oplus m_k} \xrightarrow{a_{ik}} E[t]/(t^{\lambda_i})^{\oplus m_i}$$

4. Computation of the radical

Consider $E^{\lambda_j^{m_j}} = E[t]/(t^{\lambda_j})^{\oplus m_j}$. Reduction modulo t gives a surjection $E^{\lambda_j^{m_j}} \to E^{m_j}$. There is a corresponding reduction modulo t for $M_{\lambda_j^{m_j}}(E) = \mathrm{End}_{E[t]}(E^{\lambda_j^{m_j}})$:

$$M_{\lambda_j^{m_j}}(E) \to M_{m_j}(E)$$

The radical of $M_{\lambda_j^{m_j}}(E)$ is the kernel of the above surjection, which we will denote by $R_{\lambda_j^{m_j}}(E)$. It consists of those endomorphisms of $E^{\lambda_j^{m_j}}$ whose image lies in $tE^{\lambda_j^{m_j}}$. This is becaus $R_{\lambda_j^{m_j}}(E)$ is a nilpotent two-sided ideal and the quotient $M_{m_j}(E)$ is semisimple. Let

$$R_\lambda(E) = \{(a_{ij}) \in M_\lambda(E) | a_{jj} \in R_{\lambda_j^{m_j}}(E) \text{ for } j = 1, \ldots, l\}$$

We claim that $R_\lambda(E)$ is the radical of $M_\lambda(E)$. If $a \in R_\lambda(E)$ and $b \in M_\lambda(E)$, then the entries in the matrix of ab are sums of terms of the form $a_{jk}b_{kj}$. If $k > j$ then the image of b_{kj} is contained in $tE^{\lambda_k^{m_k}}$. If $k < j$ then the image of a_{jk} is contained in $tE^{\lambda_j^{m_j}}$. If $k = j$, then since $a \in R_\lambda(E)$, the image of a_{jj} is contained iin $tE^{\lambda_j^{m_j}}$. Consequently, in all of these cases, the image of $a_{jk}b_{kj}$ is contained in $tE^{\lambda_j^{m_j}}$. Therefore, $R_\lambda(E)$ is a right ideal. A similar argument shows that $R_\lambda(E)$ is a two-sided ideal. The quotient $\frac{M_\lambda(E)}{R_\lambda(E)}$ is a semisimple ring:

$$\frac{M_\lambda(E)}{R_\lambda(E)} = M_{1^{m_1}}(E) \oplus \cdots \oplus M_{1^{m_l}}(E).$$

It therefore remains only to show that $R_\lambda(E)$ is nilpotent. For this, suppose that a and b are both in $R_\lambda(E)$. Then the matrix entries of ab are sums of terms of the form $a_{ik}b_{kj}$. If $i \geq j$, then the sort of reasoning that was used earlier shows that $a_{ik}b_{kj}$ has image contained in $tE^{\lambda_i^{m_i}}$. An inductive argument then shows that when $i \geq j$, then every product of r elements in $R_\lambda(M)$ has (i,j)th entry whose image lies in $t^{r-1}E^{\lambda_i^{m_i}}$. Therefore, the elements of $R_\lambda(M)^{\lambda_l}$ have all matrix entries below or on the diagonal zero. A product of l such elements is always zero. Therefore $R_\lambda(E)^{\lambda_l l} = 0$.

5. Principal indecomposable and simple modules

It follows from Section 4 that there are l isomorphism classes of simple $M_\lambda(E)$-modules. These are represented by $D_1, \ldots, D_l$ where D_j, as an E-vector space is isomorphic to E^{m_j}, and $a \in M_\lambda(E)$ acts on it by the image of a_{jj} in $M_{1^{m_j}}(E)$. For each $1 \leq i \leq l$ and $1 \leq r \leq m_i$, let e_{ir} denote the element of $M_\lambda(R)$ whose matrix has all entries 0, except the (i,i)th entry, which as an element of $M_{\lambda_i^{m_i}}(E)$ has matrix with all entries zero and the (r,r)th entry equal to 1. Then

$$1 = \sum_{i=1}^{l} \sum_{r=1}^{m_i} e_{ir},$$

and that the e_{ir}'s are pairwise orthogonal idempotents. There is a matrix unit in $M_{\lambda_i^{m_i}}$ which takes e_{ir} to $e_{ir'}$ for any $1 \leq r, r' \leq m_i$. Therefore, all the modules $M_\lambda(E)e_{ir}$ are isomorphic for a fixed value of i. Let $P_i = M_\lambda(E)e_{i1}$ for $i = 1, \ldots, l$. That e_{i1} is a primitive idempotent follows from the corresponding fact for $M_{\lambda_i^{m_i}}$. Since the $P_i/(R_\lambda(E) \cap P_i)$'s are pairwise non-isomorphic, so are the P_i's. It follows that $P_1, \ldots, P_l$ is a complete set of representatives for the isomorphism classes of principal indecomposable $M_\lambda(E)$-modules.

6. The Cartan matrix

It now remains to calculate $[P_i : D_j]$ for all $1 \leq i, j \leq l$. By [CR62, Theorem 54.16],

$$[P_i : D_j] = \dim_E e_{j1} M_\lambda(E) e_{i1}.$$

For any $a \in M_\lambda(E)$, the (k,l)th entry of $e_{j1} a e_{i1}$ is zero unless $j = k$ and $l = i$. The (j,i)th entry is $(a_{ji})_{11}$, the $(1,1)$th entry of a_{ji} when a_{ji} is thought of as a matrix whose (j,i)th entry is in $\mathrm{Hom}_{E[t]}(E[t]/(t^{\lambda_i}), E[t]/(t^{\lambda_j}))$. Now $\dim_E \mathrm{Hom}_{E[t]}(E[t]/(t^{\lambda_i}), E[t]/(t^{\lambda_j})) = \min\{\lambda_i, \lambda_j\}$. Therefore the Cartan matrix of $M_\lambda(E)$ is given by

$$c_{ij} = \min\{\lambda_i, \lambda_j\}, \text{ for all } 1 \leq i, j \leq l.$$

It follows that $M_\lambda(E)$ has a single block.

References

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras.* Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

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