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THE ALMOST UNRAMIFIED DISCRETE SPECTRUM FOR SPLIT SEMISIMPLE GROUPS OVER A RATIONAL FUNCTION FIELD

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BY
AMRITANSHU PRASAD

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To Kalyan Mukherjea

ABSTRACT

Let G be a split semisimple group over a finite field $\mathbf{F}_q$, let $F = \mathbf{F}_q(t)$, and let $\mathbf{A}$ denote the adèles of F. For all the irreducible representations of $G(\mathbf{A})$ occurring in the discrete part of $L^2(G(F)\backslash G(\mathbf{A}))$ which have vectors invariant under Iwahori subgroups at two places of F and maximal compact subgroups at all other places, we describe the local constituents at those two places in terms of the irreducible square integrable representations of an Iwahori Hecke algebra. We include proofs of certain well known results about the classification of principal G-bundles on the projective line which we use in our calculations.

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CHAPTER 1
INTRODUCTION

1.1 Overview

Let F be a global field and let G be a reductive group defined over F. Let $\mathbf{A}_F$ denote the ring of adèles of F. Let $L^2(G(F)\backslash G(\mathbf{A}))_\chi$ denote the space of functions on $G(F)\backslash G(\mathbf{A})$ which are square integrable modulo the center of $G(\mathbf{A})$ and which transform under the center by a fixed character χ. The adèlic group $G(\mathbf{A})$ acts on this space by right translations. By a *discrete automorphic representation* of $G(\mathbf{A})$, we mean an irreducible representation of $G(\mathbf{A})$ that occurs in the discrete part of $L^2(G(F)\backslash G(\mathbf{A}))_\chi$ for some χ. A fundamental problem in the theory of automorphic forms is to describe the discrete automorphic representations of $G(\mathbf{A})$ and the multiplicities with which they occur (see [2] and [3]). Any discrete automorphic representation ρ has a decomposition

$$\rho = \otimes_v \rho_v, \tag{1.1.1}$$

where for each valuation v of F, ρ_v is an irreducible representation of $G(F_v)$. Here F_v denotes the completion of F with respect to v. We call ρ_v the *local constituent* of ρ at v.

By Langlands' theory of Eisenstein series, discrete automorphic representations are either *cuspidal* or arise from the residues of Eisenstein series that are associated to cuspidal automorphic representations of proper Levi subgroups. For example, when F is a number field, Mœglin and Waldspurger [20], by analyzing certain normalized intertwining operators, describe the discrete automorphic representations of GL_n in terms of cuspidal representations of GL_d, where $d|n$. When G is a split classical group and F is a number field, then Mœglin [18] shows that the representations in the unramified discrete spectrum have multiplicity one, and are parameterized by those unipotent orbits in the Langlands dual group which do not intersect the Levi factor of any proper parabolic subgroup. When G is a split symplectic or orthogonal group, Mœglin [19] describes all the automorphic representations that arise from residues of Eisenstein series of unramified characters of a maximal split torus that are of the form $(a_1, \ldots, a_n) \in \mathbf{A}^n \mapsto |a_1|^{s_1} \ldots |a_n|^{s_n}$ where n is the rank of G, and the parameters $s_1, \ldots, s_n$ are arbitrary complex numbers.

In the case of function fields, the simplest non-trivial example is that of $F = \mathbf{F}_q(t)$ and $G = PGL_2$. This is studied by Efrat in [10], where he describes the local constituents of unramified discrete automorphic representations by realizing the space of unramified automorphic forms as a space of functions on a tree. Furthermore, Anspach [1] describes the unramified discrete automorphic representations for $F = \mathbf{F}_q(t)$ and $G = PSp_4$.

In this thesis, we take F to be $\mathbf{F}_q(t)$ and G to be any split semi-simple group over $\mathbf{F}_q$. Let ρ be a discrete automorphic representation whose local constituents at two places have non-zero vectors fixed by Iwahori subgroups and which are unramified at all but these two places. We describe the local constituents at these two places of any such representation ρ. Our techniques are completely independent of the group G and provide results, for the first time, for all exceptional groups.

We would like to point out that our results do not explicitly describe the local constituents at the other places of the automorphic representations whose local constituents we describe at two places.

1.2 Statement of the Main Theorem

Let $\mathbf{F}_q$ be the finite field with q elements and let G be a split semisimple group defined over $\mathbf{F}_q$. Fix a maximal $\mathbf{F}_q$-split torus T of G and a Borel subgroup B defined over $\mathbf{F}_q$ containing T. Let $F = \mathbf{F}_q(t)$. Let $\mathbf{A}$ denote the ring of adèles of F. For each valuation v of F, let F_v (resp. $\mathbf{O}_v$) denote the local field (resp. the ring of integers of the local field) of F at v. The pre-image of $B(\mathbf{F}_q)$ under the natural map $G(\mathbf{O}_v) \to G(\mathbf{F}_q)$ is an Iwahori subgroup of $G(F_v)$, which we denote by I_v. Consider the compact open subgroup

$$K' = I_\infty \times I_0 \times \prod_{v \neq \infty, 0} G(\mathbf{O}_v) \tag{1.2.1}$$

of $G(\mathbf{A})$. Let M be the space of functions in $L^2(G(F)\backslash G(\mathbf{A}))$ which are right-invariant under K'. These form a representation (r, M) of the tensor product $H_\infty \otimes H_0$ of Iwahori Hecke algebras (§5.1) at ∞ and 0. Let (r, M_d) denote the subrepresentation of M generated by vectors in the closed irreducible representations of $H_\infty \otimes H_0$ that occur in M. Our main results describe the irreducible representations of $H_\infty \otimes H_0$ that occur in (r, M_d) and, furthermore, show that they occur with multiplicity one. Let H denote the abstract extended Iwahori Hecke algebra associated to the data (G, B, T) (in the sense of §4.2). The algebra H admits automorphisms $\overline{\kappa}$ and I, both of order two (§4.4). I is commonly known as the *Iwahori-Matsumoto involution.* Also, we have an isomorphism $\phi_v : H_v \to H$ for each degree one valuation v of F (§5.1).

Theorem 1.2.2 (Main Theorem). *There is an isomorphism of $H_\infty \otimes H_0$-modules*

$$(r, M_d) \xrightarrow{\sim} \bigoplus_{(\rho, V) \in \hat{H}} (\rho \circ I \circ \phi_\infty \otimes \tilde{\rho} \circ \overline{\kappa} \circ I \circ \phi_0, V \otimes \tilde{V}).$$

Here $\hat{H}$ denotes the set of isomorphism classes of irreducible square integrable representations of H (see §4.3).

When the derived group of G is adjoint, Kazhdan and Lusztig have described the set $\hat{H}$ in [16] (restated here as Theorem 4.3.1). Moreover, in this case, $(\tilde{\rho} \circ \overline{\kappa} \circ I, \tilde{V})$ is isomorphic to $(\rho \circ I, V)$ as an H-module (Theorem 4.4.3). This simplifies the statement of the main theorem to

Theorem 1.2.3 (Main Theorem for groups of adjoint type). *When the derived group of G is adjoint, then there is an isomorphism*

$$(r, M_d) \xrightarrow{\sim} \bigoplus_{(\rho, V) \in \hat{H}} (\rho \circ I \circ \phi_\infty \otimes \rho \circ I \circ \phi_0, V \otimes V).$$

1.3 A Guide to the Reader

Chapter 2 is expository in nature. The goal is to prove Theorem 2.1.1, which is a version of the well known fact that any vector bundle on the projective line $\mathbf{P}^1$ can be decomposed into a direct sum of line bundles. This was proved in the special case of the direct image of the structure sheaf of a curve mapping to the projective line by Dedekind and Weber (who did not know about vector bundles) in [9], but their argument works for all vector bundles. Suppose that G is a group defined over $\mathbf{F}_q$ and X is any irreducible smooth curve over $\mathbf{F}_q$. Let $\mathbf{A}_X$ denote the adèles of $\mathbf{F}_q(X)$, and $\mathbf{O}_v$ the ring of integers at a place v of X. Then the double coset space

$$G(\mathbf{F}_q(X))\backslash G(\mathbf{A}_X)/\prod_v G(\mathbf{O}_v)$$

classifies the principal G-bundles on X. When $G = GL_n$ and $X = \mathbf{P}^1$, the assertion that vector bundles split is equivalent to saying that the map

$$T(\mathbf{F}_q(X))\backslash T(\mathbf{A}_{\mathbf{P}^1})/\prod_v T(\mathbf{O}_v) \to G(\mathbf{F}_q(X))\backslash G(\mathbf{A}_X)/\prod_v G(\mathbf{O}_v)$$

induced by inclusion is surjective. Here T is the group of diagonal matrices in GL_n. The line bundles on $\mathbf{P}^1$ are determined, up to isomorphism, by their degrees, and the decomposition into a direct sum of line bundles is unique up to permutations. The proof that we give is an adaptation of arguments of Godement [12] and Weil [21]. Over the complex numbers, a proof of this result may be found in [13].

Chapter 3 proves an important refinement of the results in Chapter 2, which is exploited in our computation of automorphic representations. In the case where G is GL_n, the double coset space discussed here classifies the vector bundles on $\mathbf{P}^1$ with affine flags specified at two rational places.

In Chapter 4, we describe Iwahori and Matsumoto's presentation [15] of the extended Iwahori Hecke algebra. We describe the classification of irreducible square integrable representations due to Kazhdan and Lusztig [16]. Finally, we introduce two involutions, I and $\overline{\kappa}$, which appear in our description of the discrete spectrum.

We prove that when the derived group of G is of adjoint type, then $\overline{\kappa}$ takes an irreducible representation of H to its contragredient.

Chapter 5 derives formulas (5.1.1)-(5.1.4) that describe the action of generators of the Hecke algebras H_∞ and H_0 on M in terms of a chosen basis for M. The computation is first reduced to evaluating certain integrals on groups over the local field (as opposed to the adèlic group). These integrals are computed using techniques from the Theory of Tits systems. Besides the main formulas (5.1.1)-(5.1.4), none of Chapter 5 is necessary to understand the rest of this thesis.

In Chapter 6 we describe the discrete spectrum of (r, M) in terms of $\hat{H}$. This is done by using a Peter-Weyl type spectral decomposition (Theorem 6.2.1) for a module (ν, N) over $H \otimes H$, and then relating it to the spectral decomposition of (r, M), thereby proving Theorem 1.2.2.

CHAPTER 2
REDUCTION THEORY FOR THE PROJECTIVE LINE

2.1 The Statements

Let G be a split reductive group defined over $\mathbf{F}_q$. Fix a Borel subgroup B defined over $\mathbf{F}_q$ with unipotent radical N, and a maximal $\mathbf{F}_q$-split torus T contained in B. Let $F = \mathbf{F}_q(t)$. For a valuation v of F, we denote the corresponding local field by F_v and its ring of integers by $\mathbf{O}_v$. Let $\mathbf{A}$ denote the adèles of F. For each v, fix a uniformizing element $\pi_v \in F \cap \mathbf{O}_v$. In particular, fix $\pi_\infty = t^{-1}$ and $\pi_0 = t$ as uniformizing elements at the places ∞ and 0 whose local fields are $\mathbf{F}_q((t^{-1}))$ and $\mathbf{F}_q((t))$ respectively. Let K be the maximal compact subgroup $\prod_v G(\mathbf{O}_v)$ of $G(\mathbf{A})$. Let $X_*(T)$ denote the lattice $\mathrm{Hom}(\mathbf{G}_m, T)$ of algebraic cocharacters of T. Given $\eta \in X_*(T)$, and a valuation v denote by π_v^η the element $\eta(\pi_v) \in T(F_v) \subset T(\mathbf{A})$. The main result of this chapter is the following

Theorem 2.1.1. *Every double coset in*

$$G(F)\backslash G(\mathbf{A})/K$$

has a unique representative of the form $(t^{-1})^\eta$*, where* $\eta \in X_*(T)$ *is antidominant.*

In §2.6, we will deduce Theorem 2.1.1 from the following local result which is proved in §2.5. Let $F_\bullet$ be the local fields $\mathbf{F}_q((\pi))$ of Laurent series in π with coefficients in $\mathbf{F}_q$. It contains, as its ring of integers, the discrete valuation ring $\mathbf{O} = \mathbf{F}_q[[\pi]]$, and as a discrete subring, the polynomial ring $R = \mathbf{F}_q[\pi^{-1}]$. Let $\Gamma = G(R)$.

Theorem 2.1.2. *Every double coset in*

$$\Gamma\backslash G(F_\bullet)/G(\mathbf{O})$$

has a unique representative of the form π^η*, where* $\eta \in X_*(T)$ *is antidominant.*

2.2 Normed Local Vector Spaces

Let V be a vector space defined over $\mathbf{F}_q$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis of the free $\mathbf{O}$-module $V(\mathbf{O})$ (so that $V(\mathbf{O})$ is isomorphic to the free $\mathbf{O}$-module generated by the $\mathbf{e}_i$'s). Given a vector $\mathbf{x} \in V(F_\bullet)$, we may write $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$, uniquely, with $x_i \in F_\bullet$. Define

$$\|\mathbf{x}\| = \sup\{|x_1|, \ldots, |x_n|\}. \tag{2.2.1}$$

Lemma 2.2.2. *If $g \in GL(V(\mathbf{O}))$, then $\|\mathbf{x}g\| = \|\mathbf{x}\|$.*

Proof. Let (g_{ij}) be the matrix of G with respect to the basis chosen above. Let $\mathbf{y} = \mathbf{x}g$. If $\mathbf{y} = y_1\mathbf{e}_1 + \ldots + y_n\mathbf{e}_n$, then

$$y_j = \sum_{i=1}^{n} x_i g_{ij}$$

and

$$\begin{aligned} \|\mathbf{y}\| &= \sup_{1 \le j \le n} |\textstyle\sum_{i=1}^{n} x_i g_{ij}| \\ &\le \sup_{1 \le j \le n} \sup_{1 \le i \le n} |x_i g_{ij}| \quad \text{[ultrametric inequality.]} \\ &\le \sup_{1 \le j \le n} \sup_{1 \le i \le n} |x_i| \quad \text{[since } g_{ij} \in \mathbf{O}.] \\ &= \|\mathbf{x}\|. \end{aligned}$$

Hence

$$\|\mathbf{y}\| \le \|\mathbf{x}\|.$$

We may apply the same reasoning to g^{-1} to show that

$$\|\mathbf{x}\| \le \|\mathbf{y}\|.$$

Therefore,

$$\|\mathbf{y}\| = \|\mathbf{x}\|.$$

□

Corollary 2.2.3. *The norm $\|\cdot\|$ is independent of our choice of basis of $V(\mathbf{O})$.*

Proof. The coordinates of a vector with respect to two different bases differ by a matrix with entries in $\mathbf{O}$. The argument in the proof of Lemma 2.2.2 shows that the norms with respect to two different bases are equal. □

Lemma 2.2.4. *The norm* $\|\cdot\|$ *satisfies the* ultrametric triangle inequality*, i.e., for vectors* $\mathbf{x}$*,* $\mathbf{y}$ *in* $V(F_\bullet)$*,*

$$\|\mathbf{x}+\mathbf{y}\| \leq \sup\{\|\mathbf{x}\|, \|\mathbf{y}\|\}.$$

Proof. Write $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ and $\mathbf{y} = y_1\mathbf{e}_1 + \ldots + y_n\mathbf{e}_n$.

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\| &= \sup\{|x_1+y_1|, \ldots, |x_n+y_n|\} \\ &\leq \sup\{\sup\{|x_1|, |y_1|\}, \ldots, \sup\{|x_n|, y_n|\}\} \\ &= \sup\{|x_1|, |y_1|, \ldots, |x_n|, |y_n|\} \\ &= \sup\{\|\mathbf{x}\|, \|\mathbf{y}\|\}. \end{aligned}$$

□

Lemma 2.2.5. *For a scalar* $\lambda \in F_\bullet$ *and a vector* $\mathbf{x} \in V(F_\bullet)$*,*

$$\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|.$$

Lemma 2.2.6. *If* $g \in GL(V(F_\bullet))$*, then there is a constant* $C_g > 0$*, such that for any vector* $\mathbf{x} \in V(F_\bullet)$*,*

$$\|\mathbf{x}g\| \leq C_g\|\mathbf{x}\|.$$

Proof. Suppose that g has matrix (g_{ij}), and $\mathbf{x}$ has coordinates $(x_1, \ldots, x_n)$ with respect to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then

$$\begin{aligned} \|\mathbf{x}g\| &= \sup\{|\sum_{i=1}^n x_i g_{i1}|, \ldots, |\sum_{i=1}^n x_i g_{in}|\} \\ &\leq \sup_{1\leq j\leq n} \sup_{1\leq i\leq n} |g_{ij}|\|\mathbf{x}\|. \end{aligned}$$

Therefore, we may let

$$C_g = \sup_{1\leq j\leq n} \sup_{1\leq i\leq n} |g_{ij}|.$$

□

Lemma 2.2.7. *If* $\mathbf{x} \in V(R)$ *is a non-zero vector then* $\|\mathbf{x}\| \geq 1$*.*

Proof. By Corollary 2.2.3, we may assume that the elements $\mathbf{e}_i$ of a basis used to define $\|\cdot\|$ lie in $V(\mathbf{F}_q)$. Then at least one coordinate of $\mathbf{x}$ is non-zero in R. But any non-zero element in R has norm at least one. Therefore, $\|\mathbf{x}\| \geq 1$. □

Proposition 2.2.8. *For any non-zero vector* $\mathbf{x} \in V(\mathbf{F}_q)$ *and any* $g \in GL(V(F_\bullet))$, *there is a positive constant* E *such that for all* $\gamma \in GL(V(R))$

$$\|\mathbf{x}\gamma g\| \geq E.$$

Consequently, for any subset S *of* $GL(V(R))$, *the set* $\{\|\mathbf{x}sg\| : s \in S\}$ *has a positive minimal element.*

Proof. Applying Lemma 2.2.6 to g^{-1}, and Lemma 2.2.7 to $\mathbf{x}\gamma$ (which lies in $V(R)$), we have

$$\|\mathbf{x}\gamma g\| \geq C_{g^{-1}}\|\mathbf{x}\gamma\| \geq C_{g^{-1}} > 0.$$

The second part of the assertion follows by noting that the values taken by the norm $\|\cdot\|$ are of the form q^j, where j is an integer. □

2.3 Fundamental Representations

Let $\alpha_1, \ldots, \alpha_r$ be the simple roots with respect to B in the root system $\Phi(G,T)$ of G with respect to T. Let $W = N_G(T)/T$ be the Weyl group of G with respect to T. To each simple root α_i, we associate an element s_i of order two in W in the usual way.

Given a subset D of $\{1, \ldots, r\}$, let W_D denote the subgroup of W generated by $\{s_j | j \in D\}$, and let P_D denote the parabolic subgroup BW_DB of G containing B. This group has a Levi decomposition

$$P_D = L_D U_D,$$

where L_D is a reductive group of rank $|D|$ and U_D is the unipotent radical of P_D. $L_D \cap B$ is a Borel subgroup for L_D containing the split torus T. The set of simple roots of L_D with respect to $L_D \cap B$ is $\{\alpha_j | j \in D\}$. Denote by P_i (resp., L_i, U_i) the parabolic subgroup (resp., Levi subgroup, unipotent subgroup) corresponding to the set $\{1, \ldots, i-1, i+1, \ldots, r\}$. These are the maximal proper parabolic subgroups of G containing B.

Theorem 2.3.1 (Chevalley [8]). *There exist irreducible finite dimensional representations (ρ_i, V_i) of G, vectors $\mathbf{v}_i \in V_i(\mathbf{F}_q)$ that are unique up to scaling, and characters $\Delta_i : P_i \to \mathbf{G}_m$, for $i = 1, \ldots, r$ all defined over $\mathbf{F}_q$, such that*

1. *P_i is the stabilizer of the line generated by $\mathbf{v}_i$ and $\mathbf{v}_i \rho_i(p) = \Delta_i(p)\mathbf{v}_i$ for each $p \in P_i$ for $i = 1, \ldots, r$.*

2. *The restrictions μ_i to T of Δ_i's are antidominant weights of T with respect to B, which generate $X^*(T) \otimes \mathbf{Q}$ as a vector space over the of rational numbers.*

Moreover, for any subset D of $\{1, \ldots, r\}$, the maximal parabolic subgroups of L_D are $P_i \cap L_D$, where $i \in D$, and the representations of L_D provided by the preceding assertions applied to L_D may be taken to be the restrictions of the representations (ρ_i, V_i) from G to L_D.

2.4 Ordering by Roots

Lemma 2.4.1. *Let L be a Levi subgroup of G associated to a parabolic subgroup P containing B. Then there is a canonical surjection*

$$G(F_\bullet)/G(\mathbf{O}) \xrightarrow{\Phi_L^G} L(F_\bullet)/L(\mathbf{O}).$$

If $Q = MN$ is a parabolic subgroup of G containing B and contained in P, then M is a Levi subgroup for L corresponding to the parabolic subgroup $L \cap Q$ of L, and $\Phi_M^L \circ \Phi_L^G = \Phi_M^G$.

Proof. Given $g \in G(F_\bullet)$, we may use the Iwasawa decomposition to write $g = luk$, where $l \in L(F_\bullet)$, $u \in U(F_\bullet)$ and $k \in G(\mathbf{O})$. Moreover, if $g = l'u'k'$ is another such decomposition, then, setting $l_0 = l'^{-1}l$, and $k_0 = k'k^{-1}$,

$$u'^{-1}l_0 u = k_0 \in G(\mathbf{O}).$$

On the other hand,

$$k_0 = u'^{-1}l_0 u = l_0 l_0^{-1} u'^{-1} l_0 u.$$

Since L normalizes U, $l_0^{-1}u'^{-1}l_0 \in U(F_\bullet)$, and hence, setting $u_0 = l_0^{-1}u'^{-1}l_0u \in U(F_\bullet)$,

$$l_0 = k_0u_0 \in G(\mathbf{O})U(F_\bullet) \cap L(F_\bullet).$$

Therefore $l_0u_0^{-1} = k_0 \in G(\mathbf{O}) \cap P(F_\bullet) = P(\mathbf{O})$, so that $l_0 \in L(\mathbf{O})$. This shows that $luk \mapsto l$ induces a well defined map $\Phi_L^G : G(F_\bullet)/G(\mathbf{O}) \to L(F_\bullet)/L(\mathbf{O})$. It is clear that this map is surjective. To see that $\Phi_M^L \circ \Phi_L^G = \Phi_M^G$, note that we may write $g = muk$ with $m \in M(F_\bullet)$, $u \in N(F_\bullet)$ and $k \in G(\mathbf{O})$. But $N(F_\bullet) = (N(F_\bullet) \cap L(F_\bullet))U(F_\bullet)$, so we may write $u = u_1u_2$, where $u_1 \in N(F_\bullet) \cap L(F_\bullet)$ and $u_2 \in U(F_\bullet)$. Therefore, we see that $mM(\mathbf{O}) = \Phi_M^L(mu_1) = \Phi_M^G(g)$. □

In the sequel we denote Φ_T^G simply by Φ. Define

$$\Omega_G := \{g \in G(F_\bullet) \ : \ |\alpha_i \circ \Phi(g)| \geq 1 \text{ for } i = 1, \ldots, r\}. \tag{2.4.2}$$

Proposition 2.4.3. $G(F_\bullet) = \Gamma\Omega_G$.

Proof. The rank one case. Here G has one simple root α_1, and one fundamental representation (ρ_1, V_1) and a vector $\mathbf{v}_1 \in V_1(\mathbf{F}_q)$ such that for any element p in the parabolic subgroup $B = TN$, where N is the unipotent radical of B,

$$\mathbf{v}_1\rho_1(b) = \Delta_1(b)\mathbf{v}_1, \tag{2.4.4}$$

where the character $\Delta_1 : B \mapsto \mathbf{G}_m$ (defined over $\mathbf{F}_q$) restricts to a dominant weight μ_1 on the maximal split torus T. Let $g \in G(F_\bullet)$. We wish to show that $g \in \Gamma\Omega_g$. To this end, by Proposition 2.2.8, by replacing g, if necessary by an appropriate element of Γg, we may assume that g has the property that

$$\|\mathbf{v}_1\rho_1(\gamma g)\| \geq \|\mathbf{v}_1\rho_1(g)\| \text{ for all } \gamma \in \Gamma. \tag{2.4.5}$$

Write $g = tnk$, where $t \in T(F_\bullet)$, $n \in N(F_\bullet)$ and $k \in G(\mathbf{O})$. By Theorem 2.3.1 and Lemma 2.2.2,

$$\|\mathbf{v}_1\rho(g)\| = |\Delta_1(t)|\|\mathbf{v}_1\| = |\mu_1(t)|. \tag{2.4.6}$$

Fix an isomorphism $u_{\alpha_1} : \mathbf{G}_a \to N$ defined over $\mathbf{F}_q$, and let $x \in F_\bullet$ be such that $n = u_{\alpha_1}(x)$. Choose σ in the nontrivial $T(\mathbf{F}_q)$-coset of $N_G T(\mathbf{F}_q)$. Note that if $S \in R$, then $\sigma u_{\alpha_1}(S) \in \Gamma$, therefore, using Proposition 2.2.8,

$$\begin{aligned} |\mu_1(t)| &= \|\mathbf{v}_1\rho_1(g)\| \\ &\leq \|\mathbf{v}_1\rho_1(\sigma u_{\alpha_1}(S)tu_{\alpha_1}(x))\| \\ &= \|\mathbf{v}_1\rho_1({}^{\sigma}t\sigma u_{\alpha_1}(\alpha_1(t)^{-1}(S+\alpha_1(t)x)))\| \\ &= |\mu_1(t)|^{-1}\|\mathbf{v}_1\rho_1(u_{-\alpha_1}(\alpha(t)^{-1}S+x))\|. \end{aligned}$$

Here $u_{-\alpha_1} = \sigma u_{\alpha_1}\sigma^{-1}$, and its image is the root subgroup for $-\alpha_1$. The element $u_{-\alpha_1}(\alpha(t)^{-1}S+x)$ lies in the derived group of G which is isomorphic to either SL_2 or PGL_2 in the rank one case. When the derived group of G is isomorphic to SL_2, we may take V_1 to be the right action of SL_2 on the space of 1×2-matrices by right multiplication. One may take the torus T to consist of diagonal matrices in SL_2, B the upper triangular matrices in SL_2 and $\mathbf{v}_1$ to be the vector $(0,1)$. Calculating with matrices, one may verify that

$$\|\mathbf{v}_1\rho_1(u_{-\alpha_1}(\alpha(t)^{-1}S+x))\| \leq \sup\{1, |\alpha(t)^{-1}S+x|\}.$$

Therefore,

$$\sup\{1, |\alpha_1(t)^{-1}S+x|\} \geq |\mu_1(t)|^2. \tag{2.4.7}$$

Choose S in R such that $|S+\alpha(t)x| < 1$. Then $|\alpha_1(t)^{-1}S+x| < |\alpha_1(t)|^{-1}$. Suppose that $|\alpha_1(t)^{-1}S+x| \geq |\mu_1(t)|^2$. Then $|\alpha_1(t)|^{-1} > |\mu_1(t)|^2$. This is impossible, since $\alpha_1(t)^{-1} = \mu_1(t)^2$. It follows that $|\alpha_1(t)^{-1}S+x| < |\mu_1(t)|^2$. Therefore, (2.4.7) can hold only if $1 \geq |\mu_1(t)|^2$, which is the same as $|\alpha_1(t)| \geq 1$. This completes the proof of Proposition 2.4.3 when the derived group of G is isomorphic to SL_2.

In the case where the derived group of G is isomorphic to PGL_2, choosing once again the upper triangular Borel and diagonal torus, we may take V_1 to be the subspace of the vector space of 2×2 matrices (on which PGL_2 acts by right conjugation) generated by any non-zero nilpotent upper-triangular matrix with entries in $\mathbf{F}_q$, which we may take to be $\mathbf{v}_1$. In this case, a calculation with matrices shows that

$$\|\mathbf{v}_1\rho_1(u_{-\alpha_1}(\alpha_1(t)^{-1}S+x))\| \le \sup\{1, |\alpha_1(t)^{-1}S+x|, |\alpha_1(t)^{-1}S+x|^2\}.$$

Therefore,

$$\sup\{1, |\alpha_1(t)^{-1}S+x|, |\alpha_1(t)^{-1}S+x|^2\} \ge |\mu_1(t)|^2. \tag{2.4.8}$$

As before, choose S in R such that $|S+\alpha(t)x|<1$. Suppose that $|\alpha_1(t)^{-1}S+x|^2 \ge |\mu_1(t)|^2$. Then $|\alpha_1(t)|^{-2} > |\mu_1(t)|^2$. This is impossible, since $\alpha_1(t)^{-1} = \mu_1(t)$. Therefore, (2.4.8) implies that either $1 \ge |\mu_1(t)|^2$, or $|\alpha_1(t)|^{-1} > |\mu_1(t)|^2$. In either case, it follows that $|\mu_1(t)| \le 1$, so that $\alpha_1(t) \ge 1$. This takes care of the case when the derived group of G is isomorphic to PGL_2, completing the proof of Proposition 2.4.3 in the rank one case.

The general case. Let G be a group of rank r, and $g \in G(F_\bullet)$. By modifying g on the left by an element of Γ, we may, for the purposes of this proof, assume, using the second assertion of Proposition 2.2.8, that

$$\|\mathbf{v}_1\rho_1(g)\| \le \|\mathbf{v}_1\rho_1(\gamma g)\| \text{ for all } \gamma \in \Gamma. \tag{2.4.9}$$

Note that if $\gamma \in P_1(F_\bullet)\cap\Gamma$, then $\mathbf{v}_1\rho_1(\gamma g) = \Delta_1(\gamma)\mathbf{v}_1\rho_1(g)$. Since $\Delta_1(\gamma) \in \mathbf{F}_q[\pi^{-1}]^\times$, $|\Delta_1(\gamma)| = 1$. Therefore, $\|\mathbf{v}_1\rho_1(\gamma g)\| = \|\Delta_1(\gamma)\mathbf{v}_1\rho_1(g)\|$. We may use the second assertion of Proposition 2.2.8 again, to assume, for the purposes of this proof, that

$$\|\mathbf{v}_2\rho_2(g)\| \le \|\mathbf{v}_2\rho_2(\gamma g)\| \text{ for all } \gamma \in \Gamma \cap P_1(F_\bullet). \tag{2.4.10}$$

while preserving (2.4.9). Continuing in this manner, we may assume that

$$\|\mathbf{v}_j\rho_j(g)\| \le \|\mathbf{v}_j\rho_j(\gamma g)\| \text{ for all } \gamma \in \Gamma \cap P_1(F) \cap \ldots \cap P_{j-1}(F), \tag{2.4.11}$$

for $j = 1, \ldots, r$. Therefore, it suffices to prove that

Lemma 2.4.12. *If an element $g \in G(F_\bullet)$ satisfies the inequalities (2.4.11) for each integer $1 \le j \le r$, then $g \in \Omega_G$.*

The proof of Proposition 2.4.3 in the rank one case shows that Lemma 2.4.12 is true when G is of semisimple rank one. We proceed to prove it by induction on the semisimple rank of G.

The Levi subgroup L_r has semi-simple rank $r-1$, therefore, if we write $g = luk$, where $l \in L_r(F_\bullet)$, $u \in U_r(F_\bullet)$, and $k \in G(\mathbf{O})$, then by the induction hypothesis, $|\alpha_j \circ \Phi_T^L(l)| \geq 1$, for $j = 1, \ldots, r-1$. But

$$|\alpha_j \circ \Phi(g)| = |\alpha_j \circ \Phi_T^L(l)| \geq 1.$$

Therefore,

$$|\alpha_j \circ \Phi(g)| \geq 1 \text{ for } j = 1, \ldots, r-1.$$

It remains to see that $|\alpha_r \circ \Phi(g)| \geq 1$. In order to do this, we may repeat the above argument replacing L_r by the rank one Levi subgroup $L_{\{r\}}$. □

2.5 Local Reduction Theory

In order to prove the existence part of Theorem 2.1.2, it suffices to show that every element g in Ω_G may be written as $g = \gamma\pi^\eta k$, where $\gamma \in \Gamma$, $\eta \in X_*(T)$ is antidominant and $k \in G(\mathbf{O})$. To this end, we may assume (using the Iwasawa decomposition) that we are given $g \in \Omega_G$, with $g = tn$, with $t \in T(F_\bullet)$ and $n \in N(F_\bullet)$. Since g, and hence t, is in Ω_G, $|\alpha_i(t)| \geq 1$, so that $\alpha_i(t)^{-1} \in \mathbf{O}$, for $i = 1, \ldots, r$. For each root $\alpha \in \Phi(G, T)$, let U_α denote the corresponding root subgroup. Fix an isomorphism $u_\alpha : \mathbf{G}_a \to U_\alpha$ defined over $\mathbf{F}_q$. Then for $x \in F_\bullet$, we have

$$tu_\alpha(x) = (tu_\alpha(x)t^{-1})t = u_\alpha(\alpha(t)x)t.$$

Therefore, if we write $\alpha(t)x = P + h$, where $P \in R$ and $h \in \mathbf{O}$, then

$$tu_\alpha(x) = tu_\alpha(\alpha(t)^{-1}P)u_\alpha(\alpha(t)^{-1}h) = u_\alpha(P)tu_\alpha(\alpha(t)^{-1}h).$$

Given two positive roots α and β, the commutator $[U_\alpha, U_\beta]$ is contained in the product of root subgroups $U_{\alpha'}$ where the α' are roots which can be written as positive linear combinations of α and β and are distinct from either α or β. Moreover, we may enumerate the positive roots as $\beta_1, \beta_2, \ldots$ so that if $j > i$, then β_i can not be written as a sum of β_j and any other positive roots.

Write n as $\prod_i u_{\beta_i}(x_i)$. Then

$$tn = tu_{\beta_1}(x_1) \prod_{i>1} u_{\beta_i}(x_i)$$

If we write $\beta_1(t)x_1 = P_1 + h_1$, where $P_1 \in \mathbf{F}_q[\pi^{-1}]$ and $h \in \mathbf{O}$, then

$$tn = u_{\beta_1}(P_1) t u_{\beta_1}(\beta_1(t)^{-1} h_1) \prod_{i>1} u_{\beta_i}(x_i)$$

Since $u_{\beta_1}(P_1) \in \Gamma$, $\beta_1(t)^{-1} \in \mathbf{O}$, and the image of u_{β_1} normalizes all the subsequent roots subgroups whose elements appear in the above expression, we may assume for the purpose of proving Theorem 2.1.2, that

$$tn = t \prod_{i>1} u_{\beta_i}(x_i'),$$

for $x_i' \in F_\bullet$. We may continue in this manner to reduce tn to t. It is then easy to see (using the decomposition $F_\bullet^\times = \pi^{\mathbf{Z}} \mathbf{O}^\times$) that t may be replaced by π^η for $\eta \in X_*(T)$. Since $|\alpha_i(\pi^\eta)| \geq 1$, it follows that η is antidominant, proving the existence part of Theorem 2.1.2.

We now prove the uniqueness part of Theorem 2.1.2. In order to do this, it suffices to show that if η and ν are two dominant coweights, and $\pi^\nu = \gamma \pi^\eta k$ for some $\gamma \in \Gamma$ and $k \in G(\mathbf{O})$, then $\nu = \eta$. Since the weights $\mu_1, \ldots, \mu_r$ corresponding to the fundamental representations in Theorem 2.3.1 generate the vector space $X^*(T) \otimes \mathbf{Q}$, it suffices to show that $\langle \mu_i, \nu \rangle = \langle \mu_i, \eta \rangle$ for each i. In order to do this, we need the following

Lemma 2.5.1. *For any non-zero vector* $\mathbf{v} \in V_i(F_\bullet)$ *and any antidominant coweight* $\mu \in X_*(T)$,

$$\frac{\|\mathbf{v}\rho_i(\pi^\mu)\|}{\|\mathbf{v}\|} \geq \frac{\|\mathbf{v}_i\rho_i(\pi^\mu)\|}{\|\mathbf{v}_i\|}$$

Proof. Since T is an $\mathbf{F}_q$-split torus and ρ_i is defined over $\mathbf{F}_q$, V has a decomposition (over $\mathbf{F}_q$) into root subspaces

$$V = \bigoplus_{\lambda} V_\lambda,$$

where T acts on V_λ by the character $\lambda : T \to \mathbf{G}_m$. It is easy to see that μ_i is the lowest weight of T occurring in (ρ_i, V_i), so that $\langle \mu_i, \mu\rangle \geq \langle \lambda, \mu\rangle$ for any weight λ of T occuring in (ρ_i, V_i) and any antidominant coweight μ. Given any vector $\mathbf{v} \in V(F_\bullet)$, we may write

$$\mathbf{v} = \sum x_j \mathbf{u}_j, \text{ where } x_j \in F_\bullet \text{ and } \mathbf{u}_j \in V_{\lambda_j}(\mathbf{F}_q) \text{ for each } j,$$

where $x_j \in F_\bullet$ and $\mathbf{u}_j \in V_{\lambda_j}(\mathbf{F}_q)$ for each j and the λ_j's are not necessarily distinct.

$$\begin{aligned}
\|\mathbf{v}\rho_i(\pi^\mu)\| &= \left\|\sum \lambda_j(\pi^\mu) x_j \mathbf{u}_j\right\| \\
&= \sup_j\{|\lambda_j(\pi^\mu) x_j|\} \\
&= \sup_j\{q^{-\langle \lambda_j, \mu\rangle}|x_j|\} \\
&\geq q^{-\langle \mu_i, \mu\rangle} \sup_j\{|x_j|\} \\
&= \|\mathbf{v}_i\rho_i(\pi^\mu)\|\ \|\mathbf{v}\|
\end{aligned}$$

Since $\|\mathbf{v}_i\| = 1$, this completes the proof of Lemma 2.5.1. $\square$

Lemma 2.5.1 allows us to compare $\langle \mu_i, \nu\rangle$ and $\langle \mu_i, \eta\rangle$:

$$\begin{aligned}
q^{-\langle \mu_i, \eta\rangle} &= \frac{\|\mathbf{v}_i\rho_i(\pi^\eta)\|}{\|\mathbf{v}_i\|} \\
&\leq \frac{\|\mathbf{v}_i\rho_i(\gamma\pi^\eta)\|}{\|\mathbf{v}_i\rho_i(\gamma)\|} \\
&\leq \frac{\|\mathbf{v}_i\rho_i(\gamma\pi^\eta)\|}{\|\mathbf{v}_i\|} \\
&= \frac{\|\mathbf{v}_i\rho_1(\pi^\nu)\|}{\|\mathbf{v}_i\|} \\
&= q^{-\langle \mu_i, \nu\rangle}.
\end{aligned}$$

The first inequality is Lemma 2.5.1 applied to $\mathbf{v} = \mathbf{v}_i\rho_i(\gamma)$. The second inequality follows from Lemma 2.2.7 with $\mathbf{x} = \mathbf{v}_i\rho_i(\gamma)$. Interchanging the roles of η and ν in the above arguments shows that $\langle\mu_i, \eta\rangle = \langle\mu_i, \nu\rangle$ for each i. This completes the proof of the uniqueness part of the assertion of Theorem 2.1.2.

2.6 Global Reduction Theory

If $g = (g_v)_v$ is an element of $G(\mathbf{A})$ then, since $g_v \in G(\mathbf{O}_v)$ for all but finitely many places v of F, we may assume, for the purpose of proving Theorem 2.1.1 that g is a finite product $g = g_\infty g_{v_1} g_{v_2} \dots g_{v_k}$, with $g_\infty \in G(F_\infty)$ and $g_{v_j} \in G(F_{v_j})$, $v_j \neq \infty$, for $1 \leq j \leq k$. By Theorem 2.1.2, there is a decomposition

$$g_{v_k} = \gamma_k \pi_{v_k}^{\eta_k} \kappa_k,$$

where $\gamma_k \in G(\mathbf{F}_q[\pi_{v_k}^{-1}])$, $\eta_k \in X_*(T)$, and $\kappa_k \in G(\mathbf{O}_{v_k})$. Now γ_k and $\pi_{v_k}^{\eta_k}$ are contained in $G(F)$ and in $G(\mathbf{O}_v)$ for all $v \neq \infty$. Therefore, by multiplying g on the left by $\pi_{v_k}^{-\eta_k}\gamma^{-1}$ we get an element of the subset

$$G(F_\infty) \times \prod_{j=1}^{k-1} G(F_{v_j}) \times \prod_{\text{all other } v} G(\mathbf{O}_v).$$

of $G(\mathbf{A})$.

We have now reduced g to an element with non-trivial entries at only at most $k-1$ places and ∞. We may continue in this manner until the entries at all places except ∞ are trivial. Finally, the use of Theorem 2.1.2 to $v = \infty$ gives us a representative each double coset of type asserted by Theorem 2.1.1.

The uniqueness part of the theorem follows from the corresponding assertion in the local situation, because two elements g and h of $G(F_\infty)$ lie in the same double coset if and only if $g = \gamma h k$, with $\gamma \in G(\mathbf{F}_q[t])$ and $k \in G(\mathbf{O}_\infty)$.

CHAPTER 3
THE BIRKHOFF DECOMPOSITION

3.1 The Statement

We use the notation introduced in Chapter 2. Moreover, for a degree one valuation v (such as ∞ or 0), let I_v denote the pre-image of $B(\mathbf{F}_q)$ under the natural map $G(\mathbf{O}_v) \to G(\mathbf{F}_q)$. Consider the compact, open subgroup

$$K' = I_\infty \times I_0 \times \prod_{v \neq 0,\infty} G(\mathbf{O}_v).$$

Let W denote the *Weyl group* $N_G(T)/T$ of G with respect to T. Fix a function $\phi : W \to G(\mathbf{F}_q)$ (which is not necessarily a group homomorphism) such that $\phi(w) \in B(\mathbf{F}_q)wT(\mathbf{F}_q)B(\mathbf{F}_q)$ for each $w \in W$. For each valuation v of F, our choice of a uniformizing element $\pi_v \in \mathbf{O}_v \cap F$ gives us a function

$$W \ltimes X_*(T) \to G(F_v) \hookrightarrow G(\mathbf{A})$$

which we denote by ϕ_v, defined by the formula

$$\phi_v(w\eta) = \phi(w)\pi_v^\eta.$$

Theorem 3.1.1 (Birkhoff decomposition). *The map $\phi_{t^{-1}}$ induces a bijection*

$$W \ltimes X_*(T) \tilde{\to} G(F)\backslash G(\mathbf{A})/K'.$$

K' is a subgroup of finite index in K. Theorem 2.1.1 gives us the structure of $G(F)\backslash G(\mathbf{A})/K$. We prove Theorem 3.1.1 by studying the fibers of the function

$$\Psi : G(F)\backslash G(\mathbf{A})/K' \to G(F)\backslash G(\mathbf{A})/K.$$

3.2 Structure of the Fiber

Fix a dominant coweight $\eta \in X_*(T)$. For convenience in notation, let u denote t^{-1}. Any element $k \in K$ gives us an element $G(F)u^\eta kK'$ in the fiber of Ψ over $G(F)u^\eta K$. Clearly, two elements of K give the same element in the fiber if they lie in the same double coset of $K \cap u^{-\eta}G(F)u^\eta \backslash K/K'$. Since K and K' differ only at the places ∞ and 0,

$$K/K' \;\cong\; G(\mathbf{O}_\infty)/I_\infty \times G(\mathbf{O}_0)/I_0 \;\cong\; G(\mathbf{F}_q)/B(\mathbf{F}_q) \times G(\mathbf{F}_q)/B(\mathbf{F}_q).$$

The latter identification is obtained via the natural map $q_v : G(\mathbf{O}_v) \to G(\mathbf{F}_q)$. Moreover, $k \in K$ acts on the left of $G(\mathbf{F}_q)/B(\mathbf{F}_q) \times G(\mathbf{F}_q)/B(\mathbf{F}_q)$ as componentwise left multiplication by $(q_\infty(k_\infty), q_0(k_0))$, where k_v denotes the component of k at the place v.

Let K_η denote the subgroup $K \cap u^{-\eta}G(F)u^\eta$ of K. For each root α, fix an isomorphism $u_\alpha : \mathbf{G}_a \to U_\alpha$, where U_α is the root subgroup for α.

Lemma 3.2.1. *For each adèle $x = (x_v)_v$ of F, $u_\alpha(x)$ lies in $u^\eta K_\eta u^{-\eta}$ if and only if x is a polynomial in t of degree at most $v_\infty(\alpha(u^\eta))$ (a rational function, and hence a polynomial in t is identified with the adèle whose entries at all places equal this function). In particular, if $\langle \alpha, \eta \rangle < 0$, then $u_\alpha(x)$ lies in $u^\eta K_\eta u^{-\eta}$ if and only if $x = 0$.*

Proof. The lemma follows from the observation that the following conditions are imposed upon x:

1. $x \in F \subset \mathbf{A}$.
2. $x_v \in \mathbf{O}_v$ for all $v \neq \infty$.
3. $(\alpha(u^{-\eta})x)_\infty \in \mathbf{O}_\infty$.

The first two conditions imply that x is a polynomial in t, and the third condition implies that this polynomial is of degree at most $v_\infty(\alpha(u^\eta))$. $\square$

Let D denote the set of simple reflections s in W such that $s(u^\eta) = u^\eta$. Then we may associate a parabolic subgroup P_D with unipotent radical U_D and Levi component L_D to D as in §2.3. Let L be the group $L_D(\mathbf{F}_q) \subset L_D(F) \subset L_D(\mathbf{A})$ and U the subgroup of $G(\mathbf{A})$ generated by elements $u_\alpha(x)$, where $x \in F \subset \mathbf{A}$ is a polynomial in t of degree at most $v_\infty(\alpha(u^\eta))$. It follows, from Lemma 3.2.1, that $K_\eta = u^{-\eta} L U u^\eta$. Moreover, this group acts on the first and second components of $G(\mathbf{F}_q)/B(\mathbf{F}_q) \times G(\mathbf{F}_q)/B(\mathbf{F}_q)$ (which we have identified with K/K') via its the constant term of its constituents at ∞ and 0 respectively, viewed as polynomials in the uniformizer. The constituents at ∞ and 0 of L are the same, because conjugation by u^η fixes L. On the other hand, $u^{-\eta} u_\alpha(x_0 + x_1 t + \ldots + x_{v_\infty(\alpha(u^\eta))} t^{v_\infty(\alpha(u^\eta))}) u^\eta$ acts on the first component by $u_\alpha(x_{v_\infty(\alpha(u^\eta))})$ and on the second component by $u_\alpha(x_0)$. Let $\Delta : G(\mathbf{F}_q) \to G(\mathbf{F}_q) \times G(\mathbf{F}_q)$ denote the diagonal inclusion, and j_i denote the inclusion of $G(\mathbf{F}_q)$ into $G(\mathbf{F}_q) \times G(\mathbf{F}_q)$ along the ith coordinate. We have shown that the fiber of Ψ over $G(F) u^\eta K$ is in bijective correspondence with

$$S = \Delta[L_D(\mathbf{F}_q)] j_1[U_D(\mathbf{F}_q)] j_2[U_D(\mathbf{F}_q)] \backslash [G(\mathbf{F}_q)/B(\mathbf{F}_q) \times G(\mathbf{F}_q)/B(\mathbf{F}_q)]. \qquad (3.2.2)$$

Let $\Delta : W \to W \times W$ denote the diagonal inclusion.

Proposition 3.2.3. *The map $\Delta(W_D)\backslash(W \times W) \to S$ which takes $\Delta(W_D)(w_1, w_2)$ to the double coset of $(\phi(w_1), \phi(w_2))$ in S is a bijection.*

Proof. Let

$$W^D = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in D\}.$$

This is the set consisting of the unique longest elements of the left (or right) W_D-cosets. As an $L(\mathbf{F}_q)$-space,

$$U(\mathbf{F}_q)\backslash G(\mathbf{F}_q)/B(\mathbf{F}_q) = \coprod_{w \in W^D} L(\mathbf{F}_q)/B_D(\mathbf{F}_q),$$

where $B_D = B \cap L_D$. Therefore,

$$L(\mathbf{F}_q)\backslash[U(\mathbf{F}_q)\backslash G(\mathbf{F}_q)/B(\mathbf{F}_q) \times U(\mathbf{F}_q)\backslash G(\mathbf{F}_q)/B(\mathbf{F}_q)] =$$
$$= L(\mathbf{F}_q)\backslash \coprod_{W^D \times W^D} [L(\mathbf{F}_q)/B_D(\mathbf{F}_q) \times L(\mathbf{F}_q)/B_D(\mathbf{F}_q)] = \coprod_{W^D \times W^D} W_D.$$

This shows that the sets S and $\Delta(W_D)\backslash(W \times W)$ have the same cardinality.

To show that the map in the assertion is surjective, it suffices to show that every double coset in (3.2.2) has a representative in the image of $\phi \times \phi$. For the remainder of this section, we will, for convenience, write w instead of $\phi(w)$.

Given $(g_1, g_2) \in G(\mathbf{F}_q)^2$, we may write

$$g_i = l_i u_i w_i b_i \; ; \; l_i \in L_D(\mathbf{F}_q), \; u_i \in U_D(\mathbf{F}_q), \; w_i \in W \text{ and } b_i \in B(\mathbf{F}_q), \text{ for } i = 1, 2.$$

Since L_D normalizes U_D, it follows that the double coset of (g_1, g_2) contains the element (lw_1, w_2), where $l = l_2^{-1} l_1$. Let Φ^+ (resp. Φ_D^+) denote the positive roots in $\Phi(G, T)$ (resp. $\Phi(L_D, T)$ with respect to B (resp. $B \cap L_D$). Then,

Lemma 3.2.4. *For any $w \in W$, there exists $\tilde{w} \in W_D$ such that $\tilde{w}\Phi_D^+ \subset w\Phi^+$.*

Proof of lemma. We may write $w = w_D (w^D)^{-1}$, with $w_D \in W_D$, and $w^D \in W^D$ [14, p. 123]. Moreover, $w^D \Phi_D^+ \subset \Phi^+$ [14, p. 111]. Therefore, if we set $\tilde{w} = w_D = ww^D$,

$$\tilde{w}\Phi_D^+ = ww^D \Phi_D^+ \subset w\Phi^+.$$

□

Let $\tilde{w}_i$ be the element of W_D provided by the above lemma when it is applied to $w = w_i$, for $i = 1, 2$. Use the Bruhat decomposition for L_D to write

$$\tilde{w}_2^{-1} l \tilde{w}_1 = b_1 w b_2, \text{ where } b_i \in B(\mathbf{F}_q) \cap L(\mathbf{F}_q) \text{ and } w \in W_D.$$

Then, using $\sim$ to denote the equivalence relation of belonging to the same double coset,

$$\begin{aligned}(lw_1, w_2) &= (\tilde{w}_2 b_1 w b_2 \tilde{w}_1^{-1} w_1, w_2)\\ &\sim (w b_2 \tilde{w}_1^{-1} w_1, b_1^{-1} \tilde{w}_2^{-1} w_2)\\ &= (w \tilde{w}_1^{-1} w_1 w_1^{-1} \tilde{w}_1 b_2 \tilde{w}_1^{-1} w_1, \tilde{w}_2^{-1} w_2 w_2^{-1} \tilde{w}_2 b_1^{-1} \tilde{w}_2^{-1} w_2)\\ &\sim (w \tilde{w}_1^{-1} w_1, \tilde{w}_2^{-1} w_2)\end{aligned}$$

proving surjectivity (the last step uses the fact that $\tilde{w}_i$ conjugates $B(\mathbf{F}_q) \cap L(\mathbf{F}_q)$ into $w_i B(\mathbf{F}_q) w_i^{-1}$). □

Let $X_*(T)^{++}$ denote the set of dominant coweights. Define a function $q : W \ltimes X_*(T) \to X_*(T)^{++}$ by mapping $w\eta \in W \ltimes X_*(T)$ to the unique dominant weight in the W-orbit of η. Clearly, the diagram

$$\begin{array}{ccc} W \ltimes X_*(T) & \xrightarrow{\phi_{t^{-1}}} & G(F)\backslash G(\mathbf{A})/K' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \Psi} \\ X_*(T)^{++} & \xrightarrow{\sim} & G(F)\backslash G(\mathbf{A})/K \end{array}$$

commutes. The lower horizontal arrow is a bijection. We have shown that the fibers of the vertical arrows are in bijective correspondence. Therefore, the upper horizontal arrow is also a bijection.

This completes the proof of Theorem 3.1.1.

CHAPTER 4
EXTENDED IWAHORI HECKE ALGEBRAS

4.1 The Extended Affine Weyl Group

Let G be a split semi-simple group over $\mathbf{F}_q$, with a simple root system. As before, fix a maximal split torus T and a Borel subgroup B containing T defined over $\mathbf{F}_q$. Let $F_\bullet$ be the local field of Laurent series in one variable π with coefficients in $\mathbf{F}_q$, and denote its ring of integers by $\mathbf{O}$.

The *extended affine Weyl group* of G is the group

$$\tilde{W} = N_G(T)(F_\bullet)/T(\mathbf{O}).$$

The Weyl group W of G acts on the lattice $X_*(T)$ of algebraic cocharacters of T. Moreover, the map taking a cocharacter η to $\eta(\pi) \in T(F_\bullet)$ induces an isomorphism $X_*(T) \to T(F_\bullet)/T(\mathbf{O})$. The extended affine Weyl group $\tilde{W}$ is the semidirect product $W \ltimes X_*(T)$. Let Q denote the sublattice of $X_*(T)$, known as the *root lattice*, which is generated by the set of roots $\Phi = \Phi(G,T)$ of G with respect to T. The subgroup $W_a = W \ltimes Q$ is called the *affine Weyl group* of G.

Let $X_* = X_*(T) \otimes \mathbf{R}$. Φ is a root system in the dual vector space X^*. Let $\alpha_1, \ldots, \alpha_r$ denote the set of simple positive roots with respect to the Borel subgroup B. Let s_i denote the reflection about the hyperplane $\alpha_i = 0$ in X_*. Then $s_1, \ldots, s_r$ generate W. Let $\tilde{\alpha}$ denote the *highest root*, and s_0 denote the reflection about the hyperplane $\tilde{\alpha} = 1$. Then W_a is generated by the simple reflections $s_0, s_1, \ldots, s_r$, subject to the relations that $s_i^2 = 1$ for each i, and for every pair of distinct simple reflections s_i and s_j, a braid relation:

$$s_i s_j s_i \ldots = s_j s_i s_j \ldots \ (m_{ij} \text{ factors on each side}) .$$

for some integer $m_{ij} > 1$.

Let α_0 denote the affine linear functional $1 - \tilde{\alpha}$. Let

$$C = \{\mathbf{x} \in X_* \mid \alpha_i(\mathbf{x}) > 0 \text{ for } i = 0, \ldots, r\}.$$

The closure of C is a fundamental domain for the action of W_a on X_*.

The quotient $\Pi = X_*(T)/Q$ is a finite abelian group, and $\tilde{W} = W_a \rtimes \Pi$. The group Π may be realized as a subgroup of W_a, whose elements map C into itself, acting on X_* via symmetries of C. We say that G is of *simply connected type* if Π is trivial. Since G is semi-simple, the Killing form gives a non-degenerate W-invariant bilinear form $(\cdot,\cdot)$ on the vector spaces X_* and X^*. For each $\alpha \in \Phi$ let $\check{\alpha}$ be the linear functional on X^* given by

$$\check{\alpha}(\lambda) = \frac{(2\alpha, \lambda)}{(\alpha, \alpha)}.$$

The set of coroots of G with respect to T is the subset

$$\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$$

of X_*. The sublattice of $X_*(T)$ generated by $\check{\Phi}$ in X_* is known as the *coroot lattice*. We say that G is of *adjoint type* if the coroot lattice equals the lattice $X_*(T)$.

As in §3.1, fix a map $\phi : W \to G(\mathbf{F}_q)$. Define $\tilde{\phi} : \tilde{W} \to G(F_\bullet)$ by $\tilde{\phi}(w, \eta) = \phi(w)\pi^\eta$, where π^η denotes $\pi(\eta) \in T(F_\bullet)$. Let $I_\bullet$ denote the Iwahori subgroup of $G(F_\bullet)$ corresponding to B. Then the *affine Bruhat decomposition* asserts that $\tilde{\phi}$ induces a bijection $\tilde{W} \to I_\bullet \backslash G(F_\bullet)/I_\bullet$. For the remainder of this section we will abuse notation and denote $\tilde{\phi}(w)$ simply by w for all $w \in \tilde{W}$.

4.2 The Extended Iwahori Hecke Algebra

Consider the convolution algebra H of compactly supported complex valued measures on $G(F_\bullet)$ which are left and right invariant under translation by elements of $I_\bullet$ with the convolution product;

$$\int_{G(F_\bullet)} f(g)d(\mu_1 * \mu_2)(g) = \int_{G(F_\bullet)\times G(F_\bullet)} f(g_1g_2)d(\mu_1 \otimes \mu_2)(g_1g_2).$$

This algebra is known as the *extended Iwahori Hecke algebra*.

The affine Bruhat decomposition yields a vector space isomorphism

$$\mathbf{C}[\tilde{W}] \to H$$

taking the basis element 1_w of $\mathbf{C}[\tilde{W}]$ to $1_{I_\bullet w I_\bullet} dg$, for each $w \in \tilde{W}$, where, for any subset S of $G(F_\bullet)$, 1_S denotes the characteristic function of S, and dg is the Haar measure on $G(F_\bullet)$ which assigns unit measure to $I_\bullet$. Let T_i denote the image of 1_{s_i} under the above isomorphism. Let H_a (known as the *affine Hecke algebra*) be the sub-algebra of H generated by the T_i's, for $i = 0, \ldots, r$. Then Iwahori and Matsumoto have shown that H_a has a presentation with generators T_i, $i = 0, \ldots, r$, with relations

$$T_i^2 = q + (q-1)T_i \text{ for } i = 0, \ldots, r \tag{4.2.1}$$

$$T_iT_jT_i \ldots = T_jT_iT_j \ldots \ (m_{ij} \text{ factors}), \text{ for each pair } i \neq j. \tag{4.2.2}$$

and the algebra H is an extension of H_a by $\mathbf{C}[\Pi]$, acting by

$$1_a T_i 1_a^{-1} = T_{a(i)}, \tag{4.2.3}$$

where $a(i)$ is such that $as_ia^{-1} = s_{a(i)}$, for $a \in \Pi$.

4.3 Square Integrable Representations

Given a representation (ρ, V) of H, we define its contragredient representation to be the representation $(\tilde{\rho}, \tilde{V})$, where $\tilde{V}$ is the vector space $\mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ and the action of H on $\tilde{V}$ is defined by

$$(\tilde{\mathbf{v}}\tilde{\rho}(h))(\mathbf{v}) = \tilde{\mathbf{v}}(\mathbf{v}\rho(h^{\mathrm{op}})) \text{ for all } \tilde{\mathbf{v}} \in \tilde{V},\ \mathbf{v} \in V \text{ and } h \in H.$$

Here h^{op} is the image of $h \in H$ under the anti-involution (i.e., algebra homomorphism $H \to H^{\text{op}}$) which takes each generator T_i of H to itself. Note that on the level of measures,

$$\int_{G(F_\bullet)} f(g)d\mu^{\text{op}}(g) = \int_{G(F_\bullet)} f(g^{-1})d\mu(g)$$

for every compactly supported locally constant function f on $G(F_\bullet)$.

The representation theory of extended affine Hecke algebras has been studied in terms of the *Langlands dual group* of G. This is a complex Lie group $\check{G}$ with a maximal split torus $\check{T}$ that is identified with the set of unramified complex characters of T, so that the lattice, $X^*(\check{T})$, of algebraic characters of $\check{T}$ is identified with $X_*(T)$, and such that its root system with respect to $\check{T}$ is identified with $\check{\Phi}$.

Let (ρ, V) be an irreducible representation of H and let $(\tilde{\rho}, \tilde{V})$ denote its contragredient. Fix non-zero vectors $\mathbf{v} \in V$ and $\tilde{\mathbf{v}} \in \tilde{V}$. We define the *matrix coefficient* of (ρ, V) with respect to the vectors $\mathbf{v}$ and $\tilde{\mathbf{v}}$ as the $\mathbf{C}$-linear function $H \to \mathbf{C}$ given by

$$c_{\mathbf{v},\tilde{\mathbf{v}}} : h \mapsto \langle \tilde{\mathbf{v}}, \mathbf{v}\rho(h) \rangle.$$

This corresponds, in a natural manner, to a complex valued function on $I_\bullet \backslash G(F_\bullet)/I_\bullet$, which we will also denote by $c_{\mathbf{v},\tilde{\mathbf{v}}}$. We say that an irreducible representation V is a *square integrable* representation of H if for any pair of vectors $(\mathbf{v}, \tilde{\mathbf{v}}) \in V \times \tilde{V}$, the function $c_{\mathbf{v},\tilde{\mathbf{v}}}$ is square integrable with respect to a Haar measure on $G(F_\bullet)$.

Let $\hat{H}$ denote the set of irreducible square integrable representations of H. Kazhdan and Lusztig describe $\hat{H}$ in terms of $\check{G}$ in [16]:

Theorem 4.3.1 (Kazhdan-Lusztig). *Assume that G is of adjoint type and H is the associated extended Iwahori Hecke algebra. The irreducible square-integrable representations of H are parameterized by the set of conjugacy classes of triples (s, u, σ), where s is a semisimple element of $\check{G}$, u a unipotent element of $\check{G}$ such that $sus^{-1} = u^q$ and such that both s and u are not contained in the Levi subgroup of any proper parabolic subgroup of $\check{G}$, and σ is a representation of the group of connected components of the simultaneous centralizer of s and u in $\check{G}$ which occurs in $H_*(\mathbf{B}_{s,u}, \mathbf{Q})$, where $\mathbf{B}_{s,u}$ is the variety of Borel subgroups of $\check{G}$ containing both s and u.*

4.4 Involutions

We discuss here two involutions that appear in our description of discrete automorphic representations.

The Iwahori-Matsumoto Involution

The Iwahori-Matsumoto involution is an algebra homomorphism $I : H \to H$ of order two. We first define it on the subalgebra H_a in terms of the generators T_i by the formula

$$I(T_i) = -qT_i^{-1} \tag{4.4.1}$$

Firstly, observe that (4.2.1) may be used to compute

$$T_i^{-1} = (q^{-1} - 1) + q^{-1}T_i. \tag{4.4.2}$$

In particular, T_i^{-1} is well defined. Also,

$$\begin{aligned} [I(T_i)]^2 &= (-qT_i^{-1})^2 \\ &= q^2[(q^{-1} - 1) + q^{-1}T_i]^2 \\ &= [(1 - q) + T_i]^2 \\ &= (1 - q)^2 + q + (1 - q)T_i. \end{aligned}$$

On the other hand

$$\begin{aligned} I(T_i^2) &= I[q + (q - 1)T_i] \\ &= q + (q - 1)(-q)[(q^{-1} - 1) + q^{-1}T_i] \\ &= (1 - q)^2 + q + (1 - q)T_i, \end{aligned}$$

showing that I preserves the relations (4.2.1). It is clear that I preserves the braid relations (4.2.2). Therefore, (4.4.1) gives a well defined algebra homomorphism (obviously of order two) $H_a \to H_a$. The automorphism I commutes with the action of

$\mathbf{C}[\Pi]$ on H_a, and hence lifts to an automorphism $I : H \to H$. We call the resulting automorphism the Iwahori-Matsumoto involution on H.

The Involution $\overline{\kappa}$

The finite Weyl group W of G contains a unique element w_0 of maximal length in terms of the generators $s_1, \ldots, s_r$. Define an involution $\overline{\kappa} : \tilde{W} \to \tilde{W}$ by

$$\overline{\kappa}(w\pi^{\eta}) = w_0 w w_0^{-1} \pi^{-w_0(\eta)} \text{ for all } w \in W \text{ and } \eta \in X_*(T).$$

It is easily verified that the above formula defines an automorphism on $\tilde{W}$. Moreover, if $\bar{i}$ is such that $s_{\bar{i}} = w_0 s_i w_0^{-1}$ for $i = 1, \ldots, r$, then $\overline{\kappa}(s_i) = s_{\bar{i}}$ If we view $\tilde{W}$ as a subgroup of the group of affine automorphisms of X_*, then

$$\overline{\kappa}(w)(\mathbf{x}) = w(-w_0(\mathbf{x}))$$

Since $w_0(\tilde{\alpha}) = -\tilde{\alpha}$, we have:

$$\begin{aligned} \overline{\kappa}\{\mathbf{x} | \tilde{\alpha}(\mathbf{x}) = 1\} &= \{-w_0(\mathbf{x}) | \tilde{\alpha}(\mathbf{x}) = 1\} \\ &= \{\mathbf{x} | \tilde{\alpha}(\mathbf{x}) = 1\}. \end{aligned}$$

This shows that $\overline{\kappa}(s_0) = s_0$. Since $\overline{\kappa}$ fixes C, it also maps Π into itself.

We now define an automorphism of H, which we also denote by $\overline{\kappa}$, by requiring that

$$\begin{aligned} T_i &\mapsto T_{\bar{i}} \text{ for } i = 1, \ldots, r, \\ T_0 &\mapsto T_0, \text{ and} \\ 1_a &\mapsto 1_{\overline{\kappa}(a)} \text{ for all } a \in \Pi. \end{aligned}$$

Since $\overline{\kappa}$ maps generators, to generators, it preserves the relations (4.2.1). Since it comes from an automorphism of W_a, it preserves the relations (4.2.2) and (4.2.3).

Clearly, the automorphism $\overline{\kappa}$ is of order two. It has a simple interpretation in terms of the representations of H when G is of adjoint type:

Theorem 4.4.3. *Suppose that G is of adjoint type, and (ρ, V) is an irreducible representation of H. Then $(\rho \circ \overline{\kappa}, V)$ and $(\tilde{\rho}, \tilde{V})$ are isomorphic as representations of H.*

Proof. For any dominant cocharacters η_1 and η_2,

$$(1_{I_\bullet \pi^{\eta_1} I_\bullet} dg) * (1_{I_\bullet \pi^{\eta_2} I_\bullet} dg) = 1_{I_\bullet \pi^{\eta_1+\eta_2} I_\bullet} dg.$$

These elements, therefore, generate a commutative subalgebra S of H that is canonically isomorphic to $\mathbf{C}[X_*(T)]$. By [11, Theorem 5.5], the isomorphism class of an irreducible finite-dimensional H-module is determined by the weights of S on it. The weights of S on (ρ, V) coincide with the weights of S on the normalized Jacquet module (ρ_N, V_N), where $N \subset B$ is the maximal unipotent subgroup [7, Section 3]. By [5, Lemma 4.7], it suffices to show that the weights of S on $(\rho \circ \overline{\kappa}_N, V_N)$ are the same as those on $((\tilde{\rho})_N, (\tilde{V})_N)$. By [7, Corollary 4.2.5], $((\tilde{\rho})_N, (\tilde{V})_N)$ is isomorphic to the contragredient of $(\rho_{\overline{N}}, V_{\overline{N}})$, where $\overline{N}$ is the maximal unipotent subgroup of $\overline{B}$. But μ is a weight of $(\rho_{\overline{N}}, V_{\overline{N}})$ if and only if $w^l(\mu)$ is a weight of (ρ_N, V_N). Therefore the weights of S on $((\tilde{\rho})_N, (\tilde{V})_N)$ are of the form $-w^l(\mu)$. However, $\overline{\kappa}$ induces the involution $\mu \mapsto -w^l(\mu)$ on $X_*(T)$. □

CHAPTER 5
FORMULAS FOR CONVOLUTIONS

5.1 Bases and Generators

Let M_c be the space of compactly supported complex-valued functions on $G(F)\backslash G(\mathbf{A})$ that are invariant under right translation by K' (K' is defined in (1.2.1). We endow M_c (and all other functions spaces in this chapter) with the usual L^2-norm. The complex vector space M_c has a basis consisting of the characteristic functions of the K'-orbits on $G(F)\backslash G(\mathbf{A})$. By Theorem 3.1.1, these orbits are indexed by the elements of $\tilde{W}$. Let t_w denote the indicator function of the orbit of $\phi_{t^{-1}}(w)$, for each $w \in \tilde{W}$.

For each degree one valuation v of F, let H_v denote the convolution algebra of I_v-biinvariant measures on $G(F_v)$. The choice of a uniformizing element π_v gives us an isomorphism $\phi_v : H_v \to H$. We have seen, in §4.2, that the algebra H is generated by elements T_i, $I = 0, \dots, r$ and $\mathbf{C}[\Pi]$. Let $T_i^v \in H_v$ be such that $\phi_v(T_i^v) = T_i$, for $i = 0, \dots, r$, and 1_a^v be such that $\phi_v(1_a^v) = 1_a$, for each $a \in \Pi$.

In the remainder of this chapter, we will prove the following formulas for the actions of the generators of H_∞ and H_0 described above in terms of the basis $\{t_w\}_{w\in\tilde{W}}$ of M_c:

$$t_w \cdot T_i^\infty = \begin{cases} (q-1)t_w + qt_{ws_i} \\ \quad \text{if } l(w_0ws_i) > l(w_0w), \\ t_{ws_i} \\ \quad \text{if } l(w_0ws_i) < l(w_0w) \end{cases} \qquad i = 0, \dots, r \tag{5.1.1}$$

$$t_w 1_a^\infty = t_{wa}, a \in \Pi, \tag{5.1.2}$$

$$t_w \cdot T_i^0 = \begin{cases} (q-1)t_w + qt_{\overline{\kappa}(s_i)w} \\ \text{ if } l(s_i\kappa(w_0w)) > l(\kappa(w_0w)), \\ t_{\overline{\kappa}(s_i)w} \\ \text{ if } l(s_i\kappa(w_0w)) < l(\kappa(w_0w)) \end{cases} \quad i = 0, \ldots, r \tag{5.1.3}$$

$$t_w 1_a^0 = t_{\overline{\kappa}(a)^{-1}w}, a \in \Pi \tag{5.1.4}$$

Here κ is the involution defined in Lemma 5.4.1.

5.2 Reformulation

It turns out that the calculations are easiest when, instead of K', we work with

$$\overline{K}' = I_\infty \times \overline{I}_0 \times \prod_{v \neq \infty, 0} G(\mathbf{O}_v),$$

where $\overline{I}_0 = w_0 T_0 w_0^{-1}$. Instead of M_c, we may consider the module $\overline{M}_c$ of compactly supported functions on $G(F)\backslash G(\mathbf{A})$ that are constant on $\overline{K}'$-orbits. The vector space $\overline{M}_c$ is then a module over $H_\infty \otimes \overline{H}_0$, where $\overline{H}_0$ is the convolution algebra of measures on $G(F_0)$ that are biinvariant under translations in $\overline{I}_0$.

We now describe how one may pass between M_c and $\overline{M}_c$. In what follows, we denote by (g_∞, g_0), the element $\phi_{t^{-1}}(g_\infty)\phi_t(g_0)$ of $G(\mathbf{A})$, where $g_\infty, g_0 \in G(F_\bullet)$. Then $\overline{M}_c = M_c * \delta_{(1,w_0)}$ (as a subspace of $L^2(G(F)\backslash G(\mathbf{A}))$, and $\overline{H}_0 = \delta_{w_0} * H_0 * \delta_{w_0}$, where $\delta_{(1,w_0)}$ (resp. δ_{w_0} is the unit delta measure on $G(\mathbf{A})$ at $(1, w_0)$ (resp. on $G(F_0)$ at $\phi_t(w_0)$). Set

$$\theta_w = t_{w_0w} * \delta_{(1,w_0)}, \text{ for all } w \in \tilde{W}, \tag{5.2.1}$$

$$\overline{T}_i^0 = \delta_{w_0} * T_i^0 * \delta_{w_0}, \text{ for all } i = 0, \ldots, r \text{ and} \tag{5.2.2}$$

$$\overline{1}_a^0 = \delta_{w_0} * 1_a^0 * \delta_{w_0} \text{ for all } a \in \Pi. \tag{5.2.3}$$

Then (5.1.1)-(5.1.4) are equivalent to

$$\theta_w * T_i^\infty = \begin{cases} (q-1)\theta_w + q\theta_{ws_i} \\ \text{ if } l(ws_i) > l(w), \\ \theta_{ws_i} \\ \text{ if } l(ws_i) < l(w), \end{cases} \quad i = 0, \ldots, r, \tag{5.2.4}$$

$$\theta_w 1_a^\infty = \theta_{wa}, a \in \Pi, \tag{5.2.5}$$

$$\theta_w * \overline{T}_i^0 = \begin{cases} (q-1)\theta_w + q\theta_{\kappa(s_i)w}, \\ \text{ if } l(s_i\kappa(w)) > l(\kappa(w)) \\ \theta_{\kappa(s_i)w} \\ \text{ if } l(s_i\kappa(w)) < l(\kappa(w)), \end{cases} \quad i = 0, \ldots, r, \tag{5.2.6}$$

$$\theta_w 1_a^0 = \theta_{\overline{\kappa}(a)^{-1}w}, a \in \Pi. \tag{5.2.7}$$

These formulas are equivalent to Theorems 5.4.10 and 5.4.11 put together. The remaining sections of this chapter are devoted to proving these formulas and the asserted equivalence.

5.3 Reduction to local calculations

Fix a $G(\mathbf{A})$-invariant measure on $G(F)\backslash G(\mathbf{A})$ such that $G(F)\overline{K}'$ has unit measure. Consider the subgroup Γ_∞ of $G(F)$ consisting of elements whose image (under the completion map) in $G(F_0)$ lies in $\overline{I}_0$, and whose image in $G(F_v)$ lies in $G(\mathbf{O}_v)$ for all $v \neq \infty, 0$. Fix a $G(F_\infty)$-invariant measure on $\Gamma_\infty\backslash G(F_\infty)$ such that $\Gamma_\infty I_\infty$ has unit measure.

Analogously, define $\overline{\Gamma}_0$ to be the subgroup of $G(F_0)$ consisting of elements whose image (under the completion map) in $G(F_\infty)$ lies in I_∞, and whose image in $G(F_v)$ lies in $G(\mathbf{O}_v)$ for all $v \neq \infty, 0$. Then, we also have

Lemma 5.3.1. *1. The map $\phi_{t-1} : G(F_\infty) \to G(\mathbf{A})$ induces an isometry*

$$\phi_{t-1} : \Gamma_\infty\backslash G(F_\infty)/I_\infty \to G(F)\backslash G(\mathbf{A})\overline{K}'.$$

2. *The map* $\phi_t : G(F_0) \to G(\mathbf{A})$ *induces an isometry*

$$\phi_t : \overline{\Gamma}_0 \backslash G(F_0)/\overline{I}_0 \to G(F)\backslash G(\mathbf{A})\overline{K}'.$$

Proof. It is clear from the definition of Γ_∞ that the map is well defined. It is surjective because every double coset in $G(F)\backslash G(\mathbf{A})\overline{K}'$ has a representative in $G(F_\infty)$, by Theorem 3.1.1. That it is an isometry is evident from our normalization of measures. This proves part (1). The proof of part (2) is similar. □

Define

$$\begin{aligned} \overline{M}_c^\infty &= C_c(\Gamma_\infty\backslash G(F_\infty)/I_\infty), \\ \overline{M}_c^0 &= C_c(\overline{\Gamma}_0\backslash G(F_0)/\overline{I}_0). \end{aligned}$$

The former is an H_∞-module and the latter an $\overline{H}_0$-module.

Proposition 5.3.2. *1. The map* $\phi_{t^{-1}}$ *between double cosets defined in Lemma 5.3.1 induces an isometry of* H_∞*-modules*

$$L_\infty : \overline{M}_c \to \overline{M}_c^\infty.$$

2. *The map* ϕ_0 *between double cosets defined in Lemma 5.3.1 induces an isometry of* $\overline{H}_0$*-modules*

$$L_0 : \overline{M}_c \to \overline{M}_c^0.$$

Proof. By Lemma 5.3.1, it only remains to check that L_∞ (resp. L_0) preserves the H_∞ (resp. $\overline{H}_0$)-module structure. Indeed, given $\mu_\infty \in H_\infty$ and $f \in \overline{M}_c$,

$$\begin{aligned} L_\infty(f\cdot\mu_\infty)(x_\infty) &= (f\cdot\mu_\infty)(\phi_{t^{-1}}(x_\infty)) \\ &= \int_{G(F_\infty)} f(\phi_{t^{-1}}(x_\infty g_\infty^{-1}))d\mu_\infty(g_\infty) \\ &= \int_{G(F_\infty)} L_\infty(f)(x_\infty g_\infty^{-1}))d\mu_\infty(g_\infty) \\ &= (L_\infty(f)\cdot\mu_\infty)(x_\infty), \end{aligned}$$

for each $x_\infty \in G(F_\infty)$. This completes the proof of part (1). The proof of (2) is similar. ◻

5.4 Local Calculations

For each $w \in \tilde{W}$, let $\tau_w^\infty \in \overline{M}_c^\infty$ (resp. $\tau_w^0 \in \overline{M}_c^0$) be the characteristic function of $\Gamma_\infty \phi_{t-1}(w) I_\infty$ (resp. $\overline{\Gamma}_0 \phi_t(w) \overline{I}_0$).

Lemma 5.4.1. *Let $\kappa : \tilde{W} \to \tilde{W}$ be the automorphism $w\pi^\eta \mapsto w\pi^{-\eta}$, for $w \in W$ and $\eta \in X_*(T)$. Then the following diagram commutes:*

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\ \kappa\ } & \tilde{W} \\ & {\scriptstyle \phi_{t-1}} \searrow \quad \swarrow {\scriptstyle \phi_t} & \\ & G(F). & \end{array}$$

Proof. It is clear that the diagram commutes when restricted to either W or $X_*(T)$, from which the proposition follows. ◻

A simple corollary is the following

Lemma 5.4.2. *For each $w \in \tilde{W}$,*

$$\begin{aligned} \tau_w^\infty &= L_\infty(\theta_w), \\ \tau_{\kappa(w^{-1})}^0 &= L_0(\theta_w). \end{aligned}$$

The action of H_∞

For each root $\alpha \in \Phi(G,T)$, fix an morphism $u_\alpha : \mathbf{G}_a \to G$ (defined over $\mathbf{F}_q$), which is an isomorphism onto the root subgroup corresponding to α.

Given a root $\alpha' \in \Phi(G,T)$, we may think of it as a linear functional $X_* \to \mathbf{R}$. In addition, given an integer n, denote by $\alpha' + n$ the affine linear map $\mathbf{x} \mapsto \alpha'(\mathbf{x}) + n$ from X_* to $\mathbf{R}$. The set of affine roots is defined as

$$\check{\Phi}(G,T) = \{\alpha = \alpha' + n : X_* \to \mathbf{R} \mid \alpha' \in \Phi(G,T),\ n \in \mathbf{Z}\}.$$

Let α_0 be the affine root $-\tilde{\alpha}+1$ where $\tilde{\alpha}$ is the highest positive root in $\Phi(G,T)$ with respect to our choice $\{\alpha_1,\ldots,\alpha_r\}$ of simple roots. The affine roots $\{\alpha_0,\ldots,\alpha_r\}$ are called the *simple affine roots*. We say that an affine root α is *positive*, and write $\alpha > 0$, if it can be written as a linear combination of the simple affine roots with all coefficients positive. Then for each affine root α, either $\alpha > 0$ or $-\alpha > 0$.

Given an affine root $\alpha = \alpha' + n$, define U_α to be $u_{\alpha'}(t^{-n}\mathbf{F}_q) \subset G(F_\infty)$.

Lemma 5.4.3. *1. $U_\alpha \subset I_\infty$ if and only if $\alpha > 0$.*

2. $U_\alpha \subset \Gamma_\infty$ if and only if $\alpha < 0$.

Lemma 5.4.4. *For each $i \in \{0,\ldots,r\}$,*

$$\begin{aligned} I_\infty s_i I_\infty &= s_i I_\infty \coprod \left(\coprod_{\xi\in\mathbf{F}_q^\times} u_{\alpha_i}(\xi) s_i I_\infty \right) \\ &= s_i I_\infty \coprod \left(\coprod_{\xi\in\mathbf{F}_q^\times} s_i u_{\alpha_i}(\xi) s_i I_\infty \right). \end{aligned}$$

Proof. Since $[I_\infty : I_\infty \cap s_i I_\infty s_i] = q$, $I_\infty s_i I_\infty$ consists of q right I_∞-cosets. Clearly,

$$s_i I_\infty \coprod \left(\coprod_{\xi\in\mathbf{F}_q^\times} u_{\alpha_i}(\xi) s_i I_\infty \right) \subset I_\infty s_i I_\infty.$$

Moreover, no non-trivial element of U_{α_i} fixes $s_i C_0$. Hence the right cosets appearing above are distinct. Since U_{α_i} fixes the hyperplane $\alpha_i(x) = 0$ in $\mathcal{A}$, each of the chambers $u_{\alpha_i}(\xi)C_0$ shares a face contained in this hyperplane with C_0. Now $s_i u_{\alpha_i}(\xi) s_i$ fixes $s_i C_0$, but not C_0 for $\xi \in \mathbf{F}_q^\times$. Therefore, for non-zero ξ, $s_i u_{\alpha_i}(\xi) s_i C_0$ are the alcoves sharing a face with C_0 which is contained in $\alpha_i(x) = 0$. □

Let dx denote the Haar measure on $G(F_\infty)$ which assigns unit measure to I_∞. This determines an identification of H_∞ with the convolution algebra of functions

$C_c(I_\infty\backslash G(F_\infty)/I_\infty)$. For $\phi \in \overline{M}_c^\infty$, and $f(x)$ in H_∞,

$$(\phi * f)(t) = \int_{G(F_\infty)} \phi(tx^{-1})f(x)dx. \tag{5.4.5}$$

Specifically, taking $f(x)$ to be the characteristic function $1_{I_\infty s_i I_\infty}$, so that $f(x)dx = T_i^\infty$ and $\phi = \tau_w^\infty$ to be $1_{\Gamma w I_\infty}$ (recall that w here represents $\phi_{t^{-1}}(w)$),

$$\begin{aligned}
(\tau_w^\infty * T_i^\infty)(t) &= \int_{G(F_\infty)} \tau_w^\infty(tx^{-1})T_i^\infty(x)dx \\
&= \sum_{gI_\infty \in (I_\infty s_i I_\infty)/I_\infty} \int_{gI_\infty} \tau_w^\infty(tx^{-1})dx \\
&= \tau_w^\infty(ts_i) + \sum_{\xi \in \mathbf{F}_q^\times} \tau_w^\infty(tu_{\alpha_i}(\xi)s_i) && (5.4.6) \\
&= \tau_w^\infty(ts_i) + \sum_{\xi \in \mathbf{F}_q^\times} \tau_w^\infty(ts_iu_{\alpha_i}(\xi)s_i) && (5.4.7)
\end{aligned}$$

The last two steps use Lemma 5.4.4.

Relation to lengths. From the theory of Tits systems (see, for example [6, Chapitre IV]) , we know that

Lemma 5.4.8. *If $w \in \tilde{W}$, then $l(ws_i) = l(w) + 1$ if and only if $w\alpha_i > 0$.*

Lemma 5.4.9. *Let $w \in \tilde{W}$, with $w \in W_a$. Then*

1. *$ws_iu_{\alpha_i}(\xi)s_iw^{-1} \in \Gamma$ for all $\xi \in \mathbf{F}_q^\times$ if and only if $w\alpha_i > 0$, i.e., $l(ws_i) = l(w) + 1$.*

2. *$wu_{\alpha_i}(\xi)w^{-1} \in \Gamma$ for all $\xi \in \mathbf{F}_q^\times$ if and only if $w\alpha_i < 0$, i.e., $l(ws_i) = l(w) - 1$.*

Proof. The lemma follows from lemma 5.4.8, proposition 5.4.3 and the fact that $s_iU_{\alpha_i}s_i = U_{-\alpha_i}$. ☐

Evaluation of $\tau_w^\infty * T_i^\infty$. *Case 1: $l(ws_i) = l(w) + 1$.* Then

$$\tau_w^\infty(ws_iu_{\alpha_i}(\xi)s_i) = \tau_w^\infty(ws_iu_{\alpha_i}(\xi)s_iw^{-1}w) = 1.$$

since $ws_i u_{\alpha_i}(\xi)s_i w^{-1} \in \Gamma$ by lemma 5.4.9, part 1. Substitute $t = ws_i$ in equation (5.4.6) to get:

$$\begin{aligned} \tau_w^\infty * T_i^\infty(ws_i) &= \tau_w^\infty(w) + \sum_{\xi \in \mathbf{F}_q^\times} \tau_w^\infty(ws_i u_{\alpha_i}(\xi)s_i) \\ &= q - 1. \end{aligned}$$

Substitute $t = w$ in equation (5.4.7) to get:

$$\begin{aligned} \tau_w^\infty * T_i^\infty(w) &= \tau_w^\infty(ws_i) + \sum_{\xi \in \mathbf{F}_q^\times} \tau_w^\infty(ws_i u_{\alpha_i}(\xi)s_i) \\ &= q - 1. \end{aligned}$$

Case 2: $l(ws_i) = l(w) - 1$. Then

$$\tau_w^\infty(wu_{\alpha_i}(\xi)s_i) = \tau_w^\infty(wu_{\alpha_i}(\xi)w^{-1}ws_i) = 0 \text{ for } \xi \neq 0.$$

since $wu_{\alpha_i}(\xi)w^{-1} \in \Gamma$ by lemma 5.4.9, part 2. Proceeding as in *Case 1*, we see that $\tau_w^\infty * T_i^\infty = 0$ and $\tau_w^\infty * T_i^\infty(ws_i) = 1$. This proves the first part of the following

Theorem 5.4.10. *For each $w \in \tilde{W}$, and $i = 1, \ldots, r$,*

$$\tau_w^\infty * T_i^\infty = \begin{cases} (q-1)\tau_w^\infty + q\tau_{ws_i}^\infty & \text{if} \quad l(ws_i) = l(w) + 1 \\ \tau_{ws_i}^\infty & \text{if} \quad l(ws_i) = l(w) - 1. \end{cases}$$

and for each $a \in \Pi$,

$$\tau_w^\infty * 1_a^\infty = \tau_{wa}^\infty.$$

The second formula in the statement above follows from the fact that Π normalizes I_∞. Formulas (5.2.4) and (5.2.5) follow from the above theorem, Proposition 5.3.2 and Lemma 5.4.2.

The action of $\overline{H}_0$

Similarly, we may prove

Theorem 5.4.11. *For each $w \in \tilde{W}$, and $i = 1, \ldots, r$,*

$$\tau_w^0 * \overline{T}_i^0 = \begin{cases} (q-1)\tau_w^0 + q\tau_{ws_i}^0 & \text{if} \quad l(ws_i) = l(w) + 1 \\ \tau_{ws_i}^0 & \text{if} \quad l(ws_i) = l(w) - 1. \end{cases}$$

and for each $a \in \Pi$,

$$\tau_w^\infty * 1_a^0 = \tau_{w\overline{a}}^0.$$

Formulas (5.2.6) and (5.2.7) follow from Theorems 5.4.10 and 5.4.11, Proposition 5.3.2 and Lemma 5.4.2.

CHAPTER 6
SPECTRAL DECOMPOSITIONS

6.1 The Local Module

Let H be as in §4.2. Consider the right $H \otimes H$ module (ν, N), where

$$N = L^2(G(F_\bullet)\backslash[G(F_\bullet) \times G(F_\bullet)]/(I_\bullet \times I_\bullet)),$$

and the action is given by

$$n\nu(\mu_1, \mu_2)(x_1, x_2) = \int_{G(F_\bullet)\backslash[G(F_\bullet)\times G(F_\bullet)]} n(x_1 g_1^{-1}, x_2 g_2^{-1}) d(\mu_1 \otimes \mu_2)(g_1, g_2),$$

for all $n \in N$, $\mu_1, \mu_2 \in H$, and $(x_1, x_2) \in G(F_\bullet) \times G(F_\bullet)$.

Consider the $H \otimes H$ module $(\tilde{\nu}, H)$, with action

$$h\tilde{\nu}(\mu_1, \mu_2) = \mu_2^{\mathrm{op}} * h * \mu_1.$$

The isometry

$$G(F_\bullet)\backslash[G(F_\bullet) \times G(F_\bullet)]/(I_\bullet \times I_\bullet) \to I_\bullet\backslash G(F_\bullet)/I_\bullet$$

induced by $(g_1, g_2) \mapsto g_2^{-1} g_1$ induces and isometry of $H \otimes H$-modules (ν, N) and $(\tilde{\nu}, H)$. For each $w \in \tilde{W}$, let τ_w denote the characteristic function of the double coset $G(F_\bullet)(w_0 w, 1)(I_\bullet \times I_\bullet)$. Then one may use [15, §3] and the above isomorphism to compute the action of $H \otimes H$ in terms of the basis elements $\{\tau_w\}_{w\in\tilde{W}}$:

$$\tau_w\nu(T_i,1) = \begin{cases}\tau_{ws_i} \text{ if } l(w_0ws_i) > l(w_0w)\\ q\tau_{ws_i} + (q-1)\tau_w \text{ if } l(w_0ws_i) < l(w_0w)\end{cases}, \qquad (6.1.1)$$

$$\tau_w\nu(1_a,q) + \tau_{wa}, \qquad (6.1.2)$$

$$\tau_w\nu(1,T_i) = \begin{cases}\tau_{s_iw} \text{ if } l(s_iw_0w) > l(w_0w)\\ q\tau_{s_iw} + (q-1)\tau_w \text{ if } l(s_iw_0w) < l(w_0w)\end{cases}, \qquad (6.1.3)$$

$$\tau_w\nu(1,1_b) = \tau_{b^{-1}w}. \qquad (6.1.4)$$

6.2 The Discrete Spectrum of the Local Module

The following theorem is a form of the Peter-Weyl Theorem. Since this form is not standard, we sketch a proof. Let (ν, N_d) denote the discrete part of (ν, N) (i.e., the submodule generated by eigenvectors for the center of $H \otimes H$).

Theorem 6.2.1. *The map*

$$\Phi : \bigoplus_{(\rho,V)\in\hat{H}} V \otimes \tilde{V} \to N_d$$

defined by setting

$$\Phi(\mathbf{v}\otimes\tilde{\mathbf{v}})(g_1,g_2) = \langle \mathbf{v}\rho(1_{I_\bullet g_1^{-1}I_\bullet}dx), \tilde{\mathbf{v}}\rho(1_{I_\bullet g_2^{-1}I_\bullet}dx)\rangle$$

for $(\rho,V)\in\hat{H}$, $\mathbf{v}\in V$ and $\tilde{\mathbf{v}}\in\tilde{V}$ is an isomorphism.

Proof. One may check, from the definition that $\Phi(\mathbf{v}\otimes\tilde{\mathbf{v}})$, is left invariant under the left diagonal action of $G(F_\bullet)$ and the right action of $I_\bullet \times I_\bullet$). It is square integrable because $\Phi(\mathbf{v}\otimes\tilde{\mathbf{v}})(g_2^{-1}g_1,1)$ is a matrix coefficient of a square-integrable representation of H.

Let f be an eigenvector for the center of $H \otimes H$ in (ν,N). In order to prove the surjectivity of Φ, it suffices to show that f lies in the image of Φ. Consider the module $V_f = f\nu(H\otimes\{1\})$.

Lemma 6.2.2. *The vector space V_f is finite dimensional.*

Proof. It follows from [4] that H is a finitely generated module over its center $Z(H)$. Let $h_1, \dots h_k$ be generators of H over $Z(H)$. Given $h \in H$, write $h = \sum a_i h_i$, with $a_i \in Z(H)$. Then $f\nu(h \otimes 1)$ is then a linear combination of the $f\nu(a_i h_i, 1)$'s. However, each $f\nu(a_i h_i, 1)$ is a scalar multiple of $f\nu(h_1, 1)$ since the center of $H \otimes H$ acts by scalars on f. This means that V_f is generated by the $f\nu(h_i, 1)$'s as a complex vector space. $\square$

The Hecke algebra H admits complex antilinear antiinvolution $h \mapsto h^*$, where $h^*(x) = \overline{h(x^{-1})}$. The complex vector space V_f inherits a Hermitian inner product $(\cdot, \cdot)$ from the usual Hermitian inner product on N. For f_1 and f_2 in V_f and $h \in H$,

$$(f_1\nu(h,1), f_2) = (f_1, f_2\nu(h^*, 1)),$$

It follows that the orthogonal complement of an H-submodule of V_f is also an H-module. Therefore, V_f is a semisimple H module. Write V_f as a finite direct sum

$$V_f = V_1^{\oplus m_1} \oplus \dots \oplus V_n^{\oplus m_n},$$

where $V_1, \dots V_n$ are pairwise non-isomorphic irreducible H-modules. Let

$$\overline{V}_f = V_1 \oplus \dots \oplus V_n.$$

The natural maps $V_i^{\oplus m_i} \to V_i$ induced by summation induce a map $p : V_f \to \overline{V}_f$. Similarly, we have a map $\tilde{p} : \tilde{V}_f \to \tilde{\overline{V}}_f$ on contragredients. These maps have the property that

$$\langle \mathbf{v}, \tilde{\mathbf{v}} \rangle = \langle p(\mathbf{v}), \tilde{p}(\tilde{\mathbf{v}}) \rangle \text{ for all } \mathbf{v} \in V_f \text{ and } \tilde{\mathbf{v}} \in \tilde{V}_f.$$

Let $\tilde{\mathbf{v}}$ be a vector in $\tilde{V}_f$ defined by requiring that

$$\langle \tilde{\mathbf{v}}, \mathbf{v} \rangle = \mathbf{v}(1) \text{ for all } \mathbf{v} \in V_f.$$

A vector in V_f is viewed as a complex-valued function in the above definition. Then

$$\langle \tilde{\mathbf{v}}\nu(1_{g_2^{-1}}, 1), p(f)\nu(1_{g_1^{-1}}, 1)\rangle = f(g_1, g_2),$$

for any $g_1, g_2 \in G(F_\bullet)$. Therefore, f lies in the image of Φ.

In order to see that Φ is injective, note that the $V \otimes \tilde{V}$ are irreducible and pairwise non-isomorphic modules. The kernel of Φ, being an $H \otimes H$-submodule, must consist of a direct sum of a subset of these constituent modules. But no non-trivial representation could lie in the kernel. Therefore, the kernel must be trivial. □

6.3 Comparison of Modules

We compare the $H \times H$ module (ν, N) with the $H_\infty \otimes H_0$-module (r, M). Both M and N have bases indexed by elements of the extended affine Weyl group. We use this bijection to construct an isomorphism of vector spaces. In the first part of this section we discuss how this isomorphism relates the module structures. In the second part we verify that it is in fact, an isometry.

Algebra

Define a vector space isomorphism $J : N \to M$ by requiring

$$J : \tau_w \mapsto (-q)^{l(w_0 w)} t_w, \text{ for each } w \in \tilde{W}.$$

Then, using the formulas (5.1.1)-(5.1.4), we may verify that

$$J(\tau_w \nu(T_i \otimes T_j)) = J(\tau_w) r(\phi_\infty^{-1} \circ I(T_i) \otimes \phi_0^{-1} \circ I \circ \overline{\kappa}(T_j)) \tag{6.3.1}$$

for all $w \in \tilde{W},\ 0 \leq i, j \leq r$, and

$$J(\tau_w \nu(1_a \otimes 1_b)) = J(\tau_w) r(\phi_\infty^{-1} \circ I(1_a) \otimes \phi_0^{-1} \circ I \circ \overline{\kappa}(1_b)) \tag{6.3.2}$$

for all $w \in \tilde{W},\ a, b \in \Pi$. It follows that for every $h_1, h_2 \in H$ and $n \in N$,

$$J(n\nu(h_1 \otimes h_2)) = J(n)r(\phi_\infty^{-1} \circ I(h_1) \otimes \phi_0^{-1} \circ I \circ \overline{\kappa}(h_2)). \qquad (6.3.3)$$

Measures

First, we compute the L^2-norm of $\tau_w \in N$, which is the same as the measure of the double coset $G(F_\bullet)(w_0 w, 1)(I_\bullet \times I_\bullet)$. If we normalize the measure so that $I_\bullet \times I_\bullet$ has unit measure, the measure of the double coset in question is equal to the number of right $I_\bullet \times I_\bullet$-cosets that occur in this double coset, i.e., the index $[I_\bullet : w_0 w I_\bullet w^{-1} w_0 \cap I_\bullet]$.

Lemma 6.3.4. *Let α be an affine root. Then $U_\alpha \subset w I_\bullet w^{-1}$ if and only if $w^{-1}\alpha > 0$.*

Proof. $U_\alpha \subset w I_\bullet w^{-1}$ if and only if $w^{-1} U_\alpha w \subset I\bullet$, or, $U_{w^{-1}\alpha} \subset I_\bullet$. By Lemma 5.4.3, this is equivalent to $w^{-1}\alpha > 0$. ☐

Now, as a set,

$$I\bullet = A(F_\bullet) \times \prod_{\alpha > 0} U_\alpha$$

and by Lemma 6.3.4,

$$I \bullet \cap w I_\bullet w^{-1} = A(F_\bullet) \times \prod_{\alpha > 0,\ w^{-1}\alpha > 0} U_\alpha.$$

Consequently, the index

$$[I_\bullet : I_\bullet \cap w I_\bullet w^{-1}] = q^{\#\{\alpha < 0 \text{ such that } w^{-1}\alpha > 0\}} = q^{l(w)}.$$

Therefore,

$$\|\tau_w\| = q^{l(w_0 w)} \qquad (6.3.5)$$

Now let us calculate the L^2-norm of t_w. By Proposition 5.3.2 and Lemma 5.4.2, this is the same as the norm of $\tau^\infty_{w_0 w}$. This is the measure of the double coset $\Gamma_\infty w_0 w I_\infty$. The group Γ acts transitively on the set of right I_∞ cosets in $\Gamma_\infty w_0 w I_\infty$ with finite stabilizers. Therefore, if we normalize our measure so that ΓI_∞ has unit measure, then the double coset in question has measure inverse to the cardinality of the stabilizers. Again, as a set,

$$\Gamma = A(F \cap \prod_{v \neq \infty} \mathbf{O}_v) \times \prod_{\alpha < 0} U_\alpha.$$

Therefore, by Lemma 6.3.4,

$$\#\{\gamma \cap wI_\infty w^{-1}\} = q^{\#\{\alpha<0 \text{ such that } w^{-1}\alpha>0\}} = q^{l(w)}.$$

Therefore,

$$\|t_w\| = \frac{1}{q^{l(w_0 w)}}. \tag{6.3.6}$$

Proposition 6.3.7. *The linear map J is an isometry of Hilbert spaces.*

Proof. This is evident from (6.3.5) and (6.3.6):

$$\begin{aligned} \|J(\tau_w)\|^2 &= \|(-q)^{l(w_0 w)} t_w\|^2 \\ &= q^{2l(w_0 w)} \|t_w\| \\ &= q^{2l(w_0 w)} \times q^{-l(w_0 w)} \\ &= \|\tau_w\|. \end{aligned}$$

Since the τ_w's and t_w's form orthogonal bases for N and M respectively, it follows that J is an isometry. ∎

Theorem 1.2.2 now follows from Theorem 6.2.1, equation (6.3.3) and Proposition 6.3.7.

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