Permutation Reps.

G : group

A <u>G-space</u> is a set X + an action

$$a : G \longrightarrow \underset{\substack{\uparrow \\ \text{gp. of bijections } X \to X}}{\mathrm{Aut}(X)} \quad \text{homomorphism}$$

<u>Permutation rep.</u> : $(\rho_X, K[X])$

$K[X]$ = K-valued fns. on X

$$(\rho_X(g)\, f)(x) = f(\underbrace{g^{-1}\cdot x}_{a(g^{-1}, x)}).$$

$$\rho_X(g)\, 1_x = 1_{g\cdot x}$$

<u>Partition spaces</u>: $\underline{n} := \{1, \dots, n\}$

Ordered partition: $\underline{n} = S_1 \sqcup S_2 \sqcup \dots \sqcup S_\ell$

If $\lambda_i = |S_i|$, then $n = \lambda_1 + \dots + \lambda_\ell$.

$\boxed{\lambda \vDash n}$ $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is called a <u>composition</u> of n

$\boxed{\lambda \vdash n}$ If $\lambda_1 \geq \dots \geq \lambda_\ell$, λ is called a <u>partition</u> of n

$S_1 \sqcup \dots \sqcup S_\ell$ is called a partition of shape λ.

X_λ = Set of partitions of shape λ.

$$|X_\lambda| = \frac{n!}{\lambda_1! \dots \lambda_\ell!} \quad \text{(multinomial coeff.)}$$

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X_λ is a G-space when $G = S_n$.

$g(S_1, \dots, S_\ell) = (g(S_1), \dots, g(S_\ell))$.

$g(S_i) = \{g \cdot x \mid x \in S_i\}$

Integral operators: $k \in K[X \times Y]$

$T_k : K[Y] \longrightarrow K[X]$

$$T_k f(x) = \sum_{y \in Y} k(x, y)\, f(y)$$

"integral operator with kernel k."

$k \longmapsto T_k$

$K[X, Y] \to \mathrm{Hom}_K(K[Y], K[X])$

is an isomorphism (injective + right dim.)

Suppose X and Y are G-sets.

Then $K[X]$ & $K[Y]$ are reps. of G.

Qn: For which $k \in K[X, Y]$ is $T_k \in \mathrm{Hom}_G(K[Y], K[X])$?

i.e., $T_k \circ \rho_Y(g) = \rho_X(g) \circ T_k \quad \forall\, g \in G$, i.e.,

$$\boxed{\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g) = T_k} \quad (*)$$

$$\left[\rho_X(g)^{-1} \circ T_k \circ \rho_Y(g)\ f\right](x)$$
$$= \left[T_k \circ \rho_Y(g) f\right](g\cdot x)$$
$$= \sum_{y\in Y} k(g\cdot x, y)\left(\rho_Y(g)\ f\right)(y)$$
$$= \sum_{y\in Y} k(g\cdot x, y)\ f(g^{-1}\cdot y)$$
$$= \sum_{y\in Y} k(g\cdot x, g\cdot y)\ f(y)$$

$\therefore$ $(*)$ holds iff

$$\boxed{k(g\cdot x, g\cdot y) = k(x,y)} \quad \forall\ x\in X, y\in Y, g\in G$$

<u>Thm</u>: $\dim \mathrm{Hom}_G(K[Y], K[X]) = \left| G\backslash(X\times Y)\right|$

meaning of $G\backslash(X\times Y)$: G acts on $X\times Y$ by $g\cdot(x,y) = (g\cdot x, g\cdot y)$.

$G\backslash(X\times Y)$ denotes the set of orbits for this action of G on $X\times Y$, and is called the set of <u>relative positions</u> in $X\times Y$.

<u>Example</u>: $\mathbb{R}^n \circlearrowleft \mathbb{R}^n$, $\quad \mathbb{R}^n\backslash \mathbb{R}^n\times\mathbb{R}^n \cong \mathbb{R}^n$, $\quad (\vec{x},\vec{y}) \mapsto \vec{y}-\vec{x}$

(Classical ~~definition~~ idea of rel. position of vectors)

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Example: $\overset{\text{"}X_k\text{"}}{X_{(k,n-k)}} = \{S \subset \underline{n} \mid |S| = k\} \circlearrowleft S_n$

$$S_n \backslash X_k \times X_\ell \xrightarrow{\cong} [\max\{0, \overset{k+\ell-n}{\cancel{n-k-\ell}}\}, \min\{k,\ell\}]$$

$$(S,T) \longmapsto |S \cap T|$$

integer interval

is ~~an~~ a bijection.

$$\cancel{\dim}\ |S_n \backslash X_k \times X_\ell| = \min\{k,\ell\} - \max\{0, \overset{k+\ell-n}{\cancel{n-k-\ell}}\} + 1.$$

Decomposing ~~Grassmannian~~ subset permutation reps:

Observation: $X_k \cong X_{n-k}$ (as an S_n-space)

$$S \mapsto \underline{n} - S$$

$\therefore K[X_k] \cong K[X_{n-k}]$.

So we may restrict attention to $k \le \frac{n}{2}$.

For $k,\ell \le \frac{n}{2}$, $\dim \mathrm{Hom}_{S_n}(K[X_k], K[X_\ell]) = \min\{k,\ell\}+1$

$\ell \backslash k$	0	1	2	3	…
0	1	1	1	1	…
1	1	2	2	2	…
2	1	2	3	3	…
3	1	2	3	4	…
⋮	⋮	⋮	⋮	⋮	⋱

Assume that the $K[X_i]$ are all completely reducible. (e.g. if char $K >> 0$), K alg. closed

Recall: $V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$ $\quad V_1, \dots, V_k$ distinct simples

$W = V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k}$

then $\dim_K \operatorname{Hom}_G(V, W) = \sum m_i n_i$.

$\operatorname{End}_{S_n} K[X_0]$ is 1-dim $\Rightarrow K[X_0]$ is simple.

Define $V_0 := K[X_0]$.

$\dim \operatorname{Hom}_{S_n}(K[X_0], K[X_k]) = 1 \Rightarrow V_0$ occurs exactly once in $K[X_k]$ $\forall\, k \geq 1$.

$\therefore \quad K[X_k] \cong V_0 \oplus \underbrace{K[X_k]_0}_{\text{some rep. of } S_n \text{ where } V_0 \text{ does not occur}}$

$\dim \operatorname{Hom}_{S_n}(K[X_k]_0, K[X_\ell]_0)$

$= \dim \operatorname{Hom}_{S_n}(K[X_k], K[X_\ell]) - 1$.

So for $\dim \operatorname{Hom}_{S_n}(K[X_k]_0, K[X_\ell]_0)$ we get

$\ell \backslash k$	1	2	3	
1	1	1	1	$\cdots$
2	1	2	2	$\cdots$
3	1	2	3	$\ddots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

So $K[X_1]_0$ is simple.

$V_1 := K[X_1]_0$.

Continuing in this way, we construct a family of simple representations

$$V_0, V_1, V_2, \dots \text{ of } S_n$$

such that $K[X_k] = V_0 \oplus \cdots \oplus V_k$ for all $k \leq \frac{n}{2}$.

$$\dim V_k = \binom{n}{k} - \binom{n}{k-1} \quad \forall\, k.$$

Example: For $n = 3$, get V_0 - 1 dim
V_1 - 2 dim.

Recall: $tr(\rho_X(g); K[X]) = \#\{x \in X \mid g.x = x\}$.

	$tr(\rho_{X_1}(g); K[X_1])$	$tr(\rho_1(g); V_1)$
1	3	2
(12)	1	0
(123)	0	-1

In principle, a similar method can be used to calculate the chacter of V_k for each $k \leq \frac{n}{2}$.

More generally, suppose $\lambda, \mu \models n$ (compositions)

$\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\lambda_1 + \dots + \lambda_\ell = n$

$\mu = (\mu_1, \dots, \mu_m)$, $\mu_1 + \dots + \mu_m = n$

Take $S = S_1 \amalg \dots \amalg S_\ell \in X_\lambda$ $\quad S' = S_1' \amalg \dots \amalg S_m' \in X_\mu$

What is $|S_n \backslash X_\lambda \times X_\mu|$?

Let $r_{ij} = |S_i \cap S'_j|$

$$r := \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ r_{\ell 1} & \cdots & r_{\ell m} \end{pmatrix} \begin{array}{l} \text{row sums} \\ \to \lambda_1 \\ \vdots \\ \to \lambda_\ell \end{array} = r(S, S')$$

$$\begin{array}{lccc} \text{col. sums:} & \mu_1 & \cdots & \mu_m \end{array}$$

Clearly: $(S, S') \longmapsto r(S, S')$

$$X_\lambda \times X_\mu \longrightarrow M_{\lambda\mu}$$

where $M_{\lambda\mu}$ is the set of matrices with non-neg. integer entries, row sums $\lambda_1, \ldots, \lambda_\ell$, & col. sums $\mu_1, \ldots, \mu_m$ descends to a fn

$$S_n \backslash X_\lambda \times X_\mu \xrightarrow{\bar{r}} M_{\lambda\mu}.$$

Thm: $\bar{r}$ is a bijection

Injectivity: Suppose $r(S, S') = r(T, T')$

Need to construct $\sigma : \underline{n} \to \underline{n}$ such that $\sigma \cdot S = T$ and $\sigma \cdot S' = T'$

Example: The subsets $S_k \cap S'_\ell$ and $T_k \cap T'_\ell$ have the same cardinalities for all (k, ℓ), and furthermore, partition $\underline{n}$. $\therefore$ any permutation $\sigma : \underline{n} \to \underline{n}$ which takes each $S_k \cap S'_\ell$ to $T_k \cap T'_\ell$ will do.

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~~Conversely~~, if $r \in M_{\lambda\mu}$, you prepare the subsets $\emptyset S_k \cap S'_\ell \emptyset$ as a partition of n; explicitly, start with a partition

$$\underline{n} = \coprod_k \coprod_\ell A_{k\ell}$$

where $|A_{k\ell}| = r_{k\ell}$.

Define
$$S_k = \coprod_\ell A_{k\ell}$$
$$S'_\ell = \coprod_k A_{k\ell}$$

Then $r = r(S, S')$.

<u>Theorem</u>: $\dim \mathrm{Hom}_{S_n}(K[X_\lambda], K[X_\mu]) = M_{\lambda\mu}$.

<u>Note</u>: If λ' is the composition obtained by permuting the parts of λ, then $|M_{\lambda'\mu}| = |M_{\lambda\mu}|$.

$$\therefore \dim \mathrm{Hom}_{S_n}(K[X_\lambda], K[X_{\lambda'}]) = \dim \mathrm{Hom}_{S_n}(K[X_\lambda], K[X_\lambda])$$
$$\|$$
$$\dim \mathrm{Hom}_{S_n}(K[X_{\lambda'}], K[X_{\lambda'}])$$

$\Rightarrow K[X_\lambda] \cong K[X_{\lambda'}]$.

(for completely red. reps. ~~if dim Hom~~$_G$(V, W) $= \dim \mathrm{Hom}_G(V, V) = \dim \mathrm{Hom}_G(W, W)$, then $V \cong W$)

For $n=3$, $\dim \mathrm{Hom}_{S_n}(K[X_\lambda], K[X_\mu])$ is given by:

$\mu \backslash \lambda$	3	2+1	1+1+1
3	1	1	1
2+1	1	2	3
1+1+1	1	3	6

$\dim \mathrm{End}_{S_3} K[X_{(3)}]$ is simple

$$V_{(3)} := K[X_{(3)}]$$

$V_{(3)}$ occurs once in $K[X_{(2,1)}]$ & $K[X_{(1,1,1)}]$

Let $K[X_\lambda] = V_{(3)} \oplus K[X_\lambda]_{(3)}$.

$$\dim_K \mathrm{Hom}_{S_3}(K[X_\lambda]_{(3)}, K[X_\mu]_{(3)})$$
$$= \dim_K \mathrm{Hom}_{S_3}(K[X_\lambda], K[X_\mu]) - 1$$

$\mu \backslash \lambda$	(2, 1)	(1, 1, 1)
(2, 1)	1	2
(1, 1, 1)	2	5

$V_{(2,1)} := K[X_{(2,1)}]_{(3)}$ is simple.

occurs 2 times in $K[X]_{(1,1,1)}$.

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$$\therefore K[X]_{(1,1,1)} = V_{(3)} \oplus V_{(2,1)}^{\oplus 2} \oplus K[X_{(1,1,1)}]_{(2,1)}$$

$$\dim \mathrm{End}_{S_3}\left(K[X_{(1,1,1)}]_{(2,1)}\right)$$

$$= \dim \mathrm{End}_{S_3}\left(K[X_{(1,1,1)}]_{(3)}\right) - 2^2$$

$$= 1$$

$\therefore$ $V_{(1,1,1)} := K[X_{(1,1,1)}]_{(2,1)}$ is simple.

<u>Note:</u> As an S_3-set, $X_{(1,1,1)} \cong S_3$

$\{1\} \amalg \{2\} \amalg \{3\} \to id$

So $K[S_3] = V_{(1)} \oplus V_{(2,1)}^{\oplus 2} \oplus V_{(1,1,1)}$

is the decomposition of the regular representation of S_3.

$\therefore$ S_3 has 3 simple reps., $V_{(1)}$, $V_{(2,1)}$ & $V_{(1,1,1)}$ of dimension 1, 2, and 1 respectively.

The analysis of partition representations can be used to calculate the character table.