Lecture II

Definition: (Semisimple algebra) (R-fin. dim alg)

R is semisimple if every finite-dim R-module is a sum of simple modules

Maschke's thm: $K[G]$ is semisimple if $\operatorname{char} K \nmid |G|$

Suppose $V = V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r}$

$W = V_1^{\oplus n_1} \oplus \cdots \oplus V_r^{\oplus n_r}$

$(m_i, n_i \geq 0)$, $V_1, \dots, V_r$ pairwise non-isomorphic simple R-modules.

$T \in \operatorname{Hom}_R(V, W)$, then $T = \bigoplus_{k=1}^{r} T_k$

where $T_k : V_k^{\oplus n_k} \to V_k^{\oplus m_k}$ is an R-module hom.

T_k can be expressed as an $n_k \times n_k$ matrix

$T_k = (T_{ij})_{m_k \times n_k}$ $T_{ij} \in$ ~~Hom_K~~ $\operatorname{End}_R V_k$

Write $v_k \in V_k^{\oplus n_k}$ as $\begin{pmatrix} v_1 \\ \vdots \\ v_{n_k} \end{pmatrix}$

$T_k(v_k) = \begin{pmatrix} T(v_k)_1 \\ \vdots \\ T(v_k)_{m_k} \end{pmatrix}$. Then T_k can be expressed as an $m_k \times n_k$ matrix:

$$\begin{pmatrix} T(x)_1 \\ \vdots \\ T(x)_{m_k} \end{pmatrix} = \begin{pmatrix} T_{11} & \cdots & T_{1n_k} \\ \vdots & \ddots & \vdots \\ T_{m_k 1} & \cdots & T_{m_k n_k} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_{n_k} \end{pmatrix}$$

where $T_{ij} \in \mathrm{End}_K V_k$ $\cancel{= K\,\mathrm{id}_{V_k} \text{ if}}$

K alg. closed $\Rightarrow$ T_{ij} = scalar $\cdot\, \mathrm{id}_{V_k}$, so can think of T_{ij} as a scalar $\in K$.

$$\mathrm{Hom}_R(V, W) = \bigoplus_{i=1}^{k} M_{m_k \times n_k}(K)$$

<u>Spl. case</u>: $\mathrm{End}_R V \cong \bigoplus_{i=1}^{r} M_{m_i}(K)$

iso. of algebras!

the alg. stru. here is componentwise addition & mult.

$$\boxed{\dim \mathrm{End}_R V = \sum_{i=1}^{r} m_i^2} \quad ①$$

<u>Center</u> (of an algebra): $z \in R$

$$Z(R) = \{ z \in R \mid zr = rz \ \ \forall r \in R \}$$

<u>Example</u>: $Z(M_n(K)) = K I_n$

identity matrix

$$\boxed{r = \dim Z(\mathrm{End}_R V)} \quad ②$$

$$\boxed{\dim \mathrm{Hom}_R(V, W) = \sum_{i=1}^{r} m_i n_i} \quad ③$$

Decomposing the regular R-module

Thm: If R is a finite semisimple then every simple R-module is a submodule of (L, R).

Pf: V simple, $v \in V$, $v \neq 0$.

Define $\varphi_v : R \to V$ by $\varphi_v(r) = rv \ \forall \ r \in R$.

Then $\varphi_v \in Hom_R(R, V)$ & $\varphi_v \neq 0$.

By (3), V is a submodule of R.

Let $V_1, \dots, V_k$ be pairwise non-iso simples such that

$$R = V_1^{\oplus m_1} \oplus \dots \oplus V_k^{\oplus m_k}$$

Then $End_R R = \bigoplus_{i=1}^{k} M_{m_i}(K)$

Define $\Psi : R \to End_R R$ by

$$r \longmapsto \psi_r$$

where $\psi_r(s) = sr \ \forall \ s \in R$.

If R has a multiplicative unit, r can be recovered from ψ_r by $r = \psi_r(1)$.

$\therefore \Psi$ is a vector space iso.

What happens to the algebra structures?

$$\psi_r \circ \psi_{r'}(s) = \psi_{r'}(s) r = (sr')r = \psi_{r'r}(s)$$

$\rightarrow$ order of mult. gets reversed.

<u>Opposite of an algebra</u>: $R^{opp} = R$ as a K-vector space but $r \cdot_{opp} s = s \cdot r$.

$A \mapsto A^T$ is an iso $M_n(K) \to M_n(K)^{opp}$.

~~Theorem~~ <u>Wedderburn Decomposition</u>: If K is an alg. closed field, every semisimple K-algebra is of the form $\bigoplus_{i=1}^{k} M_{m_i}(K)$ ($k = \dim Z(R)$).

Let $V \subset K^n$ linear subspace

$$M_V := \{ A \in M_n(K) \mid \overset{R}{\cancel{C}}(A) \subset V \}$$

row
column space

- Then the M_V's are the invariant subspaces of the regular $M_n(K)$-module.
- M_V is simple $\iff$ V is one dimensional.
- $M_V \cong M_V \iff$ ~~dim V =~~ $\dim V = \dim W$.

Thus if $K^n = V_1 \oplus \cdots \oplus V_n$ is a decomposition into a direct sum of one-dim subspaces,

$$M_n(K) = M_{V_1} \oplus \cdots \oplus M_{V_n}$$
$$\cong M_{V_1}^{\oplus n}$$

is a decomposition of ~~M_n~~ the reg. rep. of $M_n(K)$ into a sum of simples.

$M_{V_1} \cong$ ~~K^n~~ matrices whose ~~col.~~ row space is $(* \; 0 \ldots 0)$
$\cong K^n$ (col. vector). ($M_n(K) \cong \mathrm{End}_K K^n$)

Suppose R, S are unital algebras, $(\tilde{\rho}, M)$ an $R\oplus S$ module. Then $M = M_R \oplus M_S$, where

$M_R = \tilde{\rho}(1_R) M$ and $M_S = \tilde{\rho}(1_S) M$

are invariant subspaces. (S acts by 0 on M_R)

Conversely, given $(\tilde{\rho}_R, M_R)$ & $(\tilde{\rho}_S, M_S)$, R & S modules resp., $M_R \oplus M_S$ can be ~~written~~ is an R module under $\tilde{\rho}(r\oplus s)(m_R \oplus m_S) = \tilde{\rho}(r)m_r \oplus \tilde{\rho}(s)m_s$

Thus: $R\oplus S$ - modules $=$ (R-modules) $\times$ (S-modules)

$\{$Simple $(R\oplus S)$ - modules$\}$ $=$ $\{$simple R-modules$\}$ $\amalg$ $\{$simple S-modules$\}$.

Theorem: Let $R = \bigoplus_{i=1}^{k} M_{m_i}(K)$

~~Let~~ Write $r \in R$ as $r = r_1 + \dots + r_k$, $r_i \in M_{m_i}(K)$.

Let $V_i = K^{m_i}$

$$\tilde{\rho}_i(r) = r_i v_i \quad \forall \; v_i \in V_i$$

Then V_i is a simple R-module and the decomposition of the regular R-module is

$$R = \bigoplus_{i=1}^{k} V_i^{\oplus m_i}$$

Cor: R - semisimple K-alg., then there is an iso of algebras

$$R \cong \bigoplus_{i=1}^{k} \mathrm{End}_K V_i \qquad (*)$$

where $V_1, \dots, V_k$ are the simple R-modules.

<u>Have</u>: $1 = \varepsilon_1 + \cdots + \varepsilon_k$

where $\varepsilon_i = \mathrm{id}_{V_i}$ under the isomorphism (*).

<u>Idempotent element</u>: $\varepsilon \in R$ such that $\varepsilon = \varepsilon^2$

<u>Central element</u>: $z \in R \ni zr = rz \ \forall \ r \in R$.

$Z(R) = \{z \in R \mid z \text{ is central}\}$ is a subalg. of R.

<u>Primitive central idempotent</u>: $\varepsilon \in R$ such that

1) ε is central and idempotent

2) If $\varepsilon = \varepsilon' + \varepsilon''$, where ε' and ε'' are central idempotents, then either $\varepsilon' = 0$ or $\varepsilon'' = 0$.

<u>Note</u>: In $\bigoplus_{i=1}^{k} \mathrm{End}_K V_i$, the only primitive central idempotents are $\varepsilon_1, \ldots, \varepsilon_k$.

$\mathrm{End}_K V_i$, as a 2-sided ideal of $\overset{R:}{\bigoplus} \mathrm{End}_K V_i$ can be recovered as $R\varepsilon_i R$.

<u>Conclusion</u>: The decomposition (*) of a semisimple algebra into a sum of matrix algebras is unique.

~~Since~~ When $R = K[G]$, with $\mathrm{char}\, K \nmid |G|$, each ε_i can be viewed as a function

$$\varepsilon_i : G \longrightarrow K.$$

The rest of this lecture will be devoted to these fns.

<u>Tensor product</u>: V, W : finite dim. vector spaces.

$v_1, \dots, v_n$: basis of V

$w_1, \dots, w_m$: basis of W.

$V \otimes W$ is the vector space with basis $v_i \otimes w_j$,
$i = 1, \dots, n$, $w = 1, \dots, m$.

<u>Coordinate-free defn.</u> : There exists a bilinear map $V \times W \xrightarrow{B} V \otimes W$ such that for every vector space U and every bilinear map $D : V \times W \to U$, there exists a unique linear map $\tilde{D} : V \otimes W \to U$ such that

$$D = \tilde{D} \circ B.$$

(see Birkhoff & MacLane, Survey of Modern Alg. Section 8.10)

For example : $S \in \mathrm{End}_K V$, $T \in \mathrm{End}_K W$

Define $D : V \times W \to V \otimes W$ by

$$D(v, w) = B(S(v), T(w))$$

Get $\tilde{D} : V \otimes W \to V \otimes W$

$$S \otimes T := \tilde{D}. \quad \text{(Kronecker product)}$$

Let $V' = \mathrm{Hom}_K(V, K)$

For $\phi \in \mathrm{End}_K V$, define $\phi\, T' \in \mathrm{End}_K V'$ by
$T'\xi(v) = \xi(T(v))$.

$$(S\circ T)' = T'\circ S'.$$

<u>Contragredient rep.</u> of (ρ, V) is (ρ', V'), where $V' = \mathrm{Hom}_K(V, K)$ and

$$\rho'(g) = \rho(g^{-1})'.$$

Let (ρ, V) be a rep. of G.
Consider ~~three~~ <u>reps.</u> of $G\times G$:

1) $K[G]$ $\qquad [H(g,h)f](x) = f(g^{-1}xh)$

2) $\mathrm{End}_K V$ $\qquad E_\rho(g,h)T = \rho(h)\circ T\circ\rho(g)$

3) $V'\otimes V$ $\qquad (\rho'\otimes\rho)(g,h) = \rho'(g)\otimes\rho(h)$

<u>Intertwiners:</u>

1) $$V'\otimes V \longrightarrow K[G]$$
$$\xi\otimes v \longmapsto \underbrace{g\mapsto \xi(\rho(g)v)}_{\text{matrix coeff.}} \quad g\in G$$

injective if V is simple ($V'\otimes V$ is simple, & non-zero)

2) $$V'\otimes V \longrightarrow \mathrm{End}_K V$$
$$\xi\otimes v \longmapsto x\mapsto \xi(x)v \quad x\in V$$

isomorphism

<u>Get</u>: Let $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ be ~~iso~~ reps. of iso classes of simple reps. of G.

$$\begin{tikzcd} \bigoplus_{i=1}^{r} V_i' \otimes V_i \arrow[r, "\cong"] & K[G] \\ \bigoplus_{i=1}^{r} \mathrm{End}_K V_i \arrow[u, "\cong"] \arrow[ur, "*"'] & \end{tikzcd}$$

(injective + dim count)

Under the composition, $T \in \mathrm{End}_K V_i$ maps to ??

$$\xi \otimes v \longmapsto \xi(v)$$

<u>Ex:</u>

$$\begin{tikzcd} V' \otimes V \arrow[r] & K \\ \mathrm{End}_K V \arrow[u, "\cong"] & \end{tikzcd}$$

under the composition, $T \mapsto \mathrm{tr}\, T$.

So under $(*)$, $T \longmapsto \mathrm{tr}(\rho_i(g) \circ T)$

$\therefore$ there is an iso of $G \times G$ reps:

$$\bigoplus_{i=1}^{r} \mathrm{End}_K V_i \longrightarrow K[G]$$

$$\mathrm{id}_{V_i} \longmapsto \mathrm{tr}(\rho_i(g))$$

Wedderburn decomposition is also such an isomorphism, $\therefore$ differs on simples from this one by scalars.

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Conclusion:

$$\varepsilon_i(g) = c_i \, tr(\rho_i(g)) \qquad (*)$$

for some $c_i \in K$.

As a rep. ~~of~~ $(L, K[G])$ has decomposition

$$K[G] = \bigoplus_{i=1}^{r} \cancel{\dim V_i} \, V_i^{\oplus \dim V_i}$$

Taking trace at g:

$$tr(L(g); K[G]) = \sum d_i \, tr(\rho_i(g); V_i)$$
$$\shortparallel$$
$$|G| \, \delta_{g 1_G}$$
$$\shortparallel$$
$$|G| \, 1_{K[G]}$$

$$\therefore \quad 1_{K[G]} = \sum_{i=1}^{r} \frac{d_i}{|G_i|} \, tr(\rho_i(g); V_i)$$
$$\shortparallel$$
$$\varepsilon_1 + \cdots + \varepsilon_r$$

Linear independence + $(*)$

$$\Rightarrow \quad \varepsilon_i(g) = \frac{d_i}{|G_i|} \, tr(\rho_i(g); V_i).$$