Lecture I

1. Representations and modules:

K : field

G : group

A representation is a ~~homom~~ pair (ρ, V) where

- V is a vector space $/K$
- $\rho : G \longrightarrow GL(V)$ is a homom. of groups.

A K-algebra is a ring R whose underlying additive ~~gp.~~ is a vector space over K and for which multiplication is K-bilinear $R \times R \to R$.

K-algebra homomorphism:

- ~~algebra~~ ring homomorphism
- K-linear.

Idea: Study rep. theory via algebras and their structure (Frobenius 1897)

Group algebra: $K[G]$

Definition 1: It is the K vector space with basis $\{1_g \mid g \in G\}$ and mult. defined by extending (bilinearly)

$$1_g \cdot 1_h = 1_{gh} \quad \forall\, g, h \in G.$$

Definition 2: It is the space of all fns. $G \longrightarrow K$ (with finite support) and mult. defined by

$$f_1 * f_2(g) = \sum_{xy=g} f_1(x) f_2(y).$$

The analog of representation for an K-algebra is a module.

An R-module is a pair $(\tilde{\rho}, V)$, where (R a (K-alg.))
V is a vector space $/K$ and
$\tilde{\rho} : R \longrightarrow \mathrm{End}_K V$ is a K-algebra homom.

Given $\rho : G \to GL(V)$

define $\tilde{\rho} : K[G] \to \mathrm{End}_k(V)$ by

$$\tilde{\rho}(1_g) = \rho(g).$$

Given $\tilde{\rho} : K[G] \to \mathrm{End}_k(V)$

define $\rho : G \to GL(V)$ by

$$\rho(g) = \tilde{\rho}(1_g).$$

Qn: why is $\tilde{\rho}(1_g)$ in $GL(V)$?

~~Example~~. Regular representation :

For $r \in R$, define $\tilde{L}(r) : R \to R$, $x \mapsto rx$

$(\tilde{L}, R)$ is an R-module, called the

left regular R-module

If $R = K[G]$, this gives rise to a

rep. of G, $\Phi(L, K[G])$ called the

left regular rep. of G

④

We have

$$L(g)\,1_x = 1_{gx}.$$

<u>Exercise</u>: $(L(g)\,f)(x) = ?$

If we define $R : G \to GL(K[G])$ by

$$R(g)\,1_x = 1_{xg^{-1}}$$

then $(R, K[G])$ is also a representation of G, called the <u>right regular rep</u>

<u>Check this!</u>

2. <u>Invariant subspaces</u> and <u>irreducibility</u>

<u>Invariant subspace</u>: $W \subset V$ is an invariant subsp. of a rep. $\rho : G \to GL(V)$ if $\rho(g) W \subset W \quad \forall\, g \in G$.

(and similarly for modules)

<u>Example</u>: Constant fns. form an invar. subspace for $(L, K[G])$. So do fn. $f \in K[G]$ for which $\sum_{g\in G} f(g) = 0$.

Simple rep or module:

A rep/module is simple if it contains no non-trivial proper invariant subsp.

e.g., every 1-dim. rep/module.

Intertwiner $(\rho_1, V_1), (\rho_2, V_2)$ - reps ~~/ modules~~.

A lin. map $T: V_1 \to V_2$ is called an intertwiner if

$$\cancel{T(\rho_1(g)) = \rho_2(g)} \quad T \circ \rho_1(g) = \rho_2(g) \circ T$$

$\forall g \in G$ (similarly for modules).

We say: $T \in \mathrm{Hom}_G(V_1, V_2)$.

Schur's lemma I: K-alg. closed, V fin. dim. simple rep. of G, then every self-intertwiner $T: V \to V$ is a scalar multiple of the identity map, i.e., $\underset{\mathrm{Hom}_G(V,V)}{\mathrm{End}_G\, V} = \lambda\, \mathrm{id}_V$.

Proof: T has an eigenvalue (say λ).

Then $\ker(T - \lambda I)$ is a non-trivial invariant subspace of V.

$\therefore \ker(T - \lambda I) = V \iff T = \lambda I$.

<u>Isomorphism:</u> Representations (ρ_1, V_1) & (ρ_2, V_2) are said to be isomorphic if there is an bijective intertwiner $T: V_1 \to V_2$.

<u>Exercise:</u> T^{-1} ~~will be~~ is an intertwiner $V_2 \to V_1$.

<u>Schur's lemma II</u> If V_1 & V_2 are simple, then every non-zero intertwiner $T: V_1 \to V_2$ is an iso. Consequently, either $V_1 \cong V_2$ or $\mathrm{Hom}_G(V_1, V_2) = 0$.

<u>Corollary:</u> K alg. closed, V_1, V_2 simple, $T: V_1 \to V_2$ a non-zero intertwiner, then $\mathrm{Hom}_G(V_1, V_2) = KT$.

<u>Pf:</u> By Schur II, T is invertible.

$\therefore T^{-1} \circ S \in \mathrm{End}_G V_1$

Schur I $\Rightarrow T^{-1} \circ S = \lambda \, \mathrm{Id}_{V_1}$

$\Rightarrow S = \lambda T$. QED.

Projections and decompositions

Given a decomposition of a vector space $V = W \oplus U$,
For every $v \in V$, $\exists!\ w \in W$ and $u \in U$ such that
$$v = w + u.$$
Define $P_W, P_U \in \mathrm{End}_K V$ by $P_W(v) = w$ & $P_U(v) = u$.

Then 1. $P_W + P_U = \mathrm{id}_V$

2. $P_W^2 = P_W$

3. $P_W(V) = W$ and $P_U(V) = U$.

Recall: $P \in \mathrm{End}_K V$ is called a projection if $P^2 = P$.

In the above situtation, the projection P_W determines the decomposition $V = W \oplus U$.

More precisely, suppose P is a projection.
Let $W = P(V)$ and $U = (\mathrm{id}_V - P)(V)$.
Then $V = W \oplus U$ with $P_W = P$ and $P_U = \mathrm{id}_V - P$.

Proof: $v = P(v) + v - P(v) = P(v) + (\mathrm{id}_V - P)v$

$\Rightarrow V \subset \cancel{W \oplus U}\ W + U$.

Suppose $w \in W$. Then $w = P(v)$ for some $v \in V$.

$\therefore P(w) = P^2(v) = P(v) = w$

Similarly if $u \in U$, then $u = (\mathrm{id}_V - P)(v)$ for some $v \in V$

$\therefore (\mathrm{id}_V - P)(u) = (\mathrm{id}_V - P)^2(v) = (\mathrm{id}_V - 2P + P^2)(v)$

$= (\mathrm{id}_V - P)(v) = u$.

$\therefore$ if $x \in W \cap U$, then $P(x) = x$ and $x - P(x) = x$

$\therefore x = 0$. Hence $W \cap U = 0$.

Maschke's theorem:

If (ρ, V) is a rep. of G, char $K \nmid |G|$, $W \subset V$ is an invariant subspace, then W has an invariant complement.

Pf: Whenever $V = W \oplus U$,

$$\mathrm{id}_V = P_W \oplus P_U$$

$$x = \underset{\substack{\cap \\ W}}{P_W(x)} + \underset{\substack{\cap \\ U}}{P_U(x)}$$

Note: P_W depends on W and U.

Lemma: P_W (and hence P_U) $\in \mathrm{End}_G V$ iff U is also an invar. subsp.

Proof: Write $x = x_W + x_U$, $x_W \in W$, $x_U \in U$.

$$P_W(x_W) = x_W$$

$$\Rightarrow \rho(g) P_W(x_W) = \rho(g) x_W.$$

But W is invariant, so $\rho(g) x_W \in W$

$$\Rightarrow P_W(\rho(g) x_W) = \rho(g) x_W$$

(1) $$= \rho(g) P_W(x_W). \quad \forall g \in G$$

(8)

On the other hand

$$\cancel{\rho(g)(x_U) =} \; P_W(x_U) = 0$$

$$\therefore \quad \rho(g)\, P_W(x_U) = 0.$$

But $\quad P_W(\rho(g)(x_U)) = 0$

$$\iff \rho(g)(x_U) \in U.$$

$$\therefore \quad \rho(g)\, P_W(x_U) = P_W\, \rho(g)(x_U) \qquad (2)$$

$$\iff \rho(g)(x_U) \in U$$

(1)+(2) is the intertwining prop. for P_W.

$\therefore$ P_W is an intertwiner iff U is invar.

To complete the proof of Maschke,

define $\quad \overline{P}_W = \underset{0 \neq}{\frac{1}{|G|}} \sum_{g \in G} \rho(g)\, P_W\, \rho(g)^{-1}.$

① $\overline{P}_W$ is an intertwiner.

$\rightarrow$ easy

② $\overline{P}_W^{\,2} = \overline{P}_W$

$\rightarrow$ check for $x \in W$:

$$\overline{P}_W(x) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, \underbrace{P_W\, \underbrace{\rho(g)^{-1} x}_{\in W}}_{= \rho(g)^{-1} x}$$

$$= x \qquad (*)$$

~~$\to$ check for general $x \in V$~~

Now, if $x \in V$, then $\overline{P}_W(x) \in W$.

$\therefore \overline{P}_W^{\,2}\, x = \overline{P}_W(\overline{P}_W(x))$

$= \overline{P}_W(x)$ [by (*)].

Now set: $\overline{P}_U = id_V - \overline{P}_W$.

$\overline{P}_U^{\,2} = (id_V - \overline{P}_W)^2 = id_V - 2\overline{P}_W + \overline{P}_W = \overline{P}_U$.

Let $\overline{U} = \overline{P}_U(V)$.

~~On the other hand,~~

If $x \in W$, then $\overline{P}_W(x) = x$

(a) $\overline{P}_U(x) = 0 \quad \forall \; x \in W$

If $x \in \overline{U}$, then $x = \overline{P}_U(x')$ for some $x' \in V$.

(b) $\therefore \overline{P}_U(x) = \overline{P}_U^{\,2}(x') = \overline{P}_U(x') = x \;\forall x \in \overline{U}$.

(a) + (b) $\Rightarrow \overline{U} \cap W = 0$

$\therefore V = W \oplus \overline{U}$, which are both invariant (by the lemma).

Corollary (Complete reducibility): Every fin. dim. rep. is a sum of simples if char $K \nmid |G|$.