Lecture VII

Families of Sym. fns: $\begin{cases} \vec{m} & : \text{monomial} \\ \vec{e} & : \text{elementary} \\ \vec{h} & : \text{complete} \\ \vec{p} & : \text{power sum} \end{cases}$

$K = (K_{\mu\lambda})$: Kostka nos. $\quad J = (\delta_{\mu\lambda'})$

$X = (\chi_{\mu\lambda}) \qquad \chi_{\mu\lambda} = tr(\omega_\lambda ; V_\mu)$

Last time, we found that:

$$\vec{e} = \vec{m} K' J K = \vec{s} J K$$
$$\vec{h} = \vec{m} K' K = \vec{s} K$$
$$\vec{p} = \vec{m} K' X = \vec{s} X$$

Let $\vec{s} = \vec{m} K'$, i.e.,

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

These are the Schur fns., as interpreted by Kostka.

Thm: For any field k, $\{ s_\lambda \mid \lambda \vdash n \}$ is a basis of Λ^n_k.

Classically, symmetric fns. were studied as ~~fns.~~ ~~of~~ symmetric polys in a finite no. of variables, say $x_1, \ldots, x_m$. $f(x_1, \ldots, x_m) = f(x_1, \ldots, x_m, 0, \ldots, 0)$

If $m \geq n$, then $\{ m_\lambda \mid \lambda \vdash n \}$ form a basis of hgns. symmetric polys in m v'bles, and all transition matrices remain valid.

∴ our theorems about $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$ & $\{s_\lambda\}$ forming bases remain valid.

Cauchy's defn. of Schur fns.

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ - multi-index.

$$a_\alpha := \begin{vmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_m} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_m} \\ \vdots & & & \\ x_m^{\alpha_1} & x_m^{\alpha_2} & \cdots & x_m^{\alpha_m} \end{vmatrix} = \det(x_i^{\alpha_j})$$

Example: $\delta = (m-1, \ldots, 1, 0)$

$$a_\delta = \begin{vmatrix} x_1^{m-1} & \cdots & x_1 & 1 \\ x_2^{m-1} & \cdots & x_2 & 1 \\ \vdots & & & \\ x_m^{m-1} & \cdots & x_m & 1 \end{vmatrix} = \prod_{i<j} (x_i - x_j)$$

is the Vandermonde determinant.

For any $w \in S_m$, $a_\alpha(x_{w(1)}, \ldots, x_{w(m)}) = \varepsilon(w) a_\alpha(x_1, \ldots, x_m)$, i.e., a_α is an alternating polynomial (hgus. of degree $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$).

Note: $f(x) = \sum_\alpha c_\alpha x^\alpha$

$\forall\, w \in S_n \quad c_{w\cdot\alpha} = \varepsilon(w)\, c_\alpha$.

Suppose $\alpha_i = \alpha_j$, $w = (ij)$

$$c_\alpha = c_{w\cdot\alpha} = -c_\alpha$$

$\Rightarrow c_\alpha = 0$, so only x^α where α_i's are distinct occur, Also, rearranging the parts of α changes c_α by a fixed sign. $\therefore f$ is determined by the coefficients of partitions with distinct parts in decreasing order λ. Every such partition is of the form $\lambda+\delta$, where λ is a partition.

Theorem (Cauchy's defn. of Schur fns.)

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

Proof: The relation $\vec{e} = \vec{s} J K$ characterizes $\vec{s}$.
$\therefore$ it suffices to show that

$$a_\delta e_\mu = \sum_{\lambda \vdash n} K_{\lambda'\mu} \, a_{\lambda+\delta},$$

for which it suffices to show that the coeff. of $x^{\lambda+\delta}$ on both sides is the same $\forall \lambda$.

In $a_{\lambda+\delta}$, the only monomial with decreasing coeffs. is $x^{\lambda+\delta}$, and has coeff. 1.

$\therefore$ it suffices to show that coeff. of $x^{\lambda+\delta}$ on the LHS is $K_{\lambda'\mu}$ $\forall$ $\lambda \vdash n$.

A monomial in $a_\delta e_\mu = a_\delta e_{\mu_1} e_{\mu_2} \cdots$ is obtained by starting with a monomial of a_δ, and ~~raising μ_i of~~ increasing the powers of μ_i of the v'bles by 1 each.

~~If the resu~~ ~~At each stage the result is alternati~~

Since each $a_\delta e_{\mu_1} \cdots e_{\mu_i}$ is alternating only monomials with ~~strictly~~ distinct powers occur. But one can not go from a non-decreasing to a strictly ~~increasing~~ decreasing monomial without passing through one with two equal powers, at which stage contributions cancel.

So to obtain a strictly decreasing monomial in $a_\delta e_\mu$, we start with $x_1^{m-1} \cdots x_{m-1}$, and at the ith stage, choose $j_1 < \cdots < j_{\mu_i}$ distinct indices to increment in such a way that the resulting monomial has strictly decreasing powers.

Example: $m = 4$, $\mu = (3, 2, 2, 2, 2, 1, 1)$

Step 0: $x_1^3 \; x_2^2 \; x_3 \; x_4^0$

$\downarrow +1 \quad \downarrow +1 \quad \downarrow +1 \quad \downarrow 0$

Step 1: $x_1^4 \; x_2^3 \; x_3^2 \; x_4^0$ $\qquad \begin{array}{ccc} 1 & 1 & 1 \end{array}$

$\downarrow +1 \quad \downarrow 0 \quad \downarrow 0 \quad \downarrow +1$

Step 2: $x_1^5 \; x_2^3 \; x_3^2 \; x_4$ $\qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & & & \end{array}$

$\downarrow +1 \quad \downarrow +1 \quad \downarrow \quad \downarrow$

Step 3: $x_1^6 \; x_2^4 \; x_3^2 \; x_4$ $\qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & & \\ 3 & & & \end{array}$

$\downarrow +0 \quad \downarrow +0 \quad \downarrow +1 \quad \downarrow +1$

Step 4: $x_1^6 \; x_2^4 \; x_3^3 \; x_4^2$ $\qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 4 \\ 3 & & & \end{array}$

$$\begin{array}{cccc} x_1^6 & x_2^4 & x_3^3 & x_4^2 \\ \downarrow{+1} & \downarrow{+1} & \downarrow{0} & \downarrow{0} \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 4 \\ 3 & 5 & & \\ 5 & & & \end{array}$$

Step 5 $\quad x_1^7 \quad x_2^5 \quad x_3^3 \quad x_4^2$

$$\begin{array}{cccc} \downarrow{0} & \downarrow{+1} & \downarrow{0} & \downarrow{0} \end{array}$$

Step 6 $\quad x_1^7 \quad x_2^6 \quad x_3^3 \quad x_4^2 \qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 4 \\ 3 & 5 & & \\ 5 & 6 & & \end{array}$

$$\begin{array}{cccc} \downarrow{0} & \downarrow{0} & \downarrow{+1} & \downarrow{0} \end{array}$$

Step 7 $\quad x_1^7 \quad x_2^6 \quad x_3^4 \quad x_4^2 \qquad \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 4 \\ 3 & 5 & 7 & \\ 5 & 6 & & \end{array}$

$$= x^{(4,4,3,2)+(3,2,1,0)}$$

$\uparrow$ SSYT of shape $(4,4,3,2)'$ and type $(3,2,2,2,2,1,1)$.

The correspondence between the no. of ways to get $x^{\lambda+\delta}$ from x^δ and SSYT of shape λ' & type μ is: choose x_j at the ith stage

$$\Updownarrow$$

SSYT has ~~puts~~ an i in the jth column.

The fact that the monomial at each stage has strictly decreasing powers means that the shape of boxes with $1, \dots, i$ is a Young diagram, and this equivalent to having a weakly increasing rows & cols. Cols. are strictly increasing because i is put at most once in any column.

QED.

Corollary (~~of the proof~~):

Suppose f is a homogeneous symmetric poly. in m variables of degree n, and $m \geq n$. If

$$f = \sum_{\lambda \vdash n} c_\lambda s_\lambda$$

then c_λ is the coefficient of $x^{\lambda+\delta}$ in $f a_\delta$.

Proof:
$$f a_\delta = \sum_{\lambda \vdash n} c_\lambda a_{\lambda+\delta}$$

and we have already observed that the coeff. of $x^{\lambda+\delta}$ on the RHS is c_λ. QED.

Frobenius' formula for the chars. of S_n:

$$\operatorname{tr}(\omega_\lambda; V_\mu) = \text{coeff. of } x^{\lambda+\delta} \text{ in } p_\mu a_\delta.$$

Pf: Apply the above corollary to

$$p_\mu = \sum_{\lambda \vdash \mu} \underbrace{X_{\lambda\mu}}_{\operatorname{tr}(\omega_\lambda; V_\mu)} s_\lambda$$

QED.