Symmetric fn. = Symmetric poly. in ∞ly many v'bles.

$x_1, x_2, \ldots$ variables.

$\alpha = (\alpha_1, \alpha_2, \ldots)$ multi-index

$\shortparallel$

sequence of non-neg. integers with finitely many >0 terms.

$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots$: monomial.

Shape (α) : partition obtained by rearranging $\alpha_1, \alpha_2, \ldots$ in decreasing order. $|\alpha| = \sum \alpha_i$

<u>Homogeneous</u> <u>symmetric</u> <u>fn.</u> <u>of</u> <u>deg</u> <u>n</u>

$$f(x) = \sum_{|\alpha| = n} c_\alpha x^\alpha \quad \text{(formal sum)}$$

where $c_\alpha \in K$ depends only on shape(α).

$\longrightarrow \Lambda^n_K$: space of hgns. symm. fns. of deg n.

Note: $\alpha = (0, 0, \ldots) \Rightarrow$ shape$(\alpha) = \emptyset$ (partition of 0)

so $\Lambda^0_K = K$ (constant fns.).

If $n>0$ and ~~for~~ $\lambda \vdash n$, then there are always ∞ly many multi-indices ~~with~~ α with $\text{shape}(\alpha)=\lambda$.
So $f(x)$ is always an ∞ sum.

But it makes sense to multiply
$$f \in \Lambda_K^m, \quad g \in \Lambda_K^n$$
$$fg \in \Lambda_K^{m+n}$$
b/c $\forall$ multi-index α only finitely many monomials of f & g contribute to the coeff of x^α in fg.

Monomial symmetric fn:

$$m_\lambda = \sum_{\text{shape}(\lambda)=\alpha} x^\alpha.$$

Example: $m_\phi = 1$ $\quad m_{(2,1)} = \sum_{i,j} x_i x_j^2$

$$m_{(1)} = \sum x_i$$
$$m_{(2)} = \sum x_i^2$$
$$m_{(1,1)} = \sum_{i<j} x_i x_j$$

<u>Thm:</u> $\{m_\lambda \mid \lambda \vdash n\}$ is a basis of Λ_k^n.

<u>Pf:</u> By defn.

<u>Cor:</u> $\dim \Lambda_k^n = P(n) :=$ no. of partitions of n.

<u>Elementary Sym. Fns:</u>

$$e_n := m_{(1^n)} \quad \forall\ n \geq 0$$

$$= \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$ if $\lambda = (\lambda_1, \lambda_2, \dots)$.

<u>Thm:</u> $\forall\ n \geq 0$ & $\forall\ \lambda \vdash n$

$$e_\lambda = \sum_\mu N_{\mu\lambda} m_\mu.$$

($N_{\mu\lambda}$: no. of $\mu \times \lambda$ 0-1-matrices)

<u>Pf:</u>

$$\begin{matrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ \vdots & & & \end{matrix}$$

Each monomial in e_λ is obtained by choosing λ_1 entries from row 1, λ_2 entries from row 2 etc., & multiplying them.

The resulting monomial is x^μ iff μ_1 entries were chosen from col., μ_2 from col. 2, etc.

$\therefore$ coeff. of x^μ in e_λ is $N_{\lambda\mu} = N_{\mu\lambda}$.

QED.

Matrix notation:

Enumerate $\{\lambda \mid \lambda \vdash n\}$ in such a way that if $\lambda \leq \mu$ then λ appear before μ.

$\vec{m} = (m_\lambda)$ row vector $\in \Lambda_K^{P(n)}$

$N = (N_{\lambda\mu}) \in M_{P(n)\times P(n)}(\mathbb{Z}_{\geq 0})$.

$\vec{e} = (e_\lambda)$

Have: $\boxed{\vec{e} = \vec{m} N.}$

Dual RSK: $N_{\lambda\mu} = \sum_{\substack{\nu \leq \lambda \\ \nu' \leq \mu}} K_{\nu\lambda} K_{\nu'\mu}$.

$J = (\delta_{\lambda'\mu})$ - permutation matrix

Left mult. by J interchanges νth row & ν'th row.

Dual RSK: $N = K' J K$

$K = (K_{\lambda\mu})$ $\quad K'$ transpose of K.

Note: $K_{\lambda\mu} > 0 \Rightarrow \lambda \leq \mu$.

So K is upper Δ^{lar}.

$K_{\lambda\lambda} = 1$ so K has 1's along diag.

$\therefore \det N = \det J = \pm 1$.

So N is non-singular (even over $\mathbb{Z}$).

Theorem: $\{e_\lambda \mid \lambda \vdash n\}$ is a basis of Λ^n_K.

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Fundamental Thm of Symmetric Fns.

Historical Significance:

$$\prod_{i=1}^{n} (x - \lambda_i) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots \pm a_n.$$

then $a_i = e_i(\lambda_1, \ldots, \lambda_n, 0, \ldots)$

Since e_λ is a monomial in e_i's

we get,

Every ~~sa~~ symmetric poly. in the roots of an equation is a polynomial ~~of~~ in its coefficients.

Complete Symmetric Fns.

$$h_n = \sum_{|\alpha| = n} x^\alpha.$$

$$= \sum_{\lambda \vdash n} m_\lambda$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots \text{ if } \lambda = (\lambda_1, \lambda_2, \dots).$$

Thm: $h_\lambda = \sum_\mu M_{\mu\lambda} m_\mu$.

Pf:

$$\begin{matrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

A monomial in h_λ is obtained by choosing λ_i entries from the ith row (but this time, with multiplicity).

This is recorded in a $\lambda \times \mu$ matrix.

$$\boxed{\vec{h} = \vec{m} M} \qquad M = (M_{\lambda\mu}).$$

RSK: $M = K'K$

$\therefore \det M = 1$.

Thm: For any field K
$\{h_\lambda \mid \lambda \vdash n\}$ is a basis of Λ^n_K.

Power Sum Symmetric Fns.

$$p_n = m_{(n)} = x_1^n + x_2^n + \cdots$$
$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \text{ if } \lambda = (\lambda_1, \lambda_2, \ldots).$$

Define: $P_{\lambda\mu} = \operatorname{trace}(w_\mu ; K[X_\lambda])$.

$$w_\mu = (1 \ldots \mu_1)(\mu_1+1, \ldots \mu_1+\mu_2)(\mu_1+\mu_2+1 \ldots\ )$$

a permutation whose cycle decomposition has shape μ.

Theorem: $$p_\mu = \sum_{\lambda \vdash n} P_{\lambda\mu} m_\lambda.$$

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Pf:

$$\begin{array}{llll} x_1^{\mu_1} & x_2^{\mu_1} & x_3^{\mu_1} & \cdots \end{array} \left\{ \begin{array}{l} 1 \\ \vdots \\ \mu_1 \end{array} \right.$$

$$\begin{array}{llll} x_1^{\mu_2} & x_2^{\mu_2} & x_3^{\mu_2} & \cdots \end{array} \left\{ \begin{array}{l} \mu_1+1 \\ \vdots \\ \mu_1+\mu_2 \end{array} \right.$$

$$\begin{array}{llll} x_1^{\mu_3} & x_2^{\mu_3} & x_3^{\mu_3} & \cdots \end{array} \left\{ \begin{array}{l} \mu_1+\mu_2+1 \\ \vdots \\ \mu_1+\mu_2+\mu_3 \end{array} \right.$$

To get a monomial in P_μ, must pick one entry from each row.

The monomial is x^λ iff.

$\forall\, i$, sum of μ_i s for which x_i is picked $= \lambda_i$.

Construct $S_1 \amalg \cdots \amalg S_\ell \in X_\lambda$ as follows:

S_i = union of subsets associated to ~~parts~~ row where x_i is picked.

Parts of S are unions of cycles of ω_μ so these are precisely the ordered partitions fixed by ω_μ.

Matrix notation:

$$\boxed{\vec{p} = \vec{m}\, P}$$

$$P_{\lambda\mu} = \operatorname{trace}(\omega_\mu ; K[X_\lambda])$$

$$= \sum_\nu K_{\nu\lambda} \operatorname{trace}(\omega_\mu ; V_\nu)$$

$$= X_{\nu\mu} K_{\nu\lambda}$$

$$X_{\nu\mu} = \operatorname{trace}(\omega_\mu ; V_\nu)$$

character table of S_n !

$$\therefore\ P = K' X$$

det 1 if char $K > n$.

Since ~~char. table~~ ~~det~~ of X is non-~~zero~~ singular (chars. lin indep $\Rightarrow$ char. table non-sing.) P is non-singular.

<u>Thm:</u> Char $K > n$ then

$$\{p_\lambda \mid \lambda \vdash n\}$$

is a basis of Λ^n_K.