# LECTURE V

**Recall**: $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_m)$

$$M_{\lambda\mu} = \#\left\{ A \in M_{\ell\times m}(\mathbb{Z}) \;\middle|\; \begin{array}{ll} \sum_j a_{ij} = \lambda_i & i = 1, \ldots, \ell \\ \sum_i a_{ij} = \mu_j & j = 1, \ldots, m \end{array} \right\}$$

We proved:

$$\dim \mathrm{Hom}_{S_n}(K[X_\lambda], K[X_\mu]) = M_{\lambda\mu}$$

RSK Corresp:

$$M_{\lambda\mu} = \sum_{\nu \leq \lambda;\, \nu \leq \mu} K_{\nu\lambda} K_{\nu\mu}$$

By combinatorial resolution thm: (char $K > n$)

$$K[X_\lambda] = \bigoplus_{\mu \leq \lambda} V_\mu^{\oplus K_{\mu\lambda}}$$

$$= V_\lambda \oplus \bigoplus_{\mu < \lambda} V_\mu^{\oplus K_{\mu\lambda}}$$

can be used to define $V_\lambda$ & calculate its char.

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## Character twists

A [multiplicative] character is a homom. $\chi : G \longrightarrow K^*$

**Twist**: $(\rho, V) \rightsquigarrow (\rho \otimes \chi, V)$

$$\rho \otimes \chi(g) = \chi(g)\rho(g).$$

**Example:**

$$\begin{tikzcd} S_n \arrow[r, hook, "\text{perm. matrices}"'] \arrow[rr, bend left, "\varepsilon"] & GL_n(K) \arrow[r, "\det"] & K^* \end{tikzcd}$$

- $S_n$ is generated by transpositions
- $\varepsilon(\text{transposition}) = -1$

$\therefore \quad \varepsilon(S_n) = \{\pm 1\} \subset K^*$

**Question:** what is $V_\lambda \otimes \varepsilon$ ?

To answer this, we analyse

$$\mathrm{Hom}_{S_n}(K[X_\mu], K[X_\lambda] \otimes \varepsilon)$$

or more generally, $G \circlearrowright X$, $G \circlearrowright Y$, $\chi : G \to K^*$

What is $\mathrm{Hom}_G(K[Y], K[X] \otimes \chi)$ ?

Rep. space of $K[X] \otimes \chi$ is just $K[X]$.

So lin. maps $K[Y] \to K[X]$ are $\{T_k \mid k \in K[X \times Y]\}$

So qn. is: for which $k \in K[X, Y]$ is

$T_k \in \mathrm{Hom}_G(K[Y], K[X] \otimes \chi)$ ?

$$\chi(g) \rho_X(g) T_k \rho_X(g)^{-1} = T_k$$

Working out LHS as before we get

$$\boxed{k(gx, gy) = \chi(g) k(x, y) \quad \forall\, g \in G,\ x \in X,\ y \in Y}$$

In particular, $k(x,y)$ determines $k(g\cdot x, g\cdot y)$ $\forall$ $g \in G$. So

$$\dim \mathrm{Hom}_G(K[Y], K[X]\otimes \chi) \le |G \backslash X \times Y|$$

<u>Notation:</u> $\mathrm{Stab}_G x = \{g \in G \mid g\cdot x = x\}$

Suppose $\exists$ $g \in \mathrm{Stab}_G x \cap \mathrm{Stab}_G y \ni \chi(g) \neq 1$.

$k(x,y) = k(g\cdot x, g\cdot y) = \chi(g)\, k(x,y)$.

$\Longrightarrow k(x,y) = 0$

But if $\mathrm{Stab}_G x \cap \mathrm{Stab}_G y \subset \ker \chi$ then

$$k(g\cdot x, g\cdot y) = \chi(g)$$

is a well-defined fn. on $G\cdot(x,y)$:

If $(g\cdot x, g\cdot y) = (g'\cdot x, g'\cdot y)$, then

$g^{-1}g' \in \mathrm{Stab}_G x \cap \mathrm{Stab}_G y$

$\Rightarrow \chi(g) = \chi(g')$.

<u>Defn:</u> $(X\times Y)_\chi = \{(x,y) \mid \mathrm{Stab}_G x \cap \mathrm{Stab}_G y \subset \ker \chi\}$

<u>Theorem:</u>

$$\dim \mathrm{Hom}_G(K[Y], K[X]\otimes\chi) = |G \backslash (X\times Y)_\chi|$$

(42) Let $\lambda, \mu \vdash n$, $\underset{S_1,\dots,S_\ell}{(S}, \underset{(T_1,\dots,T_m)}{T)} \in X_\lambda \times X_\mu$.

Theorem: $(S,T) \in (X_\lambda \times X_\mu)_\varepsilon$ iff $|S_i \cap T_j| \leq 1 \;\forall\; (i,j)$. | $(S,T)$ is called a transversal pair

Pf: $g \in \mathrm{Stab}_{S_n} S \cap \mathrm{Stab}_{S_n} T$

$\Leftrightarrow g(S_i \cap T_j) = S_i \cap T_j \;\forall\; (i,j)$.

If $|S_i \cap T_j| \leq 1$, since each element of $\underline{n}$ is in $S_i \cap T_j$ for some $(i,j)$, each element of $\underline{n}$ is fixed by $g$.

$\Rightarrow g = \mathrm{id}_{\underline{n}}$.

$\therefore \mathrm{Stab}_{S_n} S \cap \mathrm{Stab}_{S_n} T = (\mathrm{id}_{\underline{n}}) \subset \ker \varepsilon$.

If $\exists\; (i,j) \ni |S_i \cap T_j| > 1$, choose ~~{r,s} ∈~~ $r \neq s$, $r, s \in S_i \cap T_j$.

The transposition $(rs) \in \mathrm{Stab}_{S_n} S \cap \mathrm{Stab}_{S_n} T$.

But $\varepsilon(rs) = 1$.

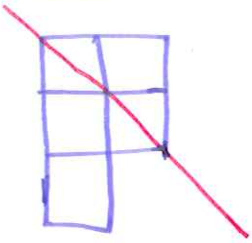
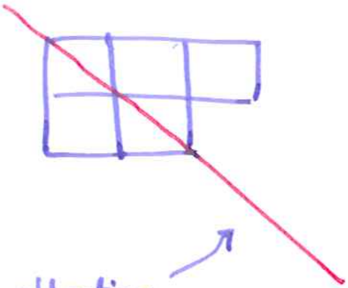
$\therefore \mathrm{Stab}_{S_n}(S) \cap \mathrm{Stab}_{S_n}(T) \not\subset \ker \chi$.

Cor: Let

$$N_{\lambda\mu} = \left\{ A \in M_{\ell \times m}(\mathbb{Z}) \;\middle|\; \begin{array}{l} a_{ij} \in \{0,1\} \;\forall\; i,j \\ \sum_j a_{ij} = \lambda_i \;\;\forall\, i = 1,\dots,\ell \\ \sum_i a_{ij} = \mu_j \;\;\forall\, j = 1,\dots,m \end{array} \right\}$$

Then $\dim \mathrm{Hom}_{S_n}(K[X_\mu], K[X_\lambda] \otimes \varepsilon) = N_{\lambda\mu}$.

## Transpose partition

$\lambda = (2,2,1)$  $\lambda' = (3,2)$ 


$\lambda'_1$ = no. of non-zero parts of $\lambda$ $(= \ell)$

$\lambda'_j$ = no. of parts of $\lambda$ which are $\geq j$.

**Lemma:** $\lambda, \mu \vdash n$, then $N_{\lambda\mu} > 0$ iff $\mu' \leq \lambda$.

**Pf:** $N_{\lambda\mu} > 0 \Rightarrow \exists$ transversal pair $(S,T) \in X_\lambda \times X_\mu$.

$Y_S$ : tableau with parts of $S$ as rows.

$Y'_T$ : tableau with parts of $T$ as cols.

**Example:** $S = \{1,2,3\} \sqcup \{4,6,7\} \sqcup \{5,8\}$

$T = \{2,5,7\} \sqcup \{4,1\} \sqcup \{3,8\} \sqcup \{6\}$.

$$Y_S = \begin{matrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 5 & 8 & \end{matrix} \qquad Y'_T = \begin{matrix} 2 & 4 & 3 & 6 \\ 5 & 1 & 8 & \\ 7 & & & \end{matrix}$$

Elements of the 1st row of $Y_S$ occur in distinct cols. of $Y'_T$.

Permute each col. of $Y'_T$ to move them to the first row:

$$\begin{matrix} 2 & 1 & 3 & 6 \\ 5 & 4 & 8 & \\ 7 & & & \end{matrix}$$

Since they fit in the first row, & freeze them.

$\lambda_1 \leq \mu'_1$.

Elements in the second row of $Y_S$ occur in distinct cols. of $Y'_T$. Move them up in their cols. without disturbing the frozen elements, and then freeze them.

$$\begin{array}{cccc} 2 & 1 & 3 & 6 \\ 7 & 4 & 8 & \\ 5 & & & \end{array}$$

They come to rest in the first two rows of $Y'_T$.

$\therefore \lambda_1 + \lambda_2 \leq \mu'_1 + \mu'_2$.

Continuing in this manner, we get

$\lambda_1 + \cdots + \lambda_i \leq \mu'_1 + \cdots + \mu'_i \;\; \forall\, i$.

or $\mu' \leq \lambda$.

<u>Converse</u>: we will use the following:

$$K_{\mu\lambda} > 0 \iff \mu \leq \lambda$$

$\Longrightarrow$ done in class

assignment $\Longleftarrow$

Starting with an SSYT of type $\lambda$ & shape $\mu'$, we will construct a transverse pair $(S, T) \in X_\lambda \times X_\mu$.

Example:

$$\begin{matrix} 1^1 & 1^2 & 1^3 & 2^4 \\ 2^5 & 2^6 & 3^7 & \\ 3^8 & & & \end{matrix}$$

$\mu = (3,2,2,1)$

$\lambda = (3,3,2)$

$S = \{1,2,3\} \amalg \{4,5,6\} \amalg \{7,8\}$

$T = \{1,5,8\} \amalg \{2,6\} \amalg \{3,7\} \amalg \{4\}$.

Lemma: $N_{\lambda\lambda'} = 1 \quad \forall \lambda \vdash n$.

Example: $\lambda = (6,5,3,3)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \in N_{\lambda\lambda'}$$

and is the only element of $N_{\lambda\lambda'}$.

Corollary: $\exists !$ simple rep. $U_\lambda$ of $S_n$ which occurs with mult. 1 in $K[X_\lambda]$ and $K[X_{\lambda'}] \otimes \varepsilon$.

Pf: $\dim \mathrm{Hom}_{S_n}(K[X_\lambda], K[X_{\lambda'}] \otimes \varepsilon) = 1$

$(\ddot{N}_{\lambda\lambda'})$

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Thm: $U_\lambda \cong V_\lambda$.

Pf: Recall: $K[X_\mu] = \bigoplus_{\nu \le \mu} V_\nu^{\oplus K_{\nu\mu}}$

and $K_{\nu\mu} > 0 \ \forall \ \nu \le \mu$.

So $V_\lambda$ is characterized by the properties:

(1) It occurs in $K[X_\lambda]$

(2) It does not occur in $K[X_\mu]$ if $\mu \not\le \lambda$.

Clearly $U_\lambda$ satisfies (1).

Also, $U_\lambda$ occurs in $K[X_{\lambda'}] \otimes \varepsilon$. Also

$\dim \mathrm{Hom}(K[X_\mu], K[X_{\lambda'}] \otimes \varepsilon) = N_{\mu\lambda'} = N_{\mu'\lambda}$

$= 0$ if $\mu \not\le \lambda$.

$\therefore$ $U_\lambda$ does not occur in $K[X_\mu]$ if $\mu \not\le \lambda$.

Corollary: $V_\lambda \otimes \varepsilon = V_{\lambda'}$.

Pf: Both occur in $K[X_{\lambda'}]$ as well as $K[X_\lambda] \otimes \varepsilon$.