LECTURE IV

Theorem: Let $(P, \leq)$ be a [finite] partially ordered set and $\{U_\lambda\}_{\lambda \in P}$ be a collection of completely reducible representations of G. Suppose $\dim \mathrm{Hom}_G(U_\lambda, U_\mu) = M_{\lambda\mu}$ $\forall\ \lambda, \mu \in P$ and there exist non-negative integers $K_{\mu\lambda}$ for $\mu \leq \lambda$ such that $K_{\lambda\lambda} = 1$ $\forall\ \lambda \in P$ and

$$M_{\lambda\mu} = \sum_{\nu \leq \lambda,\, \nu \leq \mu} K_{\nu\lambda} K_{\nu\mu}. \quad \forall\ \lambda, \mu \in P \qquad \text{(RSK)}$$

Then $\forall\ \lambda \in P$ $\exists$ simple rep. V_λ such that

$$U_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu^{\oplus K_{\mu\lambda}} \qquad \forall\ \lambda \in P. \qquad (*)$$

Proof: Take $\lambda_0 \in P$ minimal.

(RSK) $\Rightarrow$ $\dim \mathrm{End}_G U_{\lambda_0} = \underline{1}$ whence $V_{\lambda_0} := U_{\lambda_0}$ is a simple rep.

(RSK) $\Rightarrow$ $\dim \mathrm{Hom}(U_{\lambda_0}, U_\lambda) = K_{\lambda_0\lambda}$ $\forall\ \lambda \in P$,

so V_{λ_0} has multiplicity $K_{\lambda_0\lambda}$ in U_λ $\forall\ \lambda \in P$.

$\therefore$ $\exists$ representations U_λ^0 in which V_{λ_0} does not occur, and $U_\lambda = U_\lambda^0 \oplus V_{\lambda_0}^{\oplus K_{\lambda_0\lambda}}$.

Let $P^0 = P - \{\lambda_0\}$. Consider the collection $\{U_\lambda^0\}_{\lambda \in P^0}$

Then $\dim \mathrm{Hom}_G(U_\lambda^0, U_\mu^0) = \dim \mathrm{Hom}_G(U_\lambda, U_\mu) - K_{\lambda_0\lambda} K_{\lambda_0\mu}$

$$= \sum_{\substack{\lambda_0 < \nu \leq \lambda \\ \lambda_0 < \nu \leq \mu}} K_{\nu\lambda} K_{\nu\mu}.$$

The theorem therefore follows by induction on $|P|$.

<u>Semistandard</u> <u>Young</u> <u>Tableaux</u> (SSYT) of shape λ is an array $T = (T_{ij})_{\substack{1 \le i \le \ell \\ 1 \le j \le \lambda_i}}$ of positive integers weakly increasing along rows and strictly increasing along columns.

<u>Example</u>:

$$T = \begin{matrix} 1 & 1 & 1 & 3 & 4 & 4 \\ 2 & 4 & 4 & 5 & 5 & \\ 5 & 5 & 7 & & & \\ 6 & 9 & 9 & & & \end{matrix}$$

is a SSYT of shape $(6,5,3,3)$.

<u>Type</u>: $(\alpha_1, \alpha_2, \dots)$ where α_i is the no. of times i occurs in T.

In the above example: $\text{type}(T) = (3,1,1,4,4,1,1,0,2)$.

~~Lemma~~ <u>Example</u>: There are two SSYT's of shape $(2,1)$ and type $(2,1)$: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ & $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$.

<u>Kostka</u> <u>numbers</u>: $K_{\mu\lambda}$ = no. of SSYT of shape μ and type λ.

<u>Example</u>: ① If $\lambda = (n)$, then $\exists!$ SSYT of shape μ and type λ for all compositions μ of n.

② If $\lambda = (1^n)$, then <u>there are no SSYT's of shape μ and type λ unless $\mu = \lambda$.</u> : $\begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}$

Let μ and λ be partitions of n.

Lemma: $K_{\mu\lambda} > 0 \Rightarrow \mu_1 + \dots + \mu_i \geq \lambda_1 + \dots + \lambda_i$ for $i = 1, \dots, \ell$.

Proof: Each col. is strictly increasing $\Rightarrow$ in an SSYT, r must occur in the first r rows (the numbers $1, \dots, r-1$ can occur at most once in its column).

$\therefore \mu_1 + \dots + \mu_i \geq \lambda_1 + \dots + \lambda_i \quad \forall\ i.$

(the first i rows must have enough space for all occurrences of $1, \dots, i$.)

Lemma: $K_{\lambda\lambda} = 1 \quad \forall\ \lambda$.

$\forall\ i$, there is only enough space in the first i rows for the no. $1, \dots, i$.

$\therefore$ in the first row all entries must be 1

$\Rightarrow$ in the second row all entries must be 2

$\therefore$ etc.

Get an SSYT of the form

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & & \\ 3 & 3 & 3 & & & \\ 4 & & & & & \\ 5 & & & & & \end{array}$$

the only SSYT of shape and type $(6,4,3,1,1)$

Schensted's row insertion process:

P : SSYT, k : positive integer

$P \leftarrow k$ is the "insertion of k into P"

Defined as follows:

- Let r be the largest integer such that $P_{1,r-1} \leq k$. (If $P_{11} > k$, let $r = 1$)
- If $P_{1,r}$ is not in the tableaux, simply place k at the end of the first row, and stop.
- Otherwise P has at least r columns, then replace P_{1r} by k.
- Then k "bumps up" $k' = P_{1r}$ (its original value) to the second row (i.e., insert k' into the second row as described above).
- Continue until an element is inserted at the end of a row (possibly the first element of a new row).

$$\begin{array}{ccccccc}
1 & 1 & 2 & 4 & 5 & 5 & 6 \\
2 & 3 & 3 & 6 & 6 & 8 & \\
4 & 4 & 6 & 8 & & & \\
6 & 7 & & & & & \\
8 & 9 & & & & &
\end{array} \quad \longleftarrow 4$$

"4 bumps up 5"

$$\begin{array}{ccccccc}
1 & 1 & 2 & 4 & \underline{4} & 5 & 6 \\
2 & 3 & 3 & 6 & 6 & 8 & \longleftarrow 5 \\
4 & 4 & 6 & 8 & & & \\
6 & 7 & & & & & \\
8 & 9 & & & & &
\end{array}$$

"5 bumps up 6"

$$\begin{array}{ccccccc} 1 & 1 & 2 & 4 & \underline{4} & 5 & 6 \\ 2 & 3 & 3 & \underline{5} & 6 & 8 & \\ 4 & 4 & 6 & 8 & & & \\ 6 & 7 & & & & & \\ 8 & 9 & & & & & \end{array} \qquad \leftarrow 6$$

"6 bumps up 8"

$$\begin{array}{ccccccc} 1 & 1 & 2 & 4 & \underline{4} & 5 & 6 \\ 2 & 3 & 3 & \underline{5} & 6 & 8 & \\ 4 & 4 & 6 & \underline{6} & & & \\ 6 & 7 & \boxed{} & & & & \\ 8 & 9 & & & & & \end{array} \qquad \leftarrow 8$$

"8 comes to rest at the end of the 4th column"

$$\begin{array}{ccccccc} 1 & 1 & 2 & 4 & \underline{4} & 5 & 6 \\ 2 & 3 & 3 & \underline{5} & 6 & 8 & \\ 4 & 4 & 6 & \underline{6} & & & \\ 6 & 7 & \underline{8} & & & & \\ 8 & 9 & & & & & \end{array}$$

underlined entries mark the <u>insertion path</u>

~~$I(P,k)$~~ $I(P \leftarrow 4) = \{(1,5),(2,4),(3,4),(4,3)\}$

<u>Lemma</u>: The insertion path always moves (weakly) to the left, i.e., $(r,s), (r+1,t) \in I(P \leftarrow k)$ then $t \leq s$.

<u>Proof</u>. Suppose $(r,s) \in I(P \leftarrow k)$. Then either $P_{r+1,s} > P_{rs}$ (cols. are strictly increasing) or $P_{r+1,s}$ does not exist.

~~Td~~ In the first case, P_{rs} can not get bumped to the right of $P_{r+1\,s}$. ~~without violating~~

In the second case if P_{rs} got bumped to the right of $P_{r+1\,s}$ we would have a gap in $(P \leftarrow k)$.

Lemma: Let P be an SSYT, and $j \leq k$. Then $I(P \leftarrow j)$ lies strictly to the left of $I(I(P \leftarrow j) \leftarrow k)$.

(i.e., $(r,s) \in I(P \leftarrow j)$, $(r,t) \in I(I(P \leftarrow j) \leftarrow k)$ then $t > r$.)

Moreover, $I(I(P \leftarrow j) \leftarrow k)$ does not extend below the bottom of $I(P \leftarrow j)$.

Proof: Since a number can only bump a strictly larger number, k is inserted in the first row strictly to the right of j.

Since the first row of P is weakly increasing, the number bumped by k is $\geq$ the number bumped by j. Thus we can repeat this argument with the second row and so on.

The bottom element of $I(P \leftarrow j)$ was inserted at the end of its row. If $I(I(P \leftarrow j) \leftarrow k)$ were still going on, the element bumped up by it would end up at the end of the same row.

Corollary: $I(P \leftarrow k)$ is also a SSYT.

Clearly rows are always weakly increasing. Need to only check that cols. are strictly increasing. But b can only bump up a no. c strictly larger than b, which goes to a col. weakly to the left of b. So ~~if~~ c ~~occurs~~ appears below a number strictly smaller than it.

Let $A = (a_{ij})$ be a matrix with non-negative integer ~~matrix~~ entries. Associate a generalized permutation to A: i.e., a two line array

$$w_A = \begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & j_m \end{pmatrix}$$

with $i_1 \le i_2 \le \cdots \le i_m$, and if $i_r = i_{r+1}$ then $j_r \le j_{r+1}$, such that $\binom{i}{j}$ occurs as many times as the value of a_{ij}.

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ gives

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}$$

We now associate with A (or w_A) a pair of SSYT of the same shape:

Given w_A, begin with $(P(0), Q(0)) = (\emptyset, \emptyset)$.

If $(P(t), Q(t))$ are defined

(a) $P(t+1) = P(t) \leftarrow j_{t+1}$;

(b) $Q(t+1)$ is obtained from $Q(t)$ by adding a box containing i_{t+1} in such a way that $P(t+1)$ & $Q(t+1)$ have the same shape.

Example: $\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}$

i	$P(i)$	$Q(i)$
1	1	1
2	1 3	1 1
3	1 3 3	1 1 1
4	1 2 3 3	1 1 1 2
5	1 2 2 3 3	1 1 1 2 2
6	1 1 2 2 3 3	1 1 1 2 2 3
7	1 1 2 3 2 3 3	1 1 1 3 2 2 3

We denote this correspondence by $A \xrightarrow{RSK} (P,Q)$.

<u>Theorem:</u> $\forall A$, if $A \xrightarrow{RSK} (P,Q)$ then P and Q are SSYT. If $\sum_j a_{ij} = \lambda_i$ and $\sum_i a_{ij} = \mu_j$ then P is of type μ and Q is of type λ. This correspondence is a bijection from the set of all matrices ~~with~~ in $M_{\lambda\mu}$ and pairs of SSYT (P,Q) such that ~~shape $Q = \lambda$, shape $P = \mu$~~ Q has type λ, P has type μ & P & Q have the <u>same</u> shape.

<u>Corollary:</u> $|M_{\lambda\mu}| = \sum_{\substack{\nu \trianglerighteq \lambda \\ \nu \trianglerighteq \mu}} K_{\nu\lambda} K_{\nu\mu}$.

<u>Corollary:</u> For every partition λ of n, $\exists$ simple rep. V_λ of S_n such that

$$K[X_\lambda] = \bigoplus_{\mu \trianglerighteq \lambda} V_\mu^{\oplus K_{\mu\lambda}}.$$