

Theorem: Let $(P, \leq)$ be a ^[finite] partially ordered set and $\{U_\lambda\}_{\lambda \in P}$ be a collection of completely reducible representations of G . Suppose $\dim \text{Hom}_G(U_\lambda, U_\mu) = M_{\lambda\mu} \quad \forall \lambda, \mu \in P$ and there exist non-negative integers $K_{\mu\lambda}$ for $\mu \leq \lambda$ such that $K_{\lambda\lambda} = 1 \quad \forall \lambda \in P$ and

$$M_{\lambda\mu} = \sum_{\nu \leq \lambda, \nu \leq \mu} K_{\nu\lambda} K_{\nu\mu} \quad \forall \lambda, \mu \in P \quad (\text{RSK})$$

Then $\forall \lambda \in P \exists$ simple rep. V_λ such that

$$U_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu^{\oplus K_{\mu\lambda}} \quad \forall \lambda \in P. \quad (*)$$

Proof: Take $\lambda_0 \in P$ minimal.

(RSK) $\Rightarrow \dim \text{End}_G U_{\lambda_0} = 1$ whence $V_{\lambda_0} := U_{\lambda_0}$ is a simple rep.

(RSK) $\Rightarrow \dim \text{Hom}(U_{\lambda_0}, U_\lambda) = K_{\lambda_0\lambda} \quad \forall \lambda \in P$,

so V_{λ_0} has multiplicity $K_{\lambda_0\lambda}$ in $U_\lambda \quad \forall \lambda \in P$.

$\therefore \exists$ representations U_λ^0 in which V_{λ_0} does not occur, and $U_\lambda = U_\lambda^0 \oplus V_{\lambda_0}^{\oplus K_{\lambda_0\lambda}}$.

Let $P^0 = P - \{\lambda_0\}$. Consider the collection $\{U_\lambda^0\}_{\lambda \in P^0}$.

Then $\dim \text{Hom}_G(U_\lambda^0, U_\mu^0) = \dim \text{Hom}_G(U_\lambda, U_\mu) - K_{\lambda_0\lambda} K_{\lambda_0\mu}$

$$= \sum_{\substack{\lambda_0 \leq \nu \leq \lambda \\ \lambda_0 \leq \nu \leq \mu}} K_{\nu\lambda} K_{\nu\mu}.$$

The theorem therefore follows by induction on $|P|$.

③

Semistandard Young Tableaux (SSYT) of shape λ is an array $T = (T_{ij})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq \lambda_i}}$ of positive integers weakly increasing along rows and strictly increasing along columns.

Example:

$$T = \begin{array}{cccccc} 1 & 1 & 1 & 3 & 4 & 4 \\ 2 & 4 & 4 & 5 & 5 & \\ 5 & 5 & 7 & & & \\ 6 & 9 & 9 & & & \end{array}$$

is a SSYT of shape $(6, 5, 3, 3)$.

Type: $(\alpha_1, \alpha_2, \dots)$ where α_i is the no. of times i occurs in T .

In the above example: $\text{type}(T) = (3, 1, 1, 4, 4, 1, 1, 0, 2)$

Lemma Example: There are two SSYT's of shape $(2, 1)$ and type $(2, 1)$: $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$ & $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$.

Kostka numbers: $K_{\mu\lambda} = \text{no. of SSYT of shape } \mu \text{ and type } \lambda$.

Example: ① If $\lambda = (n)$, then $\exists!$ SSYT of shape μ and type λ for all compositions μ of n .

② If $\lambda = (1^n)$, then there are no SSYT's of shape μ and type λ unless $\mu = \lambda$.



$\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array}$

Let μ and λ be partitions of n .

Lemma: $K_{\mu\lambda} \geq 0 \Rightarrow \mu_1 + \dots + \mu_i \geq \lambda_1 + \dots + \lambda_i$ for $i=1, \dots, l$.

Proof: Each col. is strictly increasing $\Rightarrow$ in an SSYT, i must occur in the first i rows (the numbers $1, \dots, i-1$ can occur at most once in its column).

$$\therefore \mu_1 + \dots + \mu_i \geq \lambda_1 + \dots + \lambda_i \quad \forall i.$$

(the first i rows must have enough space for all occurrences of $1, \dots, i$.)

Lemma: $K_{\lambda\lambda} = 1 \quad \forall \lambda$.

$\forall i$, there is only enough space in the first i rows for the no. $1, \dots, i$.

$\therefore$ in the first row all entries must be 1

$\Rightarrow$ in the second row all entries must be 2

$\therefore$ etc.

Get an SSYT of the form

1	1	1	1	1	1
2	2	2	2		
3	3	3			
4					
5					

the only SSYT of shape and type $(6,4,3,1)$

Schensted's row insertion process:

P : SSYT, k : positive integer

$P \leftarrow k$ is the "insertion of k into P "

(33)

Defined as follows:

- Let r be the largest integer such that $P_{1,r-1} \leq k$. (If $P_{1,1} > k$, let $r = 1$)
- If $P_{1,r}$ is not in the tableaux, simply place k at the end of the first row, and stop.
- Otherwise P has at least r columns, then replace $P_{1,r}$ by k .
- Then k "bumps up" $k' = P_{rr}$ (its original value) to the second row (i.e., insert k' into the second row as described above).
- Continue until an element is inserted at the end of a row. (possibly the first element of a new row).

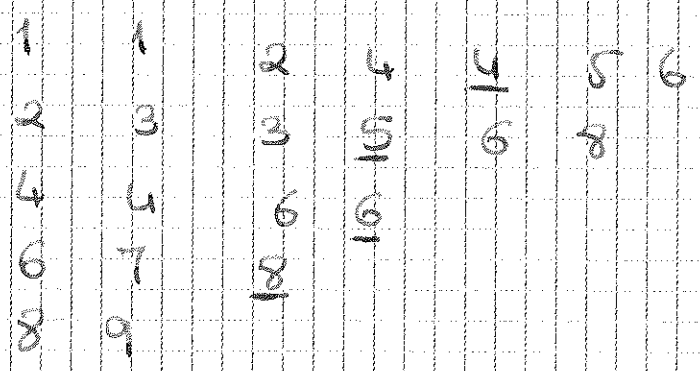
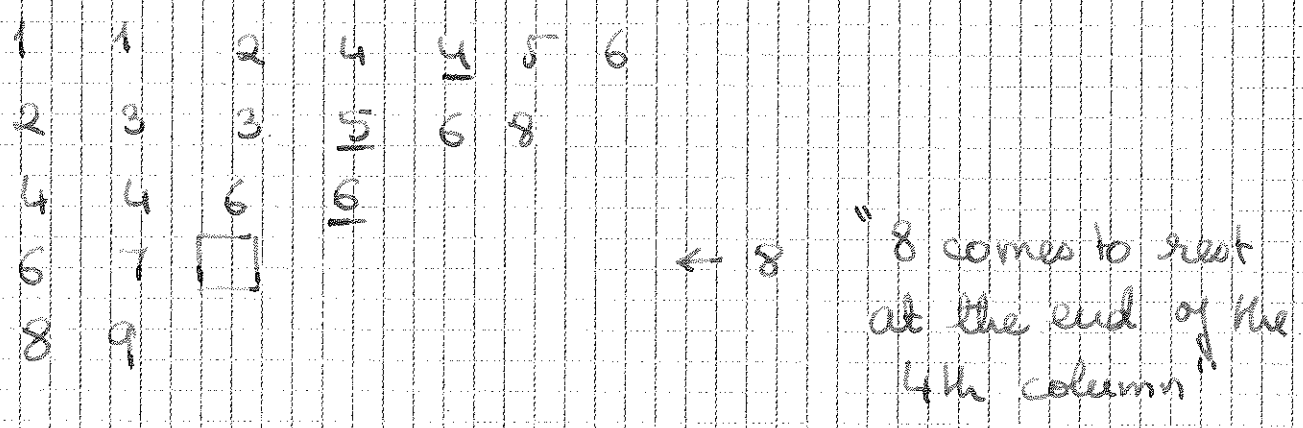
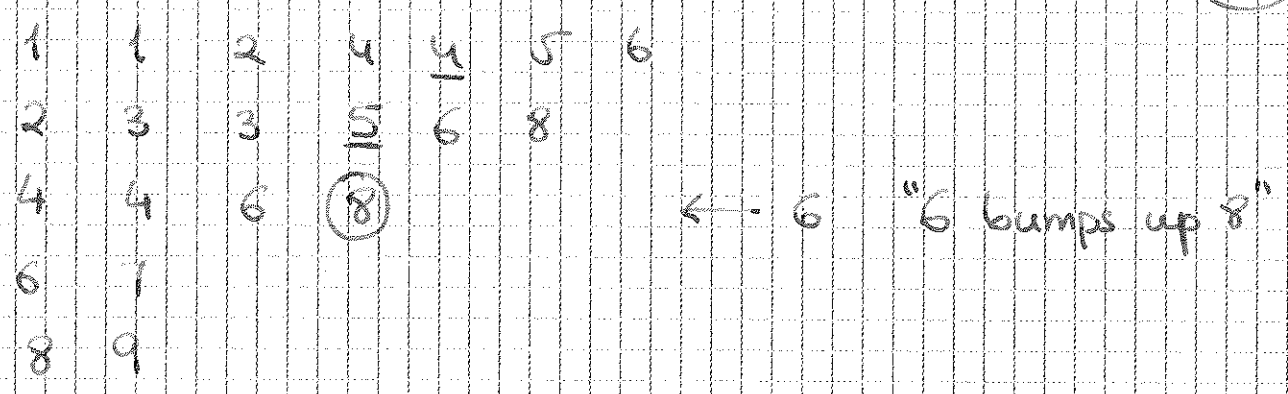
1	1	2	4	(5)	5	6		← 4
2	3	3	6	6	8			
4	4	6	8					
6	7							
8	9							

"4 bumps up 5"

1	1	2	4	<u>4</u>	5	6		
2	3	3	(6)	6	8			
4	4	6	8					
6	7							
8	9							

← 5

"5 bumps up 6"



underlined entries mark the insertion path

~~$I(P, k)$~~ $I(P \leftarrow 4) = \{(1, 5), (2, 4), (3, 4), (4, 3)\}$

Lemma: The insertion path always moves (weakly) to the left, i.e., $(r, s), (r+1, t) \in I(P \leftarrow k)$ then $t \leq s$.

Proof: Suppose $(r, s) \in I(P \leftarrow k)$. Then either $P_{r+1, s} > P_{rs}$ (cols. are strictly increasing) or $P_{r+1, s}$ does not exist.

(35)

~~If~~ In the first case, P_r s can not get bumped to the right of P_{r+1} s. ~~without violating~~

In the second case if P_r s got bumped to the right of P_{r+1} s we would have a gap in $(P \leftarrow k)$.

Lemma: Let P be an SSYT, and $j \leq k$.

Then $I(P \leftarrow j)$ lies strictly to the left of $I(I(P \leftarrow j) \leftarrow k)$.

(i.e., $(r, s) \in I(P \leftarrow j)$, $(r, t) \in I(I(P \leftarrow j) \leftarrow k)$ then $t > r$.)

Moreover, $I(I(P \leftarrow j) \leftarrow k)$ does not extend below the bottom of $I(P \leftarrow j)$.

Proof: Since a number can only bump a strictly larger number, k is inserted in the first row _{strictly} to the right of j .

Since the first row of P is weakly increasing, the number bumped by k is $\geq$ the number bumped by j . Thus we can repeat this argument with the second row and so on.

The bottom element of $I(P \leftarrow j)$ was inserted at the end of its row. If $I(I(P \leftarrow j) \leftarrow k)$ were still going on, the element bumped up

by it would end up at the end of the same row.

Corollary: $I(P \leftarrow k)$ is also an SSYT.

Clearly rows are always weakly increasing. Need to only check that cols. are strictly increasing. But b can only bump up a no. c strictly larger than b , which goes to a col. weakly to the left of b . So if c occurs appears below a number strictly smaller than it.

Let $A = (a_{ij})$ be a matrix with non-negative integer matrix entries. Associate a generalized permutation to A : i.e., a two line array

$$w_A = \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix}$$

with $i_1 \leq i_2 \leq \dots \leq i_m$, and if $i_r = i_{s+1}$ then $j_r \leq j_{r+1}$, such that $\begin{pmatrix} i \\ j \end{pmatrix}$ occurs as many times as the value of a_{ij} .

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ gives

$$w_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}$$

37

We now associate with A (or w_A) a pair of SSYT of the same shape:

Given w_A , begin with $(P(0), Q(0)) = (\emptyset, \emptyset)$.

If $(P(t), Q(t))$ are defined

- $P(t+1) = P(t) \leftarrow j_{t+1}$;
- $Q(t+1)$ is obtained from $Q(t)$ by adding a box containing i_{t+1} in such a way that $P(t+1) \not\prec Q(t+1)$ have the same shape.

Example: $\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}$

i	$P(i)$	$Q(i)$
1	1	1
2	1 3	1 1
3	1 3 3	1 1 1
4	1 2 3 3	1 1 1 2
5	1 2 2 3 3	1 1 1 2 2
6	1 1 2 2 3 3	1 1 1 2 2 3
7	1 1 2 3 2 3 3	1 1 1 3 2 2 3

We denote this correspondence by $A \xrightarrow{RSK} (P, Q)$.

Theorem: $\forall A$, if $A \xrightarrow{RSK} (P, Q)$ then P and Q are SSYT. If $\sum a_{ij} = \lambda_i$ and $\sum_i a_{ij} = \mu_j$ then P is of type μ and Q is of type λ . This correspondence is a bijection from the set of all matrices ~~with~~ in $M_{\lambda\mu}$ and pairs of SSYT (P, Q) such that ~~shape $Q = \lambda$, shape $P = \mu$~~ Q has type λ , P has type μ & $P \& Q$ have the same shape.

Corollary: $|M_{\lambda\mu}| = \sum_{\substack{\lambda \vdash n \\ \mu \vdash n}} K_{\lambda\lambda} K_{\mu\mu}.$

Corollary: For every partition λ of n , $\exists$ simple rep. V_λ of S_n such that

$$K[\lambda_2] = \bigoplus_{\mu \vdash n} V_\mu \otimes K_{\mu\lambda}.$$