REPRESENTATION THEORY

ASSIGNMENT DUE ON OCTOBER 10, 2011

K is an algebraically closed field of characteristic $> n$

(1) Let G be a finite group, with representatives $g_1, \ldots, g_n$ for conjugacy classes. Let K be an algebraically closed field whose characteristic does not divide $|G|$. Let $V_1, \ldots, V_n$ be representatives for isomorphism classes of simple representations. Let X denote the matrix whose (i, j)th entry is

$$X_{ij} = \text{trace}(g_j; V_i).$$

Let Z be the diagonal matrix with diagonal entries $z_1, \ldots, z_r$, where z_i is the cardinality of the centralizer $Z_G(g_i)$ of g_i. Let J be the matrix whose (i, j)th entry is 1 if g_i^{-1} is conjugate to g_j, and 0 otherwise. Show that

$$XZ^{-1}JX' = I.$$

(2) In the setting of the previous problem prove that, for $g, h \in G$,

$$\sum_{k=1}^{n} \text{trace}(g; V_k)\,\text{trace}(h, V_k) = \begin{cases} |Z_G(g)| & \text{if } g \text{ is conjugate to } h^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The above identities are called the *dual orthogonality relations.*

(3) Prove the *Murnaghan-Nakayama rule*, which expresses the Schur functions as a linear combination of power sum symmetric functions:

$$s_\lambda = \sum_{\nu \vdash n} \frac{p_\nu \,\text{trace}(w_\nu, V_\lambda)}{z_\mu} \text{ for all } \lambda \vdash n,$$

where $z_\mu = 1^{m_1(\mu)} m_1! 2^{m_2(\mu)} m_2! \cdots$ with $m_i(\mu)$ denoting the number of times that i occurs in the partition μ.

(4) Let R_K^n denote the space of K-valued class functions on S_n. For a representation (ρ, V) of S_n, we shall use the notation $\text{ch}(V)$ to denote the class function $g \mapsto \text{trace}(\rho(g); V)$. Let $\text{ch} : R_K^n \to \lambda_K^n$ be the linear map (known as the *characteristic map*) be defined by requiring that $\text{ch}(V_\lambda) = s_\lambda$ for each $\lambda \vdash n$. Show that $\text{ch}(K[X_\lambda]) = h_\lambda$ and $\text{ch}(K[X_\lambda] \otimes \epsilon) = e_\lambda$ for each $\lambda \vdash n$.

(5) Using the notation of the previous problem, define a bilinear from on Λ_K^n by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Show that $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, $\langle h_\lambda, e_\mu \rangle = N_{\lambda\mu}$, and that $\langle h_\lambda, h_\mu \rangle = M_{\lambda\mu}$ for all partitions λ and μ of n.

(6) Using the notation of the previous problems, show that $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$.

(7) Let $\omega : \Lambda_K^n \to \Lambda_K^n$ be the linear map defined by $\omega(h_\lambda) = e_\lambda$. Show that $\mathrm{ch}(V \otimes \epsilon) = \omega(\mathrm{ch}(V))$.

(8) If $f : S_n \to K$ is a class function, then

$$\mathrm{ch}(f) = \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\rho(w)},$$

where $\rho(w)$ denotes the partition corresponding to the cycle decomposition of w.